LINEAR ALGEBRA  

RANK-NULLITY THM  

LECTURE 18

LAST TIME: Dimension

THIS TIME: Dimensions of subspaces associated to a linear transformation

If \( V, W \) are vector spaces over a field \( F \), \( T \in \mathcal{L}(V,W) \), we have defined
\[
\ker(T) \subseteq V \quad \& \quad \text{Range}(T) \subseteq W
\]
and seen that these are subspaces.
Their dimensions are called \( \text{rank}(T) = \dim \text{Range}(T) \)
\& \( \text{null}(T) = \text{nullity}(T) = \dim \ker(T) \).

If \( A \) is a matrix & \( T \) is the associated linear map,
then \( \text{rank}(A) = \text{rank}(T) = \dim C(A) \)
\& \( \text{null}(A) = \text{null}(T) = \dim \ker(A) \).

Thm (Rank-Nullity Theorem)
If \( A \in \mathcal{M}_{m \times n}(F) \) then \( \text{rank}(A) + \text{null}(A) = n \).
More generally, whenever \( T \in \mathcal{L}(V,W) \) and \( \dim V < \infty \),
\( \text{rank}(T) + \text{null}(T) = \dim V \).

PF
We'll give one proof for matrices & then one proof for general linear maps, but note that the statement for linear maps includes the case of matrices.
So consider $A \in M_{m,n}(\mathbb{F})$.

We know that

\[
\text{rank}(A) = \dim \text{C}(A) = \# \text{ of pivots in } \text{RREF}(A)
\]

\[
\text{null}(A) = \dim \text{ker}(A) = \dim \{x \in \mathbb{F}^n : Ax = 0\}.
\]

Suppose, for simplicity, that the pivot variables of \( \text{RREF}[A:0] \) are \( x_1, \ldots, x_k \) and \( x_{k+1}, \ldots, x_n \) are free.

Then we can write the elements of \( \text{ker}(A) \) as linear combinations of vectors with coefficients \( x_{k+1}, \ldots, x_n \):

\[
\text{ker}A = \left\{ \begin{bmatrix} x_1 \\ x_k \\ \vdots \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \left[ \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix} \right] x_1 + \left[ \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix} \right] x_2 + \ldots + \left[ \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_{k+1} \\ \vdots \\ c_n \end{bmatrix} \right] x_n : x_{k+1}, \ldots, x_n \in \mathbb{F} \right\}
\]

so \( \text{null}(A) = \dim \text{ker}(A) = \# \text{ free variables in } Ax = 0 \).

Thus \( \text{rank}(A) + \text{null}(A) = \# \text{ cols. w/ pivots} + \# \text{ cols. w/o pivots} = \# \text{ cols. of } A = n \).

Now for the general case, consider \( T \in L(V, W), \dim V = m \).

Since \( \text{ker}(T) \subseteq V \), \( \text{ker}(T) \) is finite dimensional.

Let \( \mathcal{B} = (u_1, \ldots, u_k) \) be a basis of \( \text{ker}(T) \).

& extend it to \( \mathcal{B}' = (u_1, \ldots, u_k, v_1, \ldots, v_l) \) a basis of \( V \).

Claim: \( \{T(v_1), \ldots, T(v_l)\} \) is a basis of \( \text{Range}(T) \).

Once we establish the claim, we will be done.

Since then \( \text{null}(T) + \text{rank}(T) = k + l = \dim V \).
To establish the claim, we need to show that 
\[ \{T(v_1), \ldots, T(v_k)\} \text{ is lin. indep \& spans } \text{Range}(T). \]
To check lin. ind., assume  
\[ 0 = a_1 T(v_1) + \cdots + a_k T(v_k) = T(a_1 v_1 + \cdots + a_k v_k). \]

This implies  
\[ a_1 v_1 + \cdots + a_k v_k \in \ker(T) = \langle cd \rangle, \]
hence there are coefficients  \( b_1, \ldots, b_k \in \mathbb{F} \) s.t.
\[ a_1 v_1 + \cdots + a_k v_k = b_1 u_1 + \cdots + b_k u_k. \]

Writing this as  
\[ a_1 v_1 + \cdots + a_k v_k - b_1 u_1 - \cdots - b_k u_k = 0, \]
and using linear independence of \( \langle cd \rangle \), we see that
\[ a_1 = 0, \ldots, a_k = 0, b_1 = 0, \ldots, b_k = 0 \] as required.

To see that \( \{v\} \) spans \( \text{Range}(T) \), consider \( w \in \text{Range}(T) \).
There must be a \( v \in V \) such that \( T(v) = w \), and
writing  
\[ v = c_1 u_1 + \cdots + c_k u_k + d_1 v_1 + \cdots + d_k v_k \]
we find that
\[ w = T(c_1 u_1 + \cdots + c_k u_k + d_1 v_1 + \cdots + d_k v_k) = c_1 T(u_1) + \cdots + c_k T(u_k) + d_1 T(v_1) + \cdots + d_k T(v_k) = d_1 T(v_1) + \cdots + d_k T(v_k) \]
and so \( w \in \text{Range}(T) \). \( \square \)
Corollary If \( T \in \mathcal{L}(V, W) \) and \( \dim V = \dim W < \infty \), then the following are equivalent:

i) \( T \) is injective, 
ii) \( T \) is surjective, 
iii) \( T \) is an isomorphism.

**Pf**

\( T \) is injective \( \iff \) \( \text{null}(T) = \{0\} \iff \text{rank}(T) = \dim V = \dim W \iff T \) is surjective.

Corollary If \( T \in \mathcal{L}(V, W), \ S \in \mathcal{L}(W, V) \), and \( \dim V = \dim W < \infty \), then the following are equivalent:

i) \( S \circ T = \text{Id} \), 
ii) \( T \circ S = \text{Id} \), 
iii) \( S = T^{-1} \).

**Pf**

\( S \circ T = \text{Id} \Rightarrow S \) surjective \( \Rightarrow S \) invertible 
\& \( S^{-1} = S^{-1} S T = T \).

The rank of a matrix is the same as the rank of its transpose.

Indeed, given \( A \in M_{m \times n}(F) \), the span of the columns of \( A^T, \text{C}(A^T) \), is the span of the rows of \( A \).
This span doesn't change when we apply row operations to \( A \), so is also the span of the rows of \( \text{RREF}(A) \).
So its dimension equals the number of pivots in \( \text{RREF}(A) \) and this is the rank of \( A \).
Since \( \text{rank}(A) = \text{rank}(A^T) \),
the rank-nullity theorem guarantees that
for square matrices, \( \text{null}(A) = \text{null}(A^T) \).

In particular, if \( A \in \mathbb{M}_n(\mathbb{F}) \) then
\( \lambda \) is an eigenvalue of \( A \)
\( \Leftrightarrow \text{null}(A - \lambda I_n) > 0 \)
\( \Leftrightarrow \text{null}(A - \lambda I_n)^T > 0 \)
\( \Leftrightarrow \text{null}(A^T - \lambda I_n) > 0 \Leftrightarrow \lambda \) is an eigenvalue of \( A^T \).

Finally, recall that the solution set \( S \)
of a linear system of equations \( Ax = b \)
is an affine space of the form
\( S = x_0 + \text{Ker } A \), with \( x_0 \) any particular solution.
The nullity of \( A \) tells us the dimension of \( S \).