For \( A \in M_n(\mathbb{F}) \)

- \( \lambda \) is an eigenvalue of \( A \) \( \iff \) \( \ker (A-\lambda I_n) \neq \{0\} \)
- \( \iff \) \( A-\lambda I_n \) is singular \( \iff \) \( \det (A-\lambda I_n) = 0 \)

**Def:** The polynomial \( p_A(\lambda) = \det (A-\lambda I_n) \) is the characteristic polynomial of \( A \).

**Prop:** If \( A \in M_n(\mathbb{F}) \) then \( p_A(\lambda) \) is a polynomial of degree \( n \) with leading term \( (-\lambda)^n \).

**Pf:**

Let \( B = A-\lambda I_n \), so \( p_A(\lambda) = \det B = \sum_{\sigma \in S_n} sgn(\sigma) b_{\sigma(1)} \cdots b_{\sigma(n)} \)

For a given \( \sigma \in S_n \)

- let \( s_{\sigma_i} \subseteq \{1, \ldots, n\} \) be the fixed pts of \( \sigma \)
  - i.e., \( s_{\sigma_i} = \{ i : \sigma(i) = i \} \)

Then \( b_{\sigma(1)} \cdots b_{\sigma(n)} = \prod_{\sigma \in s_{\sigma_i}} (a_{\sigma_i} - \lambda) \prod_{\sigma \notin s_{\sigma_i}} a_{\lambda \sigma_i} \).

Thus each term in the sum computing \( \det (B) \) is a polynomial in \( \lambda \) of degree \( \# s_{\sigma_i} \)

The highest possible degree is \( n \) which corresponds to \( \# s_{\sigma_i} = n \)

i.e., to \( \sigma = i \), for which \( b_{i\sigma(1)} \cdots b_{i\sigma(n)} = \prod_{\sigma \in i} (a_{\sigma_i} - \lambda) = (-\lambda)^n + \text{lower order} \).
If $A \in M_n(F)$ has $n$ distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$

then $p_A(x) = \prod_{j=1}^{n}(\lambda_j - x)$.

Indeed, it is divisible by $(\lambda_j - x)$ for each $\lambda_j$,

so $\prod_{j=1}^{n}(\lambda_j - x)$ divides $p_A(x)$ but they have the same

degree, $n$, & the same leading coefficient so they must be equal.

Every field $F$ is contained in an algebraically closed field

$\bar{K}$ as a subfield (i.e. with the same operations).

The advantage of working in an algebraically closed field

is that the characteristic polynomial factors into linear terms.

E.g. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ then $p_A(x) = x^2 + 1$ has no solutions in $\mathbb{R}$

but factors into $(x+i)(x-i)$ in $\mathbb{C}$.

\textbf{Def} F alg. closed & $A \in M_n(F)$ with distinct eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$.

The characteristic polynomial of $A$ factors as $p_A(x) = (-1)^n (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k}$.

The power $m_j$ is referred to as the \textit{algebraic} multiplicity

of $\lambda_j$ as an eigenvalue of $A$.

\textbf{Eg}\textit{, The polynomial $p(x) = x^6 - 5x^5 + 6x^4 + 4x^3 - 8x^2 = x^2(x+1)(x-2)^3$

has 3 distinct roots but 6 roots if we count them with

multiplicity: 0, 0, -1, 2, 2, 2.
If \( A \) is upper triangular, the eigenvalues of \( A \) are the elements on the diagonal & the multiplicity of an eigenvalue is the number of times it appears on the diagonal.

**Remark:** The characteristic polynomial of \( A \), the eigenvalues of \( A \), and their multiplicities are all invariants of \( A \).

If \( \mathbb{F} \) is algebraically closed, any \( A \in M_n(\mathbb{F}) \) is similar to an upper triangular matrix & hence
\[
\text{tr}(A) = \lambda_1 + \ldots + \lambda_n, \quad \det(A) = \lambda_1 \ldots \lambda_n
\]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \), repeated with multiplicity.

In particular, the coeff of \( x^{n-1} \) in \( p_A(x) \) is \( (-1)^{n-1} \text{tr}(A) \)
\( B \) the coeff of \( x^0 \) in \( p_A(x) \) is \( \det(A) \).
This is true over any field \( \mathbb{F} \).

**Theorem (Cayley–Hamilton)**: If \( A \in M_n(\mathbb{F}) \) then \( p_A(A) = 0 \).

\[ p_A(A) = 0 \]

Note that we can not just plug in \( A \) into \( p_A(x) = \det(A-xI_n) \)
Instead, we will see that \( p_A(A) v = 0 \) for all \( v \in \mathbb{F}^n \).
We may assume that \( \mathbb{F} \) is algebraically closed
as otherwise we can pass to a larger alg. closed field \( \mathbb{K} \).
Note that if \( A \) & \( B \) are similar then \( p_A(A) = 0 \iff p_B(B) = 0 \)
since \( p_A(x) = p_B(x) \) & if \( B = SAS^{-1} \)
then \( p_A(B) = p_A(SAS^{-1}) = S p_A(A) S^{-1} \).
Thus it is enough to show that $p_n(A) = 0$ for $A$ upper triangular.
In this case, $p_A(x) = (a_{n-1} - x) \cdots (a_{n-n} - x)$ and $p_A(A) = (a_{n}I - A) \cdots (a_{n-n}I - A)$.
Note $(a_{j}I - A)(e_j) \in \text{span} \{e_{k}, \ldots, e_{j+1}\} \neq 0$
so inductively $(a_{j}I - A) \cdots (a_{j+k}I - A)(e_k) = 0$ for $k \in \{1, \ldots, j\}$
and hence $p_A(A) = (a_{n}I - A) \cdots (a_{n-n}I - A)$ vanishes on $(e_n, \ldots, e_n)$,
loc $p_A(A) = 0$.

**Rule:** If $A \in M_n(F)$ is invertible then $A^{-1}$ can be written
as a polynomial in $A$ of degree $\leq n-1$.

E.g., $A = \begin{bmatrix} 2 & 7 \\ 0 & 3 \end{bmatrix}$, $p_A(x) = (2-x)(3-x)$
$\implies 6 - 5x + x^2$

We know that $p_A(A) = 0 \iff A^n = 0$ but $A^2 - 5A + 6I = 0$
thus $A^2 - 5A = -6I \iff A(A - 5I) = -6I$
$\iff A \left( \frac{1}{6} (A - 5I) \right) = I$
so $A^{-1} = \frac{1}{6} (A - 5I)$. 