Math 416

Last Time: Rank Nullity Theorem
This Time: Coordinates

If \( v \in \mathbb{F}^n \) has coordinates \([v_1, \ldots, v_n] \) in the standard canonical basis of \( \mathbb{F}^n \)

we could also say that \( [v_1, \ldots, v_n] \) is the "coordinate representation of \( v \) with respect \( \{e_1, \ldots, e_n\} \)."

Def: Let \( \mathcal{B} = \{v_1, \ldots, v_n\} \) be a basis of \( V \)

For each \( u \in V \) the coordinate of \( u \) with respect to the basis \( \mathcal{B} \)

is \( (\text{unique ordered list of scalars}) \ a_1, \ldots, a_n \) s.t.

\[ u = a_1 v_1 + \cdots + a_n v_n \]

The coordinate representation \([u]_{\mathcal{B}} \in \mathbb{F}^n\) of \( u \) with respect to \( \mathcal{B} \)

is \( [u]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \)

If \( V \) is an \( n \)-dim. \( \mathbb{F} \)-vector space over \( \mathbb{F} \) by basis \( \mathcal{B} \)

representing vectors in coordinate gives an explicit isomorphism between \( V \) and \( \mathbb{F}^n \)

Indeed, the map \( C_{\mathcal{B}} : V \rightarrow \mathbb{F}^n \) is an isomorphism

\[ v \mapsto [v]_{\mathcal{B}} \]

Ex: \( V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y - 2z = 0 \right\} \) with basis \( \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \)

The vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V \) can be written as \( v = a_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \)

\[ 3a_1 + 2a_2 = 1, \quad a_2 = \frac{1}{2}, \quad a_1 = \frac{1}{6} \]

hence \( [v]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{4} \end{bmatrix} \)
If \( T \in \mathcal{L}(V,W) \) & \( V, W \) are finite dimensional
then a basis \( \{v_1, \ldots, v_n\} \) for \( V \) gives an isomorphism \( V \cong \mathbb{F}^n \)
& a basis \( \{w_1, \ldots, w_m\} \) for \( W \) give an isomorphism \( W \cong \mathbb{F}^m \).

Since we know \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \)
is given by multiplication by a matrix \( A \in \mathcal{M}_{m \times n}(\mathbb{F}) \).
These isomorphisms allow us to identify \( T \) with multiplication by a matrix.

Suppose \(Tv_j = a_{1j}w_1 + \cdots + a_{mj}w_m \).
The matrix of \( T \) with respect to \( \mathcal{B}_V \) & \( \mathcal{B}_W \) is
\[
A = \begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
a_{21} & \cdots & a_{2m} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nm}
\end{bmatrix}.
\]

We denote this matrix as \([T]_{\mathcal{B}_V, \mathcal{B}_W}^\mathcal{B}_V, \mathcal{B}_W\).

When \( V = W \) & \( \mathcal{B}_V = \mathcal{B}_W = \mathcal{B} \) we simply write \([T]_\mathcal{B}\)
& refer to it as the matrix of \( T \) with respect to \( \mathcal{B} \).

Just as before, we can find this matrix one column at a time.
The \( j \)th column of \([T]_{\mathcal{B}_V, \mathcal{B}_W}^\mathcal{B}_V, \mathcal{B}_W\) is the coordinate representation \([Tv_j]_{\mathcal{B}_W}^{\mathcal{B}_V}\).

**Lemma**

\( T \in \mathcal{L}(V,W) \), \( \mathcal{B}_V \) a basis of \( V \), \( \mathcal{B}_W \) a basis of \( W \)
& \( A = [T]_{\mathcal{B}_V, \mathcal{B}_W} \).

Then, for any \( v \in V \), \([Tv]_{\mathcal{B}_W} = A[v]_{\mathcal{B}_V}\).

**Proof**

If \( \mathcal{B}_V = \{v_1, \ldots, v_n\} \), \( \mathcal{B}_W = \{w_1, \ldots, w_m\} \),
then \( a_{ij} \) are defined by
\[
Tv_j = a_{1j}w_1 + \cdots + a_{mj}w_m.
\]

So if \( v \in V \) has \([v]_{\mathcal{B}_V} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \) (i.e. \( v = c_1v_1 + \cdots + c_nv_n \))
\[
Tv = T \left( \sum_{j=1}^n \xi_j v_j \right) = \sum_{j=1}^n \xi_j \left( \sum_{i=1}^m a_{ij} w_i \right) = \sum_{i=1}^m \left( \sum_{j=1}^n \xi_j a_{ij} \right) w_i.
\]

So the \( i \)th coordinate of \( Tv \) with respect to \( \mathcal{B}_W \) is the same as that of \( AV_i \)
Given a matrix $A \in M_{m \times n}(F)$
It defines a linear map $L(F^n, F^m)$
by matrix vector multiplication
It also defines a linear map, after choosing bases,
between any $n$-dim vector space over $F$
& any $m$-dim vector space over $F$
If $V$ is an $n$-dim vector space over $F$
& $W$ is an $m$-dim vector space over $F$
We can define a map
$T_{A, B, C, D}: V \rightarrow W$ by $[T_{A, B, C, D} v]_B = A [v]_A$

Note that we can not leave out any part of
this notation:
The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for example
gives $m$ different linear maps
if we use the standard basis $S = \{e_1, e_2, e_3\}$
or the standard basis in the opposite order $S' = \{e_3, e_2, e_1\}$
a follows

$T_{A, B, C, D}$:

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$D: \mathcal{P}_n(\mathbb{R}) \rightarrow \mathcal{P}_{n-1}(\mathbb{R})$

If we use the bases $\{1, x, \ldots, x^n\}$ for $\mathcal{P}_n(\mathbb{R})$
and $\{1, x, \ldots, x^{n-1}\}$ for $\mathcal{P}_{n-1}(\mathbb{R})$

$D$ is multiplication by the $n \times (n+1)$ matrix

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

Thus let $V$ & $W$ be finite dimensional vector spaces
and $T \in \mathcal{L}(V, W)$

there are bases $\mathcal{B}_V$ of $V$ & $\mathcal{B}_W$ of $W$
such that $A = [T]_{\mathcal{B}_V, \mathcal{B}_W}$

is given by $a_{ij} = \begin{cases} 1 & \text{if } i = j \leq m \\ 0 & \text{else} \end{cases}$