Math 416

Last Time: Similar Matrices
This Time: Triangularization

Given $T \in \mathcal{L}(V)$ we'd like to find a basis $\mathcal{B}$ for $V$
that makes $[T]_{\mathcal{B}}$ as simple as possible.

Given $A \in M_n(\mathbb{F})$ we'd like to find the simplest
matrix that is similar to $A$.

The simplest matrices are diagonal.

$\mathcal{L}(V)$ diagonalizable $\iff$ there's a basis of $V$ consisting of eigenvectors of $T$
$A \in M_n(\mathbb{F})$ $\iff$ $A$ diagonal.

Upper triangular matrices are almost as good as diagonal:
$A \in M_n(\mathbb{F})$ upper triangular.

1) If there are no zeros on diagonal, $A$ is in REF/upper triangular.
2) If there are zeros on diagonal, the first col of first zero has no pivot.
3) Eigenvectors of $A = \text{Diagonal elements of } A$.

Not every matrix has eigenvalues.
$e.g.$ $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$.
$[0 \ 1][a \ b] = \begin{bmatrix} -a \\ b \end{bmatrix}$
$\Rightarrow -a = \lambda a \Rightarrow q = \lambda(-1a) \Rightarrow (\lambda^2 + 1)a = 0$
$\Rightarrow a = \lambda b$.

Since $\lambda^2 + 1 = 0$ has no solutions in $\mathbb{R}$, $a = 0 \land b = 0$.

Def: A field $\mathbb{F}$ is algebraically closed if every polynomial
$p(x) = a_0 + a_1 x + \ldots + a_n x^n$ (coefficients in $\mathbb{F}$ and $a_0 \neq 0$)
can be factored in the form
$p(x) = b(x-c_1) \cdots (x-c_n)$ for some $b, c_1, \ldots, c_n \in \mathbb{F}$.

Thm (FTA) $\mathbb{C}$ is algebraically closed.
Prop Let $F$ be alg. closed, $V$ an $n$-dim vs. over $F$, $T \in L(V)$. Then at least one eigenvvalue

**pf**

Let $n = \dim V \land v \in V, v \neq 0$. Let $T^k(v) = \underbrace{(T \circ T \circ \cdots \circ T)}_{k \text{ times}}(v)$.

The list $\{v, Tv, T^2v, \ldots, T^nv\}$ has, $n+1$ vectors, so is lin. dependent.

So there are constants $a_0, \ldots, a_n \in F$ s.t.

$$a_0v + a_1Tv + \ldots + a_nT^nv = 0$$

Let $p(x) = a_0 + a_1x + \ldots + a_nx^n$ be the polynomial with coeff $a_0, \ldots, a_n$.

Since $F$ is algebraically closed, we can factor $p$:

$$p(x) = b(x-c_1)(x-c_2)\cdots(x-c_n)$$

Therefore:

$$(a_0I + a_1T + \ldots + a_nT^n) = b(T-c_1I)\cdots(T-c_nI)$$

Hence, $(T-c_iI)\cdots(T-c_nI)v = 0 \land v \neq 0$.

If $(T-c_nI)v = 0$ then $v$ is an eigenvector of $T$, eigenvalue $c_n$.

Else, let $v' = (T-c_nI)v \neq 0$.

$(T-c_1I)\cdots(T-c_{n-1}I)v' = 0$.

If $(T-c_{n-1}I)v' = 0$ then $v'$ is an eigenvector of $T$, eigenvalue $c_{n-1}$.

Else, let $v'' = (T-c_{n-1}I)v'$.

Continuing in this way, we eventually find that one of the $c_i$'s must be an eigenvalue of $T$.

**Example** $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is $M_2(\mathbb{C})$, $v = [1] \neq 0$.

$$\{v, Av, A^2v\} = \{[1], [-1], [-1]\}$$

is lin. dependent.

Indeed, $A^2v = [-1] = -v$ so $(A^2 + I)v = 0$ factoring $x^2 + 1 = (x+i)(x-i)$.

We can write $(A-iI)(A+iI)v = 0$, $(A+iI)v = [-1+i \ i] \neq 0$.

We're guaranteed that $\{(A-iI)\ [-1+i \ i] = 0$ and indeed

$$\begin{bmatrix} -1+i \\ 1+i \end{bmatrix} = \begin{bmatrix} 1 \ -1+i \end{bmatrix} \begin{bmatrix} -1+i \\ 1+i \end{bmatrix}$$
Lemma: \( T \in \mathcal{L}(V), \mathcal{B} = \{v_1, \ldots, v_n\} \) a basis of \( V \)
\[ [T]_{\mathcal{B}} \text{ upper triangular} \iff T(v_j) \in \langle v_1, \ldots, v_j \rangle \quad \forall j \]

Thm: If \( V \) is finite-dimensional over \( \mathbb{F} \), \( T \in \mathcal{L}(V) \)
There is a basis \( \mathcal{B} \) of \( V \) s.t. \( [T]_{\mathcal{B}} \) is upper triangular.

Proof:
Proof by induction over \( n = \dim V \)
For \( n = 1 \), it is obvious, since every \( 1 \times 1 \) matrix is upper triangular.
Suppose the theorem is known for spaces of \( \dim < n \).
We know that \( T \) has an eigenvector.
Let \( v \in V \) be an eigenvector for \( T \) w/ eigenvalue \( \lambda \).
Write \( U = \text{Range } (T - \lambda I) \).
If \( u \in U \) then \( T(u) \in U \) because \( T(u) = (T - \lambda I)(u) + \lambda u \in U \).

\( \lambda \) is an eigenvalue \( \iff T - \lambda I \) not injective \( \iff T - \lambda I \) not surjective
\( \iff \dim U < n \).

Let \( m = \dim U \). By inductive hypothesis, there is a basis \( \mathcal{B}_u = \{u_1, \ldots, u_m\} \) of \( U \) s.t. \( T \) restricted to \( U \) is upper triangular.
\( \iff T(v_j) \in \langle u_1, \ldots, v_j \rangle \quad \forall j \).

Extend \( \mathcal{B}_u \) to a basis \( \mathcal{B}_v = \{u_1, \ldots, u_m, v_1, \ldots, v_k\} \) of \( V \).
For any \( j \in \{1, \ldots, k\} \), \( T(v_j) = (T - \lambda I)(v_j) + \lambda v_j \in \langle u_1, \ldots, u_m, v_j \rangle \in \langle u_1, \ldots, u_m, v_1, \ldots, v_k \rangle \).

Hence \( [T]_{\mathcal{B}_v} \) is upper triangular.

Consequence: If \( \mathbb{F} \) is algebraically closed, \( A \in M_n(\mathbb{F}) \) \& \( p(x) \) is a polynomial in \( \mathbb{F} \)
the eigenvalues of \( p(A) \) are of the form \( p(\lambda) \) \& \( \lambda \) an eigenvalue of \( A \).
Indeed, we can reduce to the upper triangular case & then \( \lambda \) is easy to see.