Last Time: QR decomposition

This Time: Singular Values

If $V, W$ fin dim & $T \in \mathcal{L}(V, W)$

it is always possible to choose bases of $V \& W$

so that $[T]_{v,w}$ has $1's$ on diagonal & $0's$ elsewhere.

Thus this tells us nothing about $T$

DemANDING orthogonal bases when $V \& W$ are inner product spaces

makes the diagonal values (which now won't be just 1)

more significant: we call them the singular values of $T$.

LemMA. $V, W$ fin dim inner product spaces, $T \in \mathcal{L}(V, W)$

If $e \in V$ is a unit vector s.t. $\|T\|_F = \|T(e)\|

$\& u \in V$ is orthogonal to $e$

then $\langle Tu, Te \rangle = 0$

PF. Let's write $\sigma = \|T\|_F$ so that $\|T(v)\| \leq \sigma \|v\| \forall v \in V$

Suppose $\langle u, e \rangle = 0$ ($\& \sigma > 0$)

For any $a \in V$ we have $\|T(au)\|^2 \leq \sigma^2 \|au\|^2 = \sigma^2 (1 + |a|^2 \|u\|^2)$

On the other hand, $\|T(au)\|^2 = \langle T(au), T(au) \rangle$

$= \|T(e)\|^2 + 2Re(\langle a \langle Tu, Te \rangle \rangle) + |a|^2 \|u\|^2$

$\geq \sigma^2 + 2Re(\langle a \langle Tu, Te \rangle \rangle)$

So we've shown that $2Re(\langle a \langle Tu, Te \rangle \rangle) \leq \sigma^2 \|u\|^2 |a|^2 \forall a \in V$

Take $a = \langle Tu, Te \rangle \epsilon$ for $\epsilon > 0$ to be determined.

so this becomes $2 |\langle Tu, Te \rangle \epsilon|^2 \leq \sigma^2 \|u\|^2 |\langle Tu, Te \rangle \epsilon|^2 \forall \epsilon > 0$

i.e. $0 \leq |\langle Tu, Te \rangle \epsilon|^2 \left( \sigma^2 \|u\|^2 \epsilon^2 - 2 \epsilon \right) \forall \epsilon > 0$

but the only way this can hold for all positive $\epsilon$

is if $|\langle Tu, Te \rangle \epsilon|^2 = 0$
Theorem (Singular Value Decomposition) \( V, W \) are the inner product spaces. Let \( T \in L(V, W) \) be a linear map of rank \( r \).

There are orthonormal bases \( \{v_1, \ldots, v_m\} \) of \( V \) and \( \{w_1, \ldots, w_n\} \) of \( W \)

and numbers \( \sigma_1 \geq \ldots \geq \sigma_r > 0 \) such that \( T(v_j) = \sigma_j w_j \) if \( 1 \leq j \leq r \)

and \( T(v_j) = 0 \) if \( j > r \).

The numbers \( \sigma_1, \ldots, \sigma_r \) are called the singular values of \( T \).

The vectors \( \{v_1, \ldots, v_m\} \) are called the right singular vectors of \( T \),

and the vectors \( \{w_1, \ldots, w_n\} \) are called the left singular vectors of \( T \).

Proof: We will build the bases \( \{v_1, \ldots, v_m\} \) and \( \{w_1, \ldots, w_n\} \) one vector at a time.

First let \( S_1 = \{u \in V : \|u\| = 1\} \)

and let \( v_1 \in S_1 \) be a unit vector s.t. \( \|Tv_1\| = \|T\|_{op} \).

If this value is zero then \( T = 0 \) and the theorem holds.

Otherwise, let \( \sigma_1 = \|Tv_1\| \) and \( w_1 = \frac{1}{\sigma_1} Tv_1 \in W \) so \( \sigma_1 v_1 = \sigma_1 w_1 \),

and \( \|w_1\| = 1 \).

Next let \( S_2 = \{u \in V : u \perp v_1, \|u\| = 1\} \).

And let \( v_2 \in S_2 \) be a unit vector s.t. \( \|Tv_2\| = \max_{u \in S_2} \|Tu\| \).

If this value is zero then \( T \) vanishes on \( \text{span} \{v_1, v_2^\perp\} \)

so we can extend \( \{v_1, v_2\} \) to orthonormal bases

and the theorem will hold.

