Linear Algebra  Coordinates  Lecture 19

Last Time: The rank-nullity theorem
\[ \text{ker}(T) \oplus \text{im}(T) = \text{dim}(V) \]
This Time: The coordinates of a vector in a given basis

When we write a vector \( x \in \mathbb{F}^n \) as \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \)

what we are saying is that \( x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n \),
where \( e_1, e_2, \ldots, e_n \) is the standard (or canonical) basis of \( \mathbb{F}^n \).

One way of thinking about this is that \( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \)

is the "coordinate representation of \( x \) with respect to the standard basis."

which brings up the possibility of representing \( x \) using a different basis.

**Def** Let \( \mathcal{B} = (v_1, \ldots, v_n) \) be a basis of \( V \). For each \( u \in V \),
the coordinates of \( u \) with respect to the basis \( \mathcal{B} \) is the unique list of scalars \( a_1, \ldots, a_n \) s.t. \( u = a_1 v_1 + \ldots + a_n v_n \).
The coordinate representation \( [u]_{\mathcal{B}} \in \mathbb{F}^n \) of \( u \) with respect to \( \mathcal{B} \)
is \( [u]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \).

If \( V \) is an \( n \)-dimensional vector space over \( \mathbb{F} \) with basis \( \mathcal{B} \),
representing vectors in coordinates gives an explicit isomorphism \( C_{\mathcal{B}}: V \to \mathbb{F}^n \) between \( V \) and \( \mathbb{F}^n \).
\( C_{\mathcal{B}}(u) = [u]_{\mathcal{B}} \)
E.g., the vector space \( V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y - z = 0 \} \) is a 2-dimensional subspace of \( \mathbb{R}^3 \), so it is isomorphic to \( \mathbb{R}^2 \). One basis for it is \( B = \{ [1, 2], [0, 1] \} \) and, since we can write any vector in \( V \) as a linear combination of these two vectors in a unique way, this gives us a way to represent vectors in \( V \) by vectors in \( \mathbb{R}^2 \), we just keep track of the coefficients.

For example, \( v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in V \) can be written \( v = (\frac{1}{2}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\frac{1}{2}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

hence \( [v]_B = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \).

If \( T \in L(V, W) \) and \( V, W \) are finite dimensional, then a basis \( B_V = (v_1, ..., v_n) \) for \( V \) gives an isomorphism \( V \cong \mathbb{F}^n \) and a basis \( B_W = (w_1, ..., w_m) \) for \( W \) gives an isomorphism \( W \cong \mathbb{F}^m \). Since we know that every linear map in \( L(\mathbb{F}^n, \mathbb{F}^m) \) is given by multiplication by a matrix \( A \in \mathbb{M}_{m \times n}(\mathbb{F}) \), these isomorphisms allow us to represent \( T \) as matrix multiplication. Suppose for each \( v_j \in B_V \) we have \( T(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m \). The matrix of \( T \) with respect to \( B_V \) and \( B_W \) is \( A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m} \). We denote this matrix by \( [T]_{B_V, B_W} \).

When \( V = W \) and \( B_V = B_W = B \) we simply write \( [T]_B \) & refer to it as the matrix of \( T \) with respect to \( B \).
Just as before, we can find the matrix one column at a time. To find the $j$th column of $[T]_{\mathcal{B}_W,\mathcal{B}_W}$, we take the $j$th vector in $\mathcal{B}_W$, $v_j$, apply $T$, and then express the result $T(v_j)$ using the basis $\mathcal{B}_W$, $[T(v_j)]_{\mathcal{B}_W}$.

**Lemma.** Given $T \in \mathcal{L}(V,W)$, $\mathcal{B}_V$ a basis for $V$, $\mathcal{B}_W$ a basis for $W$, let $A = [T]_{\mathcal{B}_V,\mathcal{B}_W}$.

For any $u \in V$ we have $[T(u)]_{\mathcal{B}_W} = A[u]_{\mathcal{B}_V}$.

**Pf:** Denote $\mathcal{B}_V = (v_1, \ldots, v_n)$ and $\mathcal{B}_W = (w_1, \ldots, w_m)$ and recall that the entries $a_{ij}$ of $A$ satisfy $T(v_j) = a_{ij} w_i + \ldots + a_{mj} w_m$.

