Last time we proved that if
\[ V = \text{span} \{ v_1, \ldots, v_m \} \quad \text{and} \quad \{ w_1, \ldots, w_n \} \subseteq V \] is linearly independent then \( m \geq n \). It follows that any two bases of \( V \) must have the same number of elements.
We call this number the dimension of \( V \).

E.g., the canonical basis of \( \mathbb{F}^n \), \( (e_1, \ldots, e_n) \), has \( n \) elements, so \( \dim \mathbb{F}^n = n \).

\[ \mathbb{P}_n(\mathbb{R}) = \text{polynomials of degree at most } n \text{ with real coefficients} \]
has basis \( (1, x, \ldots, x^n) \) and hence has dimension \( n+1 \).

Note that for a matrix \( A \in \mathcal{M}_{m \times n}(\mathbb{F}) \)
the dimension of its column space, \( \text{col}(A) \), is equal to the number of pivots in \( \text{REF}(A) \).

Clearly, if
\[ V = \text{span} \{ v_1, \ldots, v_m \} \quad \text{and} \quad \{ w_1, \ldots, w_n \} \subseteq V \] is linearly independent then \( n \leq \dim V \leq m \).
Interestingly, if \( U_1, U_2 \) are subspaces of \( V \) and \( \dim U_1 + \dim U_2 > \dim V \), then \( U_1 \cap U_2 \neq \{0\} \).

Indeed, let's check the contrapositive:
If \( U_1, U_2 \) are subspaces of \( V \) s.t. \( U_1 \cap U_2 = \{0\} \), then \( \dim U_1 + \dim U_2 \leq \dim V \).

**Proof**

Let \( (v_1, \ldots, v_r) \) be a basis of \( U_1 \)

and \( (w_1, \ldots, w_s) \) be a basis of \( U_2 \).

We'll check that \( \{ v_1, \ldots, v_r, w_1, \ldots, w_s \} \) is linearly independent.

Suppose \( \sum_{i=1}^{r} a_i v_i + \sum_{k=1}^{s} b_k w_k = 0 \) then \( \sum_{i=1}^{r} a_i v_i = -\sum_{k=1}^{s} b_k w_k \in U_1 \cap U_2 \)

implies \( \sum_{i=1}^{r} a_i v_i = 0 \) & \( \sum_{k=1}^{s} b_k w_k = 0 \), and hence \( a_i = 0, b_k = 0 \) for all \( i \) and \( k \).

It turns out that the dimension is the only thing that distinguishes finite dimensional vector spaces, up to isomorphism.

Then let \( V \) and \( W \) be finite-dimensional vector spaces over \( \mathbb{F} \).

\( V \) and \( W \) are isomorphic if and only if \( \dim V = \dim W \).

**Proof**

\( \Rightarrow \) If \( T \in \mathcal{L}(V, W) \) is an isomorphism & \( (v_1, \ldots, v_m) \) is a basis of \( V \), then \( (T(v_1), \ldots, T(v_m)) \) is a basis of \( W \) & \( \dim V = \dim W \).

\( \Leftarrow \) If \( \dim V = \dim W \), \( (v_1, \ldots, v_m) \) is a basis of \( V \), and \( (w_1, \ldots, w_n) \) is a basis of \( W \) then there is \( T \in \mathcal{L}(V, W) \) such that \( T(v_i) = w_i \forall i \).

Since \( T \) sends a basis of \( V \) to a basis of \( W \), it is an isomorphism.
In particular, every finite dimensional vector space $V$ is isomorphic to $\mathbb{F}^n$, with $m = \dim V$.

If $\dim V = m$ and $V = \langle v_1, \ldots, v_m \rangle$

then $(v_1, \ldots, v_m)$ is a basis of $V$.

Indeed, we have shown that any list of vectors that spans $V$ contains a sublist that is a basis of $V$. Since a basis has exactly $m$ elements, it must be the whole list.

Thus, if $V$ is finite dimensional and $\mathcal{B} \subseteq V$ is a linear list then $\mathcal{B}$ can be extended to a basis of $V$.

Proof: Suppose $\mathcal{B} = (v_1, \ldots, v_n)$.

If $\langle v_1, \ldots, v_n \rangle = V$ then $\mathcal{B}$ is a basis and we’re done.

Otherwise, pick $v_n \in V \setminus \langle v_1, \ldots, v_n \rangle$ and add it to $\mathcal{B}$.

Note that $\mathcal{B}$ is still linearly independent.

We can keep adding vectors like this until $\mathcal{B}$ has length $\dim V$, at which point it will be a basis of $V$. 

Thm If \( V \) is finite dimensional and \( U \subseteq V \) is a subspace, then
\[ i) \ U \text{ is finite dimensional}, \]
\[ ii) \ \dim U \leq \dim V, \text{ and} \]
\[ iii) \ \text{if} \ \dim U = \dim V \text{ then} \ U = V. \]

Pf
\[ i) \ \text{If} \ U = \{0\} \text{ then we are done, otherwise pick} \ u \in U \setminus \{0\}. \]
\[ \text{If} \ U = \langle u \rangle \text{ then we are done, otherwise pick} \ u \in U \setminus \langle u \rangle. \]
Continuing in this way we build successively larger linear subsets of \( U \).
The size of these sets is bounded above by \( \dim V \), so the process must stop after finitely many steps. At that point the linear set must also span \( U \).
\[ ii) \ \text{If} \ \{u_1, \ldots, u_n\} \subseteq U \subseteq V \text{ is linearly independent then} \ n \leq \dim V. \]
\[ iii) \ \text{If in} \ (ii), \ n = \dim V, \text{ then} \ (u_1, \ldots, u_n) \text{ is a basis of} \ V. \]

Ex. Every subspace of \( \mathbb{R}^3 \) must have dimension 0, 1, 2, or 3.
So the subspaces of \( \mathbb{R}^3 \) are either
the origin, a line through the origin, a plane through the origin, or all of \( \mathbb{R}^3 \).