Textbook: Linear Algebra by Meckes & Meckes

Linear systems of equations show up in all areas of science and their study has a long history. A Babylonian clay tablet from around 300 BCE says roughly:

There are two fields whose total area is 1800 square yards. One produces grain at the rate of \( \frac{2}{3} \) of a bushel per square yard while the other produces grain at the rate of \( \frac{1}{2} \) a bushel per square yard. If the total yield is 1100 bushels, what is the size of each field?

One of the earliest surviving mathematical texts from China, “Jiu zhang suanshu” or “Nine chapters on the mathematical art”, written before 263 CE (maybe around 200 BCE), starts its 8th chapter with the following problem:

A combination of 3 bundles of high-quality grain, 2 bundles of medium-quality grain, and 1 bundle of low-quality grain will yield 39 barrels of flour. If we combine 2 bundles of high-quality grain, 3 bundles of medium-quality grain, and 1 bundle of low-quality grain we obtain 34 barrels of flour. Finally, combining 1 bundle of high-quality grain, 2 of medium, and 3 of low, we obtain 26 barrels of flour. How much flour can be obtained from 1 bundle of each type of grain?
In modern algebraic notation, we might write this as:

\[
\begin{align*}
3h + 2m + 1l &= 39 \\
2h + 3m + 1l &= 34 \\
1h + 2m + 3l &= 26
\end{align*}
\]

and then the problem asks us to find the values of \( h, m, l \).

The method we use today to solve these systems, called Gaussian elimination in honor of Carl Friedrich Gauss (1777-1855), is the same method used in ancient China! Eventually, it was realized that it is better to think of a linear system of equations (such as this one) as a single equation for a vector:

\[
\begin{pmatrix}
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
h \\
m \\
l
\end{pmatrix}
=
\begin{pmatrix}
39 \\
34 \\
26
\end{pmatrix},
\]

with the coefficients set up in a rectangular array known as a matrix. For a while, “determinant” were seen as the main way to study these systems. In fact, the word matrix comes from the Latin for womb since James Joseph Sylvester (1814-1897), who introduced the term, viewed matrices as the wombs of determinants. His friend, Arthur Cayley (1821-1895), understood that matrices were interesting in their own right and figured out how to multiply and invert them. This led to thinking of a matrix as inducing a natural map from vector to vectors and so the concept of linear transformation and the subject of linear algebra.
Babylonian question
Let $x$ be the area (in square yards) of the first field
And $y$ be the area of the second field

First we are told that the total area is 1800 sq yd
which we can write $x + y = 1800$

This is a linear equation in two variables
And it has infinitely many solutions
(since we can take a fractional number of square yards)
Although the 'real-world' situation requires $x \geq 0$, $y \geq 0$,
this is not built into the equation.
We should just keep this in mind when interpreting our solutions.

Following René Descartes (1596-1650)
we can draw a picture of all of the solutions
as a line in the $x$-$y$ plane

Every point in the $x$-$y$ plane corresponds to a pair of areas (Area of 1st field, Area of 2nd field)
The points on the line correspond to solutions of our equation.
Secondly, we are told that the 1st field produces grain at a rate of $\frac{1}{3}$ bushel per sq yard & the 2nd field $\frac{1}{2}$ together they yield 1100 bushels which we can write $\frac{1}{2}x + \frac{1}{2}y = 1100$

Putting these together we have a linear system of equations:

$$
\begin{align*}
    x + y &= 1800 \\
    \frac{2}{3}x + \frac{1}{2}y &= 1100
\end{align*}
$$

This is easy to solve.

Let's start by multiplying the 2nd eq by 6 to make the constants nicer:

$$
\begin{align*}
    x + y &= 1800 \\
    4x + 3y &= 6600
\end{align*}
$$

Next, let's subtract 3 times the 1st eq from the 2nd:

$$x = 6600 - 5400 = 1200$$

After substituting into the 1st eq we get $y = 600$

So the solution is $x = 1200, y = 600$

Graphically we have

Thinking about this system as the intersection of two lines makes it easy to see that there were three possibilities: 1 solution, 0 solutions, or infinitely many solutions
Chinese question

\[ 3h + 2m + l = 39 \]
\[ 2h + 3m + l = 34 \]
\[ h + 2m + 3l = 26 \]

This is also a linear system of equations:

3 equations of 3 unknowns \((h, m, l)\)

The solution is \(h = \frac{37}{4}, m = \frac{17}{4}, l = \frac{11}{4}\)

To check this, we don't need to solve the system.

It is enough to plug in & verify.

(The book calls this the Rat Poison Principle.)

Eg. in the last equation:
\[
\frac{33}{4} + 2\left(\frac{17}{4}\right) + 3\left(\frac{11}{4}\right) = \frac{1}{4}\left(37 + 34 + 33\right) = \frac{104}{4} = 26
\]

Here each equation represents a plane in \(\mathbb{R}^3\)

In general, a linear system of \(m\) equations in \(n\) variables over \(\mathbb{R}\) is a set of equations of the form

\[
a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \\
\vdots \\
a_{m1} x_1 + \ldots + a_{mn} x_n = b_m
\]
A solution of a linear system is a set of real numbers $c_1, \ldots, c_n$ such that

$$a_{11}c_1 + \ldots + a_{1n}c_n = b_1$$

$$\vdots$$

$$a_{m1}c_1 + \ldots + a_{mn}c_n = b_m$$

The system is linear in that the variables are only ever multiplied by real numbers or added together (so for example we don't have $x^2$).