So, if $u$ has coordinates $[u]_{\mathcal{B}_V} = [\begin{array}{c} c_1 \\ \vdots \\ c_n \end{array}]$, i.e., $u = c_1 v_1 + \ldots + c_n v_n$, then $T(u) = T(c_1 v_1 + \ldots + c_n v_n)$

$$= \sum_{j=1}^n c_j T(v_j) = \sum_{j=1}^n c_j (\sum_{i=1}^m a_{ij} w_i) = \sum_{j=1}^n (\sum_{i=1}^m a_{ij} c_j) w_i.$$

This says that $[T(u)]_{\mathcal{B}_W} = \begin{bmatrix} \sum_{i=1}^m a_{1j} c_j \\ \vdots \\ \sum_{i=1}^m a_{mj} c_j \end{bmatrix}$, and this is $A[\begin{array}{c} c_1 \\ \vdots \\ c_n \end{array}]$.

Similarly, starting with a matrix $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, for every choice of basis $\mathcal{B}_V$ of an $n$-dimensional vector space $V$ and a choice of basis $\mathcal{B}_W$ of an $m$-dimensional vector space $W$, we get a linear map $T_A: \mathcal{L}(V,W) \rightarrow \mathcal{L}(W)$ by prescribing $[T_A]_{\mathcal{B}_W,\mathcal{B}_W} = A$.

If $V=W$ and $\mathcal{B}_V = \mathcal{B}_W = \mathcal{B}$ we denote this map by $T_A$. 


Note that we can not leave out any part of the notation $T_{A,\mathcal{B},\mathcal{B}^*}$.

For example, starting with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
& the standard basis $\mathcal{B} = (e_1, e_2)$
we get a different map than with $\mathcal{B}^* = (e_2, e_1)$.  

\[ T_{A,\mathcal{B}} = T_{A,\mathcal{B},\mathcal{B}} \quad \rightarrow \quad \text{reflects across } x\text{-axis} \]

\[ T_{A,\mathcal{B}^*} = T_{A,\mathcal{B}^*,\mathcal{B}^*} \quad \rightarrow \quad \text{reflects across } y\text{-axis} \]

\[ T_{A,\mathcal{B}^*,\mathcal{B}^*} \quad \rightarrow \quad \text{rotates } \frac{\pi}{2} \text{ counterclockwise} \]

E.g., the derivative defines a linear map $D : \mathcal{P}_n(\mathbb{R}) \rightarrow \mathcal{P}_{n-1}(\mathbb{R})$.

If we pick bases of $\mathcal{P}_n(\mathbb{R})$ & $\mathcal{P}_{n-1}(\mathbb{R})$, this map will be given by matrix multiplication.

If we use the standard bases of monomials
$(1, x, x^2, \ldots, x^n)$ for $\mathcal{P}_n(\mathbb{R})$ & $(1, x, \ldots, x^{n-1})$ for $\mathcal{P}_{n-1}(\mathbb{R})$
then D is given by the $n \times (n+1)$ matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. 
Thm: Let $V$ & $W$ be finite dimensional vector spaces and $T \in \text{End}(V,W)$.
There are bases $\mathcal{B}_V$ of $V$ & $\mathcal{B}_W$ of $W$ such that the matrix
$A = [T]_{\mathcal{B}_V, \mathcal{B}_W}$ is given by
$A_{ij} = \begin{cases} 1 & \text{if } i = j \leq \text{rank}(T) \\ 0 & \text{otherwise} \end{cases}$

**PF**

Recall from last class that if $(u_1, \ldots, u_k)$ is a basis of $\ker(T)$ & we extend it to a basis $(u_1, \ldots, u_k, v_1, \ldots, v_k)$ of $V$, then $(T(v_1), \ldots, T(v_k))$ is a basis of $\text{Range}(T)$.
If we extend this to a basis of $W$,
$(T(v_1), \ldots, T(v_k), w_1, \ldots, w_n),$
then the bases
$\mathcal{B}_V = (v_1, \ldots, v_k, u_1, \ldots, u_k),$
$\mathcal{B}_W = (T(v_1), \ldots, T(v_k), w_1, \ldots, w_n),$
have the property required.