Linear Algebra

Linear Maps

Last Time: Vector Spaces, e.g. \( \mathbb{R}^n \), \( \mathbb{F} \), 8-solutions of a homogeneous system of equations

This Time: Maps between vector spaces

One of the great insights of 20th century mathematics is that to properly understand a structure (e.g. vector spaces) what we need to study are the maps that preserve that structure (e.g. 'linear maps').

This insight goes by the name of category theory.

The structure of a vector space is given by vector addition
A scalar multiplication.

Let \( V \) and \( W \) be vector spaces over \( \mathbb{F} \)
A function \( T: V \to W \) is a linear map (or linear transformation or linear operator)

i) Additivity: \( \forall v, v_2 \in V, \ T(v_1 + v_2) = T(v_1) + T(v_2) \)

ii) Homogeneity: \( \forall v \in V, a \in \mathbb{F}, \ T(av) = aT(v) \)

We refer to \( V \) as the domain of \( T \) & \( W \) as the codomain.

The set of all linear maps between \( V \) and \( W \) is denoted \( \mathcal{L}(V, W) \).

If \( V=W \), then we abbreviate \( \mathcal{L}(V) = \mathcal{L}(V, V) \).
E.g., we can use linear maps to recognize that
two vector spaces are really "the same".
Consider \( T : \mathbb{F}^n \rightarrow M_2(\mathbb{F}) \),
\[ T \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_3 \\ a_2 \\ a_4 \end{bmatrix}. \]

\( T \) is a linear map:
1) \( T \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \right) = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \end{bmatrix} \)
2) \( T \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right) + T \left( \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \end{bmatrix} \)
3) \( cT \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right) = T \left( \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ ca_4 \end{bmatrix} \right) = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ ca_4 \end{bmatrix} = c \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = c \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \)

This lets us compare the two vector spaces.
What shows that they are the same is that \( T \) has an inverse
\( S : M_2(\mathbb{F}) \rightarrow \mathbb{F}^4 \) that is also linear:
\[ S \left( \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}. \]

\( S \) and \( T \) are mutually inverse, i.e., \( ST(\mathbf{u}) = \mathbf{u} \neq \mathbf{u} \in \mathbb{F}^4 \)
and \( TS(\mathbf{w}) = \mathbf{w} \neq \mathbf{w} \in M_2(\mathbb{F}) \).

An invertible linear map is called an isomorphism.

E.g., \( T : \mathbb{F}^4 \rightarrow \mathbb{F}^2, \)
\[ T \left( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \]
is linear, but not invertible.

E.g., for any vector space \( V \), the identity map \( I : V \rightarrow V, I(\mathbf{v}) = \mathbf{v} \)
is linear (and an isomorphism).
E.g., reflection across a line in $\mathbb{R}^2$ (or, e.g., a plane in $\mathbb{R}^3$) is a linear transformation.

2) Additivity:

$\mathbb{R}(v+w) = \mathbb{R}(v) + \mathbb{R}(w)$

3) Homogeneity:

$a\mathbb{R}(v) = \mathbb{R}(av)$

E.g., let $A = [a_{ij}]_{1 \leq i \leq m \atop 1 \leq j \leq n}$ be an $m \times n$ matrix over $F$. A acts as a linear operator $F^n \rightarrow F^m$ by matrix-vector multiplication.

If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then $Av = \begin{bmatrix} \sum_{j=1}^n a_{1j}v_j \\ \vdots \\ \sum_{j=1}^n a_{mj}v_j \end{bmatrix}$.

For example, if $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $Av = \begin{bmatrix} av_1 + bv_2 + cv_3 \\ dv_1 + ev_2 + fv_3 \end{bmatrix}$.

Note the signs $2(3 \cdot 2) - 2$ and $-1$.

The vector $Av$ is a linear combination of the columns of $A$ by coefficients given by the entries of $v$:

$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{j=1}^n v_j a_j$
Matrix-vector multiplication is linear:

i) \( A(v+w) = \sum_{j=1}^{n} (v_j+w_j) a_j = \sum_{j=1}^{n} v_j a_j + \sum_{j=1}^{n} w_j a_j = A(v) + A(w) \),

ii) \( A(cv) = \sum_{j=1}^{n} (cv_j) a_j = c \sum_{j=1}^{n} v_j a_j = c A(v) \).

Note that if \( v = e_j \in F^n \) (\( e_j \) is the vector in \( F^n \) with 1 in the \( j \)th coordinate & 0 in other coord.'s)

then \( A v = A e_j = a_j = \) the \( j \)th column of \( A \).

E.g., If \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \), \( v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \), then \( A v = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \).

In fact any reordering of the coordinates is a linear map.

E.g., The \( n \times n \) identity matrix \( I_n \in M_n(F) \) is the \( n \times n \) matrix

with 1's on the diagonal & 0 elsewhere:

\( I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \text{diag}(1,\ldots,1) \).

As a linear map we have \( I_n v = v \) \( \forall v \in F^n \).

E.g., If \( A = \text{diag}(d_1,\ldots,d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \),

then as a linear map \( A v = A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} d_1 v_1 \\ \vdots \\ d_n v_n \end{bmatrix} \).
E.g., Revisiting the example from ancient China

\[
\begin{align*}
2h + 2m + l &= 39 \\
2h + 3m + l &= 3h \\
h + 2m + 3l &= 26
\end{align*}
\]

We recognize this as

\[
\begin{bmatrix}
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
h \\
m \\
l
\end{bmatrix} =
\begin{bmatrix}
39 \\
3h \\
26
\end{bmatrix}
\]
or \(Ax = b\).

We can write any system of linear equations concisely as a single vector-valued equation in this way.

\[
\begin{align*}
\begin{bmatrix}
a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\
\vdots & \quad \vdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m
\end{bmatrix}
\quad \Leftrightarrow \quad
\begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_n
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
\vdots \\
b_m
\end{bmatrix}
\quad \Leftrightarrow \quad
A\begin{bmatrix}x_1 \ldots x_n\end{bmatrix} = b
\end{align*}
\]

\(m \times n \quad n \quad m\).