Last Time: Inner Product  
This Time: Cauchy-Schwarz

Recall $\mathbb{F}$ is now either $\mathbb{R}$ or $\mathbb{C}$.

A subset $V$ is a vector space over $\mathbb{F}$ and $\langle \cdot, \cdot \rangle$ is an inner product. $\langle v, w \rangle \in \mathbb{F}$ for $v, w \in V$.

The norm of a vector $\|v\| = \sqrt{\langle v, v \rangle}$

And we say that two vectors are orthogonal if their inner product vanishes.

Last time we finished 7.

Lemma: Given $v, w \in V$ we can find $u \in V, a \in \mathbb{F}$, s.t. $a \perp w = 0$ and $v = aw + u$.

Thm (Cauchy-Schwarz Inequality)

For any $v, w \in V$, $|\langle v, w \rangle| \leq \|v\| \|w\|$. Equality if and only if $v$ and $w$ are colinear.

Pf

Easy if $w = 0$.

Otherwise write $v = aw + u$, s.t. $a \perp w$. Use Pythagorean Thm:

$$\|v\|^2 = \|aw + u\|^2 = |a|^2 \|w\|^2 + \|u\|^2 = \frac{\langle v, w \rangle^2 + \|u\|^2}{\|w\|^2} \geq \frac{|\langle v, w \rangle|^2}{\|w\|^2}$$

Thm (Triangle Inequality)

Equality holds if and only if $v$ and $w$ are colinear, $v = cw$, $\|c\| > 0$.

Pf

$$\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle$$

$$= \|v\|^2 + 2 \Re \langle v, w \rangle + \|w\|^2 \leq \|v\|^2 + 2 |\langle v, w \rangle| + \|w\|^2$$

$$
\leq \|v\|^2 + 2 \|v\| \|w\| + \|w\|^2 = (\|v\| + \|w\|)^2
$$
For any $u, v, w \in V$, $\|u - v\| \leq \|u - w\| + \|w - v\|$

We think of this as saying:

distance between $u$ & $v$ is less than or equal to the sum of the distance between $u$ & $w$
& $w$ & $v$

**Example:** If we identify $\mathbb{M}_{m,n}(\mathbb{C}) \cong \mathbb{C}^{mn}$

the standard inner product becomes $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \overline{b}_{ij}$

Recall that $[A^*]_{kj} = \overline{a}_{jk}$ so $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} [A]_{ij} [B^*]_{kj} = \sum_{i=1}^{n} [A^* A]_{ii}$

$= \text{tr}(A A^*)$

Called the **Frobenius** inner product

& $\|A\|_{F} = \sqrt{\text{tr}(A A^*)}$ the **Frobenius norm**

The Cauchy–Schwarz inequality takes the form

$|\langle A, B \rangle| \leq \|A\| \|B\|$

$\Rightarrow \text{tr}(A A^*) \leq (\text{tr}(A A^*)) (\text{tr}(A A^*))$
Example: Let $c_1, \ldots, c_n > 0$ be fixed positive numbers. Define an inner product on $\mathbb{R}^n$ by $\langle x, y \rangle = \sum_{j=1}^{n} c_j x_j y_j$.

The dot product corresponds to $c_j = 1$ for $j$.

The standard basis vectors are still orthogonal, but they are not unit vectors, $\|e_k\| = \sqrt{\langle e_k, e_k \rangle} = \sqrt{c_k}$.

If $n = 2$, the unit vectors are given by:

\[
\begin{align*}
&c_1 = c_2 = 1 & &c_1 = \frac{1}{2}, c_2 = 1 & &c_1 = 1, c_2 = \frac{1}{4}
\end{align*}
\]

Cauchy-Schwarz: $|\langle x, y \rangle| \leq \|x\| \|y\|$

becomes $\left| \sum_{j=1}^{n} c_j x_j y_j \right| \leq \sqrt{\sum_{j=1}^{n} c_j x_j^2} \sqrt{\sum_{j=1}^{n} c_j y_j^2}$

Example: Let $V = C_c([0,1])$, let $h: [0,1] \to \mathbb{R}$ be any function.

Define $\langle f, g \rangle = \int_{0}^{1} f(x) g(x) h(x) \, dx$ for $f, g \in C_c([0,1])$.

This is an inner product.
Example. Let $\ell^2$ denote the square-summable sequences over $\mathbb{R}$.
\[ a = (a_1, a_2, \ldots) \in \ell^2 \iff \sum_{j=1}^{\infty} a_j^2 < \infty \]

Define $\langle a, b \rangle = \sum_{j=1}^{\infty} a_j b_j$.

Let's check that this makes sense (i.e., the series converges).
For each $n \in \mathbb{N}$ we can use Cauchy-Schwarz on $\mathbb{R}^n$ to see that
\[ \sum_{j=1}^{n} |a_j b_j| \leq \sqrt{\sum_{j=1}^{n} a_j^2} \sqrt{\sum_{j=1}^{n} b_j^2} \leq \sqrt{\sum_{j=1}^{\infty} a_j^2} \sqrt{\sum_{j=1}^{\infty} b_j^2} \]

The sequence on the left is non-decreasing and bounded above so it must converge and its limit satisfies
\[ \sum_{j=1}^{\infty} |a_j b_j| \leq \sqrt{\sum_{j=1}^{\infty} a_j^2} \sqrt{\sum_{j=1}^{\infty} b_j^2} \]

Hence the series $\sum_{j=1}^{\infty} a_j b_j$ converges absolutely.

The other properties of an inner product space are easy to check.

The space $\ell^2$ is defined similarly, a sequence $a = (a_1, \ldots)$ of $\mathbb{C}$
\[ a = (a_1, a_2, \ldots) \in \ell^2 \iff \sum_{j=1}^{\infty} |a_j|^2 < \infty \]

The corresponding inner product is $\langle a, b \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}$.

Note that, as above, we could also take $\langle a, b \rangle = \sum_{j=1}^{\infty} a_j b_j c_j$

with $c_1, c_2, \ldots$ any fixed sequence of positive real numbers.