LAST TIME: Bases
This Time: Bases & Linear Maps

Recall that \( \{v_1, \ldots, v_n \} \) is a basis of \( V \)

(1) The span of \( \{v \} \) is \( V \) & (2) \( \{v \} \) is linearly independent

Every \( u \in V \) can be written in \( \{v_1, \ldots, v_n \} \)

as a linear combination of \( \{v_1, \ldots, v_n \} \).

From this characterization we see that a linear map

is determined by what it does to the elements of a basis.

Thus suppose \( \{v_1, \ldots, v_n \} \) is a basis for \( V \)

& let \( \{w_1, \ldots, w_m \} \subseteq W \) be any vectors in \( W \)

Then there is a unique linear map \( T \in \mathbb{L}(V, W) \) s.t. \( T(v_i) = w_i \) \( \forall i \).

pf

First let's show that there is such a map.

Given \( u \in V \), write it as a lin comb of \( \{v_1, \ldots, v_n \} \),

\( u = \sum_{j=1}^{n} c_j v_j \)

& define \( T(u) \in W \) to be \( T(u) = \sum_{j=1}^{n} c_j w_j \).

This sends \( v_j \) to \( w_j \), so we just need to check that it is linear.

Given \( u_1, u_2 \in V \), cf F we need to show that \( T(u_1 + c_1 u_2) = T(u_1) + c_1 T(u_2) \).

Write \( u_1 = \sum_{j=1}^{n} a_j v_j \) \( u_2 = \sum_{j=1}^{n} b_j v_j \).

Then the (unique) way to write \( u_1 + c_1 u_2 \) as a lin comb of \( \{v_1, \ldots, v_n \} \)

is \( u_1 + c_1 u_2 = \sum_{j=1}^{n} (a_j + c_1 b_j) v_j \).

& we have \( T(u_1 + c_1 u_2) = \sum_{j=1}^{n} (a_j + c_1 b_j) w_j \)

\( = \sum_{j=1}^{n} a_j w_j + c_1 \sum_{j=1}^{n} b_j w_j = T(u_1) + c_1 T(u_2) \).
This shows that there is a linear map \( T \) s.t. \( T(v_j) = v_j \) \( \forall j \).

To see that it is the only map \( T \) with this property, notice that if \( S \in \mathcal{L}(V, W) \) has this property and \( u \in V \) is equal to \( \frac{1}{j} c_j v_j \) we have

\[
S(u) = S\left( \sum_{j=1}^{n} c_j v_j \right) = \sum_{j=1}^{n} c_j S(v_j) = \sum_{j=1}^{n} c_j v_j = T(u) \quad \|
\]

The short-hand for this construction is to say: "We demand \( T(v_j) = v_j \) \( \forall j \) and then we extend by linearity.

One way of showing that two linear maps \( S, T \in \mathcal{L}(V, W) \) are the same is to find a basis \( \{v_1, ..., v_n\} \) of \( V \) s.t. \( S(v_j) = T(v_j) \) \( \forall j \).

Thus let \( T \in \mathcal{L}(V, W) \) and let \( \{u_1, ..., u_n\} \) be a basis of \( V \).

\( T \) is an isomorphism if and only if \( \{T(u_1), ..., T(u_n)\} \) is a basis of \( W \).

**Proof:**

(\( \Rightarrow \)) Assume \( T \) is an isomorphism.

We want to show that \( \{T(u_1), ..., T(u_n)\} \) is a basis of \( W \).

In fact, we want to show that:

(i) \( \{T(u_1), ..., T(u_n)\} \) spans \( W \).

(ii) \( \{T(u_1), ..., T(u_n)\} \) is linearly independent.

(i) Given \( u \in W \), since \( T \) is surjective, there is a \( \bar{u} \in V \) s.t. \( T(\bar{u}) = u \).

Write \( u = \sum_{j=1}^{n} c_j v_j \) and then \( u = T(\bar{u}) = \sum_{j=1}^{n} c_j T(v_j) \).

And so \( \{T(v_1), ..., T(v_n)\} \) spans \( W \).

(ii) Assume \( \sum_{j=1}^{n} a_j T(v_j) = 0 \) we need to show \( a_j = 0 \) \( \forall j \).

And indeed \( \sum_{j=1}^{n} a_j T(v_j) = T(\sum_{j=1}^{n} a_j v_j) = 0 \Rightarrow \sum_{j=1}^{n} a_j v_j \in \ker T = \{0\} \).

So \( \sum_{j=1}^{n} a_j v_j = 0 \) and since \( \{v_1, ..., v_n\} \) is linearly independent \( \Rightarrow a_j = 0 \) \( \forall j \).
\( \iff \) Assume that \( \{T(v_1), \ldots, T(v_n)\} \) is a basis of \( W \)

We want to show that \( T \) is an isomorphism.

1. We want to show that \( (a) T \) is injective \( (b) T \) is surjective.

(a) If \( T(u) = 0 \) we want to show \( u = 0 \)

Write \( u = \sum_{j=1}^{n} c_j v_j \) so \( T(u) = \sum_{j=1}^{n} c_j T(v_j) = 0 \)

Then since \( \{T(v_1), \ldots, T(v_n)\} \) is lin. indep. \( c_j = 0 \) for \( j = 1, \ldots, n \)

(b) Given \( w \in W \) we want to find \( u \) s.t. \( T(u) = w \)

Write \( w = \sum_{j=1}^{n} a_j T(v_j) \) & let \( u = \sum_{j=1}^{n} a_j v_j \)

Then \( T(u) = T(\sum_{j=1}^{n} a_j v_j) = \sum_{j=1}^{n} a_j T(v_j) = w \)

**Lemma**: Suppose \( \mathbf{v} \in \langle v_1, \ldots, v_m \rangle \) & that \( \{w_1, \ldots, w_n\} \) is lin. ind.

Then \( m \geq n \)

**Proof**

For each \( j = 1, \ldots, n \), \( w_j \in \langle v_1, \ldots, v_m \rangle \) so there are scalars \( a_{ij} \) s.t.

\[ w_j = \sum_{i=1}^{m} a_{ij} v_i \]

\[ \begin{align*}
A x &= 0 \\
\text{Let } A &= \begin{bmatrix} a_{ij} \end{bmatrix} & \text{consider the homogeneous system}
\end{align*} \]

If \( x \) is a solution then \( \sum_{j=1}^{n} a_{ij} x_j = 0 \)

& so \( 0 = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} x_j \right) v_j = \sum_{j=1}^{n} x_j \left( \sum_{i=1}^{m} a_{ij} v_j \right) = \sum_{j=1}^{n} x_j w_j \)

but \( \{w_1, \ldots, w_m\} \) is lin. ind. so \( x_j = 0 \) for \( j \)

Therefore \( A x = 0 \) has \( x = 0 \) as its only solution so \( m \geq n \)
It follows that if \( \{v_1, \ldots, v_n\} \) & \( \{w_1, \ldots, w_n\} \) are both bases of \( V \), then \( n = m \).

We call this number the dimension of \( V \) & we say that \( V \) is \( n \)-dimensional.

(If \( V = \emptyset \), we say that \( \dim V = 0 \)).

If \( V \) is not finite dimensional, we say that it is infinite dimensional.