Recall that two vector spaces $V, W$ are "the same" if there is a linear map $T \in \mathcal{L}(V, W)$ that is invertible. We call $T$ an isomorphism & say that $V \cong W$ are isomorphic.

**Def** If $V \cong W$ are vector spaces of norms, we say that they are **isometric** if there is an isomorphism $T \in \mathcal{L}(V, W)$ s.t. $\| T(v) \| = \| v \|$ for $v \in V$.

We call such a map an **isometry**.

**Ex** $M_2(\mathbb{C}) \cong \mathbb{C}^4$ are isomorphic, e.g. via $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

This map is also an isometry between the Frobenius norm on $M_2(\mathbb{C})$ & the standard norm on $\mathbb{C}^4$ since $\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 = \left\| \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\|^2$.

Note that any map satisfying $\| T(v) \| = \| v \|$ is automatically injective, since $\ker T = \{ v \mid \| v \| = 0 \} = \{ 0 \}$.
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Then if $V, W$ are inner product spaces, $T \in \mathcal{L}(V, W)$ is an isomorphism if and only if $\langle T(v), T(v) \rangle = \langle v, v \rangle \neq 0, v, v \in V$.

If $\langle T(v), T(v) \rangle = \langle v, v \rangle \neq 0, v, v \in V$

then $\|T(v)\|^2 = \langle T(v), T(v) \rangle = \langle v, v \rangle = \|v\|^2 \neq 0, v \in V$ so $T$ is an isometry.

If $T$ is an isometry (i.e., $\|T(v)\| = \|v\| \neq 0, v \in V$)

then we want to show $T$ preserves the inner product.

First consider the case $F = \mathbb{R}$

so that $\langle v, v \rangle = \frac{1}{4} (\|v + v\|^2 - \|v - v\|^2) \neq 0, v, v \in V$

and similarly on $W$.

Then we have

$\langle T(v), T(v) \rangle = \frac{1}{4} (\|T(v) + T(v)\|^2 - \|T(v) - T(v)\|^2)$

$= \frac{1}{4} (\|v + v\|^2 - \|v - v\|^2)$

$= \frac{1}{4} (\|v + v\|^2 - \|v - v\|^2) = \langle v, v \rangle$

For the case $F = \mathbb{C}$ we can use instead

$\langle v, v \rangle = \frac{1}{4} (\|v + iv\|^2 - \|v - iv\|^2 + i \|v + iv\|^2 - i \|v - iv\|^2) \neq 0, v \in V$.

A convenient alternate way of writing this is:

$\langle T(v), T(v) \rangle = \langle v, v \rangle \iff \langle T(v), w \rangle = \langle v, T^{-1}(w) \rangle$

$\neq 0, v, v \in V$, $w \in W$.

If we know $\langle T(v), T(v) \rangle = \langle v, v \rangle \neq 0, v, v \in V$

then $\langle T(v), w \rangle = \langle T(v), T(T^{-1}(w)) \rangle = \langle v, T^{-1}(w) \rangle \neq 0, v \in V$, $w \in W$.

If we know $\langle T(v), w \rangle = \langle v, T^{-1}(v) \rangle \neq 0, v \in V$, $w \in W$

then $\langle T(v), T(v) \rangle = \langle v, T^{-1}(v) \rangle = \langle v, v \rangle \neq 0, v, v \in V$.

Example: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation counterclockwise by $\theta$

$\langle R_\theta v, w \rangle = \langle v, R_\theta^{-1} w \rangle$

if rotating $v$ by $\theta$ counterclockwise & measuring angle with $w$

= measuring angle between $v$ & $w$ rotated clockwise by $\theta$. 


Recall that a linear map \( T: V \to W \) is an isomorphism

\[ \iff \text{it sends bases of } V \text{ to bases of } W \]

Then \( V, W \) fin dimensional inner product spaces \( T \in \mathcal{L}(V, W) \) then

(a) there is an orthonormal basis \( \{v_1, \ldots, v_n\} \) of \( V \)

\[ \text{s.t. } \{T(v_1), \ldots, T(v_n)\} \text{ is an orthonormal basis of } W \]

(b) \( T \) is an isometry

(c) whenever \( \{v_1, \ldots, v_n\} \) is an orth basis of \( V \),

\[ \{T(v_1), \ldots, T(v_n)\} \text{ is an orth basis of } W \]

\[ \begin{align*}
\text{Pf} \\
(a) \Rightarrow (b) \\
\text{Suppose } \{v_1, \ldots, v_n\} \text{ orth basis of } V \text{ & } \{T(v_1), \ldots, T(v_n)\} \text{ orth basis of } W \\
\text{Given u \in V we want to show } ||T(u)|| = ||u|| \\
||u||^2 = \left\| \sum_{j=1}^{n} \langle u, v_j \rangle v_j \right\|^2 = \left\| \sum_{j=1}^{n} \langle u, v_j \rangle v_j \right\|^2 = \sum_{j=1}^{n} || \langle u, v_j \rangle v_j ||^2 = \sum_{j=1}^{n} || v_j ||^2 \\
||T(u)||^2 = \left\| \sum_{j=1}^{n} T(\langle u, v_j \rangle v_j) \right\|^2 = \left\| \sum_{j=1}^{n} \langle u, v_j \rangle T(v_j) \right\|^2 = \sum_{j=1}^{n} || \langle u, v_j \rangle T(v_j) ||^2 = \sum_{j=1}^{n} ||v_j||^2 \\
= ||u||^2
\end{align*} \]

(b) \( \Rightarrow \) (c)

If \( \{v_1, \ldots, v_n\} \) is an orth basis of \( V \)

\[ \langle T(v_j), T(v_k) \rangle = \langle v_j, v_k \rangle \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases} \]

\& \( \{T(v_1), \ldots, T(v_n)\} \) is a basis since \( T \) is an isomorphism

(c) \( \Rightarrow \) (a) \]
Corollary 4.16

For any inner product spaces $V$ and $W$, if $V$ and $W$ are isometric, then $\dim V = \dim W$.

Proof:

If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are orthonormal bases of $V$ and $W$, respectively, then we can find a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ such that $T(v_i) = w_i$. This is necessarily an isometry.

Proof:

If $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are orthonormal bases of $V$ and $W$, respectively, then $T : \mathbb{F}^n \to \mathbb{F}^n$ is an isometry if and only if the columns of the matrix $[T]_{v_i \to w_j}$ form an orthonormal basis of $\mathbb{F}^n$.

Proof:

Let $\{v_1, \ldots, v_n\}$ be such that the columns of $[T]_{v_i \to w_j}$ are the column vectors $[T(v_i)]_{w_j \to w_j}$. Then $T$ is an isometry if and only if $\{T(v_1), \ldots, T(v_n)\}$ form an orthonormal basis of $W$.