Last Time: Bases & Linear Maps
This Time: Dimension

Last time we proved that if \( \{v_1, \ldots, v_m, w_1, \ldots, w_n\} \subseteq V \) are s.t. \( \{v_1, \ldots, v_m\} \) spans \( V \) & \( \{w_1, \ldots, w_n\} \) is l. i. in \( V \), then \( m \leq n \).

It follows that if \( \{v_1, \ldots, v_m\} \) & \( \{w_1, \ldots, w_n\} \) are bases of \( V \), then \( m \geq n \) & \( m \leq n \) so \( m = n \).

Thus, this common value is the dimension of \( V = \dim V \).

Example: The canonical basis of \( \mathbb{F}^n \), \( \{e_1, \ldots, e_n\} \), has length \( n \), so \( \dim \mathbb{F}^n = n \).
- \( \mathbb{P}_n(\mathbb{R}) \) = polynomials \( f \) real coefficients of degree at most \( n \) has basis \( \{1, x, \ldots, x^n\} \) & hence has dimension \( n + 1 \).

Clearly, if \( \{v_1, \ldots, v_m\} \subseteq V \) then \( \dim V \leq m \).
& if \( \{w_1, \ldots, w_n\} \) is l. i. in \( V \), then \( \dim V \geq n \).

Interestingly, if \( U_1, U_2 \) are subspaces of \( V \),
then \( \dim U_1 + \dim U_2 > \dim V \) then \( U_1 \cap U_2 \neq \{0\} \).

Indeed, let's check the contrapositive: If \( U_1, U_2 \) are subspaces of \( V \) s.t. \( U_1 \cap U_2 = \{0\} \) then \( \dim U_1 + \dim U_2 \leq \dim V \).

Let \( \{v_1, \ldots, v_p\} \) be a basis of \( U_1 \) & \( \{w_1, \ldots, w_q\} \) be a basis of \( U_2 \).
Let's check that \( \{v_1, \ldots, v_p, w_1, \ldots, w_q\} \) is l. i. in \( V \).
If \( \sum_{i=1}^{p} a_i v_i + \sum_{j=1}^{q} b_j w_j = 0 \) for scalars \( a_1, \ldots, a_p, b_1, \ldots, b_q \),
then \( \sum_{i=1}^{p} a_i v_i = -\sum_{j=1}^{q} b_j w_j \in U_1 \cap U_2 = \{0\} \Rightarrow \sum_{i=1}^{p} a_i v_i = 0 \Rightarrow a_i = 0 \forall i, \sum_{j=1}^{q} b_j w_j = 0 \Rightarrow b_j = 0 \forall j \).
Thus, \( \{v_1, \ldots, v_p, w_1, \ldots, w_q\} \) is a linearly independent subset of \( V \).

It turns out that the dimension is the only thing that distinguishes vector spaces up to isomorphism.

Then let \( V \) and \( W \) be finite-dimensional vector spaces over \( \mathbb{F} \).

Then \( \dim V = \dim W \) if and only if \( V \) is isomorphic to \( W \).

If \( \dim V = \dim W = n \), pick bases \( \{v_1, \ldots, v_n\} \) of \( V \)

and \( \{w_1, \ldots, w_n\} \) of \( W \).

Define \( T : \mathcal{L}(V,W) \) by demanding \( T(v_i) = w_i \) \( \forall i \)

and extending by linearity.

Since \( T \) sends a basis to a basis, it is an isomorphism.

On the other hand, since an isomorphism sends a basis to a basis, isomorphic spaces must have the same dimension.

In particular, every \( n \)-dimensional vector space is isomorphic to \( \mathbb{F}^n \).

1) Note that for a matrix \( A \),

the dimension of its column space is the number of pivots in its \( \text{RREF} \).

2) If \( \dim V = n \) and \( V = \langle v_1, \ldots, v_n \rangle \)

then \( \{v_1, \ldots, v_n\} \) is a basis of \( V \).

Indeed, we have shown that any list that spans contains a sublist that is a basis.

Counting dimensions, here it must be the entire list.
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Theorem If $V$ is a finite-dimensional vector space & $S \subseteq V$ is a linearly independent list then $S$ can be extended to a basis.

Proof Suppose $S = \{v_1, \ldots, v_m\}$

If $V = \langle v_1, \ldots, v_m \rangle$ then $S$ is a basis & we're done.

Otherwise we can find $v_{m+1} \in V \setminus \langle v_1, \ldots, v_m \rangle$ & the list $\{v_1, \ldots, v_{m+1}\}$ is linearly independent.

We can keep adding vectors like this until we've found a list of length $\dim V$ of linearly independent vectors.

In particular, if $\dim V = n$, any list of $n$ vectors in $V$ that is linearly independent must be a basis of $V$.

Theorem Suppose $V$ is a finite dimensional vector space & $U \subseteq V$ is a subspace.

i) $U$ is finite dimensional
   a) $\dim U \leq \dim V$
   b) If $\dim U = \dim V$ then $U = V$

Proof

i) If $U = \{0\}$ we're done.

Otherwise pick $u \in U \setminus \{0\}$ so that $\{u\}$ is linearly independent.

If $U = \langle u \rangle$ we're done.

Otherwise pick $v \in U \setminus \langle u \rangle$ so that $\{u, v\}$ is linearly independent.

In this way we can keep building a linearly independent list of vectors in $U$ & since $U \subseteq V$ any such list $T$ can have at most $\dim V$ vectors, so at some point we must have $U = \langle u_1, \ldots, u_k \rangle$

i) $\{u_1, \ldots, u_k\} \subseteq V$ is linearly independent so $k \leq \dim V$

ii) If $k < \dim V$ then $V = \langle u_1, \ldots, u_k \rangle = U$

Ex. In $\mathbb{R}^3$ subspaces are $\{0\}$, line through $0$, plane through $0$, all of $\mathbb{R}^3$.