Otherwise, let \( \sigma_2 = \|Tv_2\| \) and \( w_2 = \frac{1}{\sigma_2} T(v_2) \in W \).

So \( T(v_2) = \sigma_2 w_2 \), \( \|w_2\| = 1 \) and by Lemma \( \langle v_2, v_1 \rangle = 0 \)

so \( \langle T(v_2), T(v_1) \rangle = 0 \)

and \( \langle w_1, w_2 \rangle = 0 \).
If we assume we've constructed \( \{v_1, \ldots, v_k\} \subseteq V \) orthonormal
s.t. \( \text{span} \{T(v_1), \ldots, T(v_k)\} \) are orthogonal
& we've set \( \sigma_1 = \|T(v_1)\| \geq \sigma_2 = \|T(v_2)\| \geq \cdots \geq \sigma_k = \|T(v_k)\| > 0 \)

We define
\[ S_{k+1} = \{ u \in \text{span} \{v_1, \ldots, v_k\} : \|u\| = 1 \} \]
& pick \( v_{k+1} \in S_{k+1} \) such that \( \|T(v_{k+1})\| = \max_{u \in S_{k+1}} \|T(u)\| \)

If this value is zero the rank \( T = k \)
& we can complete the basis \( \{v_1, \ldots, v_k, w_1, \ldots, w_{k+1}\} \)
to orthonormal bases in any way & the theorem holds

Otherwise we set \( \sigma_{k+1} = \|T(v_{k+1})\| \), \( \gamma_{k+1} = \frac{\sigma_{k+1}}{\sigma_k} T(v_{k+1}) \)
& we keep going
Eventually this process will terminate.

The singular value decomposition of \( T \)
consists of the bases \( \{v_1, \ldots, v_k, w_1, \ldots, w_{k+1}\} \)
& the singular values \( \{\sigma_1, \ldots, \sigma_k\} \)

There was a lot of choice in the bases, but it turns
out that we always get the same numbers.
Then $V, W$ are finite inner product spaces, $T: L(V, W)$ of rank $r$

Suppose there are orthonormal bases

\[ \{v_1, \ldots, v_n\} \text{ of } V \]
\[ \{w_1, \ldots, w_m\} \text{ of } W \]

and that there are real scalars $\sigma_1, \ldots, \sigma_r > 0$ such that

\[ T(v_j) = \sum_{i=1}^{r} \sigma_i w_j \quad \text{for } j = 1, \ldots, n \]

\[ T(w_j) = \sum_{i=1}^{r} \sigma_i v_j \quad \text{for } j = 1, \ldots, m \]

Then $\sigma_i = \sigma_i$ for $i = 1, \ldots, r$

\[ \sigma_i = \sigma_i \quad \forall i \]

**Proof**

We first show that $\sigma_i, \sigma_i$ have to be equal to $\|T\|\|u\|$

And that $\sigma_i = \sigma_i$, $\sigma_1, \ldots, \sigma_r$

Let

$$ U = \{ u \in V : \| T(u) \| = \| T \| \| u \| \} $$

Given $u \in U$ write $u = \sum_{j=1}^{n} \langle u, v_j \rangle w_j$

so

$$ T(u) = T \left( \sum_{j=1}^{n} \langle u, v_j \rangle v_j \right) = \sum_{j=1}^{n} \langle u, v_j \rangle T(v_j) = \sum_{j=1}^{n} \langle u, v_j \rangle \sigma_j w_j $$

Hence by Pythagorean theorem:

$$ \| T(u) \|^2 = \sum_{j=1}^{n} | \langle u, v_j \rangle |^2 \sigma_j^2 \leq \sigma_1^2 \sum_{j=1}^{n} | \langle u, v_j \rangle |^2 \leq \sigma_1^2 \| u \|^2 $$

But since we know $\| T(u) \| = \| T \| \| u \|$ we must have $\| T \| \| u \| = \sigma_1$

And we must have $\langle u, v_j \rangle = 0$ if $\sigma_j \neq \sigma_1$, $u \in \text{span} \{ v_k : \sigma_k = \sigma_1 \}$

Let $U = \text{span} \{ v_k : \sigma_k = \sigma_1 \}$

Arguing in the same way $U = \text{span} \{ w_k : \sigma_k = \sigma_1 \}$

Applying the argument to $T|_{U^+}$ inductively we get the theorem.