Math 416

Last Time: Computing determinants
This Time: Characteristic Polynomials

A is an eigenvalue of $A$ if $A - \lambda I_n$ is singular

$\Leftrightarrow \det(A - \lambda I_n) = 0$

We call $p_A(\lambda) = \det(A - \lambda I_n)$ the characteristic polynomial of $A$

Prop A $\in M_n(\mathbb{F})$ if $p_A(\lambda)$ is a poly of degree $n$ w/ leading term $(-x)^n$

Let $B = A - \lambda I_n$ so $p_B(\lambda) = \det B = \sum_{\sigma \in S_n} \text{sgn} \sigma b_{1\sigma(1)} \cdots b_{n\sigma(n)}$

Given $\sigma \in S_n$

Let $\sigma_0 \in \{1, \ldots, n\}$ be the fixed pt of $\sigma$, i.e., $\sigma_0 = \{i : \sigma(i) = i\}$

Then $b_{1\sigma(1)} \cdots b_{n\sigma(n)} = \prod_{i=1}^{n} (a_{i\sigma_0} - x) \prod_{i=1}^{n} a_{i\sigma(i)}$

so each term is a polynomial of degree $= \# \sigma_0$

The highest possible degree is $n$ which corresponds to $\# \sigma_0 = n = i$

Hence it is $(-x)^n$

If $A \in M_n(\mathbb{F})$ has distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ then

$p_A(\lambda) = \prod_{j=1}^{n} (\lambda_j - \lambda)$

Indeed, it is divisible by $(x - \lambda_j)$ for each $\lambda_j$

has degree $n$ & leading coefficient $(-1)^n$
Every field \( F \) is contained in an algebraically closed field \( \overline{F} \) as a subfield (i.e., with the same operations).

The advantage of working over an alg. closed field is that the characteristic polynomial factors into linear terms.

For example, if \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) the \( p_A(x) = x^2 + 1 \) has no solutions in \( \mathbb{R} \) but equals \( (x-i)(x+i) \) in \( \mathbb{C} \).

**Def:** If alg. closed, \( A \in M_n(F) \) w/ distinct eigenvalues \( \{ \lambda_1, \lambda_2, \ldots, \lambda_k \} \).

The char. poly of \( A \) factors as \( p_A(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_k)^{m_k} \).

The power \( m_j \) refers to \( \lambda_j \) as an eigenvalue of \( A \).

For example: \( p(x) = x^6 - 5x^5 + 6x^4 + 4x^3 - 8x^2 = x^2(x+1)(x-2)^3 \)

has 3 distinct roots, but 6 roots if we count the multiplicity \( 0, 0, -1, 2, 2, 2 \).

If \( A \in M_n(F) \) is upper triangular, the multiplicity of an eigenvalue is \( \# \) times it appears on diagonal.

**NB:** The char. poly of \( A \), the eigenvalues of \( A \) & their multiplicities are all invariants of \( A \).

If \( F \) is alg. closed, any \( A \in M_n(F) \) is similar to an upper triangular matrix & hence:

\( \text{tr}(A) = \lambda_1 + \cdots + \lambda_n \) & \( \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n \)

where \( \{ \lambda_1, \ldots, \lambda_n \} \) are the eigenvalues of \( A \) counted \#multiplicity.
Let $F$ be any field & $A \in M_n(F)$

- the coeff of $x^{n-1}$ in $p_A(x)$ is $(-1)^{n-1} \det A$
- $x^0$ (the constant term) is $\det A$

If $F$ is algebraically closed then

$$p_A(x) = (\lambda_1 - x) \cdots (\lambda_n - x)$$
so $x^0 = p_A(0) = \lambda_1 \cdots \lambda_n = \det A$

& the $x^{n-1}$ term is $\lambda_1 (-x)^{n-1} + \lambda_2 (-x)^{n-2} + \cdots + \lambda_n (-x)^0 = (-1)^{n-1} (\det A) x^{n-1}$

If $F$ is not alg. closed then choose $K$ alg. closed containing $F$ as a subfield

$p_A(x)$ factors in $K$, so we can argue as above to see that $p_A(0) = \det A$ & the coeff of $x^{n-1}$ is $(-1)^{n-1} (\det A)$

but $p_A(x)$ is the same when we use $F$ as when we use $K$.

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**Thm (Cayley-Hamilton)** If $A \in M_n(F)$ then $p_A(A) = 0$

**pf**

Note that we can not just plug in $A$ into $p_A(x) = \det(A-xI)$

Instead, we'll see that $p_A(A)v = 0$ for all $v \in F^n$

We may assume that $F$ is alg. closed

since otherwise we can pass to a larger alg. closed field $K$.

Note that $p_A(A) = 0 \Leftrightarrow p_B(B) = 0$ for any $B$ similar to $A$.

Since $B = SAS^{-1} \Rightarrow p_A(A) = S p_B(B) S^{-1}$

$A$ is similar to an upper triangular $T$ so we need to check $p_T(T) = 0$

$p_T(x) = (t_{11} - x) \cdots (t_{nn} - x)$ so $p_T(T) = (t_{11}I - T) \cdots (t_{nn}I - T)$

Note $\langle t_{ij} I - T, e_j \rangle \neq 0$ for $j > i$

so inductively $(t_{ii}I - T) \cdots (t_{ij}I - T) (e_k) = 0$ for $k \in \{1, \ldots, j\}$

hence $p_T(T) = (t_{ii}I - T) \cdots (t_{jj}I - T)$ vanishes on $\{e_i, \ldots, e_n\}$

$p_T(T) = 0$