Linear Algebra: Triangulation

Lecture 22

Last Time: Similar matrices
This Time: The "best" matrix associated to a linear map

The simplest matrices are diagonal.

If \( \mathbb{L}(V) \) is diagonalizable,

\( \Rightarrow \) there is a basis of \( V \) consisting of eigenvectors of \( \mathbb{T} \).

\( \mathbb{A} \in M_n(\mathbb{F}) \) is diagonalizable

\( \Rightarrow \) there is a basis of \( \mathbb{F}^n \) consisting of eigenvectors of \( \mathbb{A} \).

In some ways, an upper triangular matrix is as simple as a diagonal one.

1) if \( \mathbb{A} \) is upper triangular then

i) if there are no zeros on the diagonal, then \( \mathbb{A} \) is in \( \mathbb{R} \text{E} \) \( \mathbb{F}^{t} \text{A} \) pivot in every column.

ii) if there are zeros on the diagonal, then the column containing the first zero on the diagonal does not have a pivot.

iii) eigenvalues of \( \mathbb{A} \) = diagonal entries of \( \mathbb{A} \).

There are matrices that do not have eigenvalues.

E.g., \( \mathbb{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2 (\mathbb{R}) \)

If \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix} \Rightarrow \lambda = \lambda \mathbb{A} \mathbb{B} = \lambda (-\mathbb{A}) = -\lambda \mathbb{A} \)

\( \lambda^2 + 1 \mathbb{A} + \mathbb{B} = 0 \) & since \( \lambda^2 + 1 \neq 0 \) \( \forall \lambda \in \mathbb{R} \Rightarrow \mathbb{A} = 0, \mathbb{B} = 0 \).
Def A field $\mathbb{F}$ is algebraically closed if every polynomial of degree $n > 0$

can be factored in the form

$$p(x) = b(x-c_1) \cdots (x-c_n)$$

for some $b, c_1, \ldots, c_n \in \mathbb{F}$.

Thm (Fund. Thm of Algebra) $\mathbb{C}$ is an algebraically closed field.

Prop If $V$ is a vector space of positive finite dim. over an algebraically closed field $\mathbb{F}$, every $T \in \mathcal{L}(V)$ has an eigenvalue.

Pf Let $n = \dim V > 0$, let $v \in V, v \neq 0$.

The list $(v, T(v), T^2(v), \ldots, T^n(v))$ has $n+1$ vectors, so it is lin. dep.

There are constants $a_0, \ldots, a_n \in \mathbb{F}$ not all zero s.t.

$$a_0 v + a_1 T(v) + \ldots + a_n T^n(v) = 0.$$

Suppose $a_k$ is the last coeff not equal to zero (i.e., $a_{k+1} = 0$).

Let $p(x) = a_0 + a_1 x + \ldots + a_k x^k$.

Then $p(x)$ has degree $k$ and $p(T)(v) = a_0 v + a_1 T(v) + \ldots + a_k T^k(v) = 0$.

Since $\mathbb{F}$ is alg. closed we can factor $p$.

$$p(x) = b(x-c_1) \cdots (x-c_k)$$

for some $b, c_1, \ldots, c_k \in \mathbb{F}$.

Hence $0 = a_0 v + a_1 T(v) + \ldots + a_k T^k(v) = b(T-c_1 I)(T-c_2 I) \cdots (T-c_k I)(v)$.

If $(T-c_k I)(v) = 0$ then $v$ is an eigenvector of $T$ and eigenvalue $c_k$.

Otherwise, let $v' = (T-c_k I)(v) \neq 0$.

This satisfies $(T-c_1 I) \cdots \underline{(T-c_{k-1} I)}(v') = 0$.

Continuing in this way we eventually have $w \neq 0$ s.t. $(T-c I)(w) = 0$.

Hence at least one of the $c_i$ is an eigenvalue of $T$. \(_\square\)
E.g., $A = \begin{bmatrix} i & -1 \\ -2 & i \end{bmatrix} \in M_2(\mathbb{C})$. Let's start with $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The set $\{v, Av, A^2v\} = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}\}$ is lin. dep.

Indeed, $v + A^2v = 0 = (I + A^2)(v)$.

The polynomial $1 + x^2$ factors as $(x-i)(x+i)$

so $(A-i\text{Id})(A+i\text{Id})(v) = 0$.

Either $(A-i\text{Id})(v) = 0$ or $(A+i\text{Id})(v) = v \neq 0$

$v$ is an eigenvector $\Rightarrow$ eigenvector $-i$

or $(A-i\text{Id})(v) = 0$ so $v'$ is an eigenvector $\Rightarrow$ eigenv. $i$

In this case $(A+i\text{Id})[1] = \begin{bmatrix} -i & 1+i \\ 1+i & -i \end{bmatrix} \neq 0$

& $A[\begin{bmatrix} -i \\ 1 \end{bmatrix}] = \begin{bmatrix} -1 \\ -i \end{bmatrix} = i[\begin{bmatrix} 1 \\ -i \end{bmatrix}]$ as expected.

**Rank:** $T \mathbb{C}^2(v) A \mathbb{C}^2(v) = \{v_1, \ldots, v_n\}$ is a basis of $V$

The $[T]_{\mathbb{C}^n}$ upper triangular $\Rightarrow T(v_j) \in \langle v_1, \ldots, v_{j-1} \rangle \forall j$.

**Thm:** If $V$ is a fin. dim. vector space over an alg. closed field $F$ & $T \in \mathcal{L}(V)$

there is a basis $\mathcal{B}$ of $V$ s.t. $[T]_{\mathcal{B}}$ is upper triangular.

**Pf:** Induction over $n = \dim V$. For $n = 0$ or 1 the theorem is obvious.

Suppose the theorem is known for spaces of dimension less than $n$.

From the proposition we know that there is $v \neq 0$ s.t. $T(v) = \lambda v$.

Let $U = \text{Range } (T-\lambda \text{Id})$

Note that $T$ restricts to a linear map $U \to U$.

Indeed, if $v \in U$ then $T(v) = (T-\lambda \text{Id})(v) + (\lambda \text{Id})(v)$.

On the other hand, since $\lambda$ is an eigenv. $T-\lambda \text{Id}$ is not injective.

By the rank-nullity theorem, $T-\lambda \text{Id}$ is not surjective, so $\text{null } U < n$. 
Since \( T|_{U} \in L(U) \) and \( \dim U < n \)
by inductive hypothesis we can find \( \mathcal{B}_U = \{u_1, \ldots, u_n\} \)
a basis of \( U \) s.t. \( T(u_j) \in \langle u_1, \ldots, u_j \rangle \) \( \forall j \).

Extend \( \mathcal{B}_U \) to a basis \( \mathcal{B} = \{u_1, \ldots, u_m, v_1, \ldots, v_k\} \) of \( V \).

Note that, for any \( j \in \{1, \ldots, k\} \),

\[
T(v_j) = (I - \lambda I) v_j + \lambda I v_j \in \langle u_1, \ldots, u_m, v_j \rangle \subseteq \langle u_1, \ldots, u_m, v_1, \ldots, v_k \rangle
\]

\( \in U \)

hence \( [T]_{\mathcal{B}} \) is upper triangular.

**Corollary** If \( F \) is alg. closed, \( A \in M_n(F) \), \( p \) is a polynomial
with coefficients in \( F \).

The eigenvalues of \( p(A) \) are of the form \( p(\lambda) \) where \( \lambda \) is an eigenvalue of \( A \).