Math 416

Last Time: Cauchy-Schwarz
This Time: Orthogonal Bases

Def A set of vectors \( \{v_k\} \) (possibly infinite) is called orthonormal if:

\[
\langle v_k, v_j \rangle = \begin{cases} 
0 & \text{if } k \neq j \\
1 & \text{if } k = j
\end{cases}
\]

(so they are orthogonal vectors, each of length 1)

If \( S = \{v_1, \ldots, v_n\} \) is an orthonormal list & a basis of \( V \) we call it an orthonormal basis.

Example: The standard basis is orthonormal.
1) The list \( \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) is an orthonormal basis of either \( \mathbb{R}^2 \) or \( \mathbb{C}^2 \).

It is easy to find the coordinate representation of a vector in an orthonormal basis:

If \( S = \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( V \) & \( w \in V \), then we know that \( w = a_1 v_1 + \cdots + a_n v_n \) for some \( a_1, \ldots, a_n \).

Since \( S \) is orthonormal, we have for each \( j = 1, \ldots, n \):

\[
\langle w, v_j \rangle = \sum_{k=1}^{n} a_k \langle v_k, v_j \rangle = a_j
\]

Thus \( [w]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle w, v_1 \rangle \\ \vdots \\ \langle w, v_n \rangle \end{bmatrix} \).
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It is also quick to find the matrix representing a linear map.

If \( T: V \to W \) recall that the \( j \text{th} \) column of \( [T]_{\beta_V, \beta_W} \)

is \([T(v_j)]_{\beta_W}\).

So if \( \beta_V = \{v_1, \ldots, v_n\} \) and \( \beta_W = \{w_1, \ldots, w_m\} \) are orthonormal,

then \([T(v_j)]_{\beta_W} = \begin{bmatrix} \langle T(v_j), w_1 \rangle \\ \vdots \\ \langle T(v_j), w_m \rangle \end{bmatrix}\)

hence the \( ij \text{th} \) entry of \([T]_{\beta_V, \beta_W}\) is \( \langle T(v_j), w_i \rangle \).

**Theorem:** If \( \beta = \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( V \)

then for any \( u, w \in V \) we have

\[
\langle u, w \rangle = \left( \sum_{j=1}^{n} \langle u, v_j \rangle \langle v_j, w \rangle \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle u, v_j \rangle \langle \overline{w}, v_k \rangle = \sum_{j=1}^{n} \langle u, v_j \rangle \langle \overline{w}, v_j \rangle = \langle u, w \rangle
\]

as standard inner product of \([u]_\beta \) and \([w]_\beta \).

So \( \| u \|^2 = \langle u, u \rangle = \sum_{j=1}^{n} \langle u, v_j \rangle \langle \overline{v_j}, u \rangle = \sum_{j=1}^{n} |\langle u, v_j \rangle|^2 \).
Given every finite dimensional inner product space has an orthonormal basis: here's how to find one:

Gram-Schmidt process:

i) Start with any basis \( (u_1, \ldots, u_n) \) of \( V \)

ii) Define \( v_1 = u_1 / ||u_1|| \)

iii) For \( j = 2, \ldots, n \) iteratively define \( v_j \) by

\[
\tilde{v}_j = u_j - \sum_{k=1}^{j-1} \frac{\langle u_j, v_k \rangle}{||v_k||^2} v_k \quad \text{and} \quad v_j = \frac{\tilde{v}_j}{||\tilde{v}_j||}
\]

Then \( \{v_1, \ldots, v_n\} \) is an orthonormal basis of \( V \)

Moreover, for each \( j \in \{1, \ldots, n\} \)

\[
\langle v_1, \ldots, v_j \rangle = \langle u_1, \ldots, u_j \rangle
\]

Proof: We prove by induction that \( \{v_1, \ldots, v_j\} \) is orthonormal and \( \langle v_1, \ldots, v_j \rangle = \langle u_1, \ldots, u_j \rangle \)

For \( j = 1 \), note that \( u_1 \neq 0 \) so \( v_1 \) makes sense, \( ||v_1|| = 1 \), \( \langle v_1, v_1 \rangle = \langle u_1, u_1 \rangle \)

Assume this is true for \( \{v_1, \ldots, v_{j-1}\} \)

Since \( \{u_1, \ldots, u_n\} \) is linearly independent, \( <u_j, u_{j-1}> = <v_j, v_{j-1}> \)

This implies \( v_j \neq 0 \), so \( v_j \) makes sense and is a unit vector.

Since \( \{v_1, \ldots, v_{j-1}\} \) is orthonormal we have, for any \( l \in \{1, \ldots, j-1\} \)

\[
\langle v_j, v_l \rangle = \frac{1}{||v_j||} \left[ \langle u_j, \sum_{k=1}^{j-1} \langle u_j, v_k \rangle v_k \rangle - \sum_{k=1}^{j-1} \langle u_j, v_k \rangle <v_k, v_l> \right]
\]

\[
= \frac{1}{||v_j||} \left[ \langle u_j, v_l \rangle - \langle u_j, v_l \rangle \right] = 0
\]

And so \( \{v_1, \ldots, v_j\} \) is orthonormal.

Finally, since \( <v_1, \ldots, v_j> = <u_1, \ldots, u_j> \) and \( v_j \in \langle v_1, \ldots, v_{j-1} \rangle \)

we have \( <v_1, \ldots, v_j> \subseteq <u_1, \ldots, u_j> \)

but \( \dim <v_1, \ldots, v_j> = j = \dim <u_1, \ldots, u_j> \) so \( <v_1, \ldots, v_j> = <u_1, \ldots, u_j> \)
Example: Consider the space $\mathcal{S}_2(\mathbb{R})$ with inner product
\[ \langle f, g \rangle = \int_0^1 f(x) g(x) \, dx \]

The basis $\{1, x, x^2\}$ is not orthonormal.

For instance,
\[ \langle 1, x \rangle = \int_0^1 x \, dx = \frac{1}{2} \neq 0 \]

Using Gram-Schmidt, we can find an orthonormal basis: $\{v_1, v_2, v_3\}$

\[ \tilde{v}_1 = \frac{1}{\|1\|}, \quad \|1\| = \langle 1, 1 \rangle = \int_0^1 1 \, dx = 1 \text{ so } v_1 = 1 \]

\[ \tilde{v}_2 = x - \langle 1, x \rangle 1 = x - \frac{1}{2}, \quad \|x - \frac{1}{2}\| = \sqrt{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle} = \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{12} \]

So
\[ v_2 = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \sqrt{\frac{12}{x - \sqrt{3}}} \]

\[ \tilde{v}_3 = x^2 - \langle x^2, 1 \rangle 1 - \langle x^2, \sqrt{\frac{12}{x - \sqrt{3}}} \rangle \left( \sqrt{\frac{12}{x - \sqrt{3}}} \right) = x^2 - x + \frac{1}{6} \]

\[ \|x^2 - x + \frac{1}{6}\| = \int_0^1 (x^2 - x + \frac{1}{6})^2 \, dx = \frac{1}{180} \quad \text{so } v_3 = \sqrt{\frac{180}{x^2 - x + \frac{1}{6}}} \]

So $\{1, \sqrt{\frac{12}{x - \sqrt{3}}}, \sqrt{\frac{180}{x^2 - x + \frac{1}{6}}} \}$ is an orthonormal basis.