Linear Algebra  Computing Determinants  Lecture 36

Last Time: There is 1 & only 1 function \( D_n : M_n (F) \to F \)
that is alternating, multilinear & normalized so that \( D(1) = 1 \).

This Time: Properties of the determinant.

Laplace expansion of the determinant along a row:
For any \( i \in \{1, \ldots, n\} \),
\[
\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).
\]

Prop: If \( A \) is upper triangular then \( \det(A) = a_{nn} \).
Proof: We'll show this by induction on \( n \). It's clear if \( n = 1 \).
Assuming it's true for upper triangular matrices in \( M_{n-1} (F) \),
we can expand along the last row of \( A \):
\[
\det A = \sum_{j=1}^{n} (-1)^{n+j} a_{nj} \det(A_{nj}) = (-1)^n a_{nn} \det(A_{nn}) = a_{nn} (a_{11} \ldots a_{nn}).
\]

Prop: For any \( A \in M_n (F) \), \( \det(A) = \det(A^T) \).
Proof: Define \( \xi : M_n (F) \to F \) by \( \xi(A) = \det(A^T) \).
We'll show by induction that \( \xi \) is a determinant function.
For \( n = 1 \), \( A = A^T \) so this is clear.
\( \xi \) is multilinear: Let \( A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \), suppose \( a_i = a + c_i, b_i, c_i \in F \).
\[
\xi(A) = \det(A^T) = \sum_{j=1}^{n} (-1)^{n+j} a_{jk} \det((A^T)_{kj}) = \sum_{j=1}^{n} (-1)^{n+j} (b_j + c_j) \det((A^T)_{kj})
\]
\[
= \sum_{j=1}^{n} (-1)^{n+j} b_j \det((A^T)_{kj}) + c \sum_{j=1}^{n} (-1)^{n+j} c_j \det((A^T)_{kj})
\]
\[
= \xi \left( \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right) + c \xi \left( \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \right)
\]

\( \xi \) is normalized: \( \xi(I) = \det(I^T) = \det(I) = 1 \).
\( \xi \) is alternating: if \( n=2 \), \( \det\left( \begin{bmatrix} a & c \\ c & a \end{bmatrix} \right) = \det(\begin{bmatrix} a & c \\ a & c \end{bmatrix}) = ac-cc = 0 \).

Suppose \( n \geq 3 \), \( a_k = a_k \neq i \neq k, k' \).

\[
\xi(A) = \det(A^T) = \sum_{j=1}^{n} (-1)^{a_{ji}} \det(A^T)_{ji} = \sum_{j=1}^{n} (-1)^{a_{ji}} a_{ji} \det(A_{ji})^T = \eta \sum_{j=1}^{n} (-1)^{a_{ji}} a_{ji} \det(A_{ji}) = 0
\]

Hence by uniqueness \( \xi(A) = \det(A) \).

It follows that the determinant is also an alternating multilinear function of the rows of a matrix.

Also we have a Laplace expansion along columns:

For any \( j \in \{1, \ldots, n\} \), \( \det(A) = \xi_j \sum_{j=1}^{n} (-1)^{a_{ji}} a_{ij} \det(A_{ij}) \)

Another consequence is that we can compute the determinant using row operations:

Given \( A \in \mathbb{M}_n(\mathbb{F}) \):

1. Use row operations \( R_1 \) & \( R_3 \) to convert \( A \) into upper triangular \( B \)
2. Let \( k \) be the number of times two rows were switched
3. \( \det A = (-1)^k b_{11} \cdots b_{nn} \)

\( \text{Def} \) A permutation of \( \{1, \ldots, n\} \) is a bijection \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \). That is, \( (\sigma(1), \ldots, \sigma(n)) \) is the same set as \( (1, \ldots, n) \) possibly in a different order. The set of all permutations of \( \{1, \ldots, n\} \) is denoted \( S_n \). The identity permutation is denoted \( i \).
Every permutation $\sigma \in S_n$ can be represented by a matrix $A_\sigma \in M_n(\mathbb{F})$. The entries of $A_\sigma$ are $a_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise}. \end{cases}$

For example, for $\sigma \in S_4$ given by $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 4$, $\sigma(4) = 3$,

$$A_\sigma = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

Define $\text{sign}(\sigma) = \det A_\sigma \in \{-1, 1\}$. This satisfies $\text{sign}(\sigma) = \text{sign}(\sigma^{-1})$ and $\text{sign}(\sigma \circ \rho) = \text{sign}(\sigma) \cdot \text{sign}(\rho)$.

**Prop** For $A \in M_n(\mathbb{F})$, $\det A = \sum_{\sigma \in S_n} (\text{sign } \sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}$.

**Proof** Using multilinearity,

$$\det A = \det \left[ \sum_{\sigma \in S_n} \frac{1}{2^n} a_{\sigma(1)} e_{\sigma(1)} \cdots e_{\sigma(n)} \right] = \sum_{\sigma \in S_n} \frac{1}{2^n} a_{\sigma(1)} \cdots a_{\sigma(n)} \det \left[ e_{\sigma(1)} \cdots e_{\sigma(n)} \right].$$

Since the determinant is alternating, this sum can be taken over distinct indices, i.e., over permutations, i.e.,

$$\det A = \sum_{\sigma \in S_n} a_{\sigma(1)} \cdots a_{\sigma(n)} \det \left( e_{\sigma(1)} \cdots e_{\sigma(n)} \right).$$

The $(ij)$th entry of $\left( e_{\sigma(1)} \cdots e_{\sigma(n)} \right)$ is $1$ if $i = \sigma(j)$, $\sigma(i)$, otherwise it is $0$. Since $\det A_{\sigma^{-1}} = \det A_\sigma = \text{sign}(\sigma)$, we get the formula we wanted.