MATH 416  LU Decomposition  Feb 19

LAST TIME: 1) Given any $A \in M_{m \times n}(F)$ there is a $B \in M_{m \times n}(F)$ in $\text{RREF}$ & elementary matrices $E_1, ..., E_n \in M_{m \times n}(F)$ s.t. $A = E_n \cdots E_2 E_1 B$.

2) A invertible $\iff$ $\text{RREF}(A) = I_m$

In that case, $\text{RREF}[A | I_m] = [I_m | A^{-1}]$.

THIS TIME: The LU decomposition

Here $L$ stands for a lower triangular matrix & $U$ for an upper triangular matrix.

A matrix $A \in M_{m \times n}(F)$ is lower triangular
if $a_{ij} = 0$ whenever $i > j$.

A matrix $A \in M_{m \times n}(F)$ is upper triangular
if $a_{ij} = 0$ whenever $i < j$.

E.g., The elementary matrices $P_{i,j}$ is lower triangular if $i < j$ & upper triangular if $i > j$.

For example, $P_{5,3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $P_{5,1,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

If we're given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we wanted to use row operations to make it upper triangular, i.e., to get $d = 0 = b = 0$,

then in the simplest case we would only need to use row operation $R_2 \leftarrow R_2 + \lambda R_1$ repeatedly and add a multiple of a row to a lower row.

In that case, we would have $P_{c_i,j, k} \cdots P_{c_i, i, j, k} A = U$, $i_2 > j_2$ & $k 

Solving for $A$, we find $A = (P_{c_i,j, k} \cdots P_{c_i, i, j, k})^{-1} A = P_{c_i,j, k} \cdots P_{c_i, i, j, k} U$. 

$L$
Algorithm: To find the LU decomposition of a matrix $A \in M_{n \times n}(\mathbb{F})$:

i) Try to reduce $A$ to an upper triangular matrix using only
row operations $R_i \leftrightarrow R_j$ only ever adding a multiple of row $j$ to row $i$ if $i > j$.
If this succeeds, call the result $U$.
If it fails, then $A$ does not have an LU decomposition.

ii) If the previous step succeeded, define $L \in M_{n \times n}(\mathbb{F})$ to be the lower triangular matrix with $1$'s on the diagonal
& $c$ in the $(i,j)$ entry if during the elimination process
$c$ times row $j$ was added to row $i$.
Then $A = LU$.

Ex. Find the LU decomposition, if it exists, of $A = \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ -7 & -1 & 14 & -5 & -1 \\ 15 & 9 & -21 & 12 & 2 \\ 3 & 9 & -21 & 2 & 1 \end{bmatrix}$.

\[
\begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ -7 & -1 & 14 & -5 & -1 \\ 15 & 9 & -21 & 12 & 2 \\ 3 & 9 & -21 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ 0 & 2 & -1 & 1 & -1 \\ 0 & 4 & 2 & 2 & 4 \\ 0 & 8 & -4 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ 0 & 2 & -1 & 1 & -1 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}
\]

Hence $A = P_{p_3,2,1} \bar{P}_{p_2,3,1} P_{p_3,2,1} P_{p_4,2,1} U = \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ 0 & 2 & -1 & 1 & -1 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$.

Given $A = LU$, we solve $Ax = b \iff LUx = b$ by setting $y = Ux$ so that $Ly = b$.

It is easy to solve $Ly = b$ & once we know $y$ then it is easy to solve $Ux = y$ to find $x$.

This is usually the preferred method for solving linear systems.
E.g., with $A$ as above, let's solve $Ax = \begin{bmatrix} 4 \\ -10 \\ 20 \\ 13 \end{bmatrix}$.

First we find the unique solution to $Ly = b$, which here is:

$$\begin{cases} y_1 = 4 \\ -3y_1 + y_2 = -10 \\ 5y_2 + 2y_3 + y_4 = 20 \\ y_1 + 4y_2 + y_4 = 13 \end{cases} \Rightarrow \begin{cases} y_1 = 4 \\ y_2 = 2 \\ y_3 = -4 \\ y_4 = 1 \end{cases}$$

Next we solve $Ux = y$ (with $x_5$ as free variable),

$$\begin{cases} 3x_1 + x_2 - 5x_3 + 2x_4 = 4 \\ 2x_1 - x_2 + 3x_4 - x_5 = 2 \\ 4x_2 - 4x_4 - x_5 = -4 \\ x_4 + 5x_5 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -1 + \frac{3}{5}x_5 \\ x_2 = \frac{3}{5}x_5 \\ x_3 = -1 - x_5 \\ x_4 = 1 - 5x_5 \end{cases}$$

and so the set of solutions is $\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \\ -1 \\ -5 \\ 1 \end{bmatrix} : t \in \mathbb{R} \}$.

Sometimes, a matrix does not have an LU decomposition. For example, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $l_1l_2U_1U_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. If we allow ourselves to switch to order of the rows first, then we can always find an LU decomposition.

Then for every $A \in M_{n \times n}(\mathbb{F})$, there is a matrix $P \in M_{n \times n}(\mathbb{F})$ which is a product of elementary matrices of the form $R_{i,j}$, an upper triangular matrix $U \in M_{n \times n}(\mathbb{F})$ & a lower triangular $L \in M_{n \times n}(\mathbb{F})$ such that $PA = LU$.

This is the LU (pivoting) decomposition of $A$. 
\textbf{Pf} Any $A$ can be put into upper triangular form by using row operations of type 1 & 3, so all we need to check is that it is possible to carry out the row exchanges first & then the row operations of type 1.

This follows from $R_{i,j} P_{c,k,l} = \begin{cases} P_{c,k,l} R_{i,j} & \text{if } \{i,j\} \cap \{k,l\} = \emptyset \\ P_{c,i,k} R_{j,l} & \text{if } i \neq k, j \neq l \\ P_{c,k,j} R_{l,i} & \text{if } i \neq j, k \neq l \\ P_{c,i,k} R_{j,l} & \text{if } i = k, j = l \end{cases}$