Office hour today 2-3 pm
Last Time: Linear Independence
This Time: Basis

**Def:** A vector space $V$ is finite-dimensional if there is a finite list of vectors $\{v_1, \ldots, v_n\}$ in $V$ that span $V$, $V = \langle v_1, \ldots, v_n \rangle$

Of course we prefer not to include redundant vectors:

**Def:** A list $\{v_1, \ldots, v_n\}$ is a basis of $V$ if it is linearly independent & $V = \langle v_1, \ldots, v_n \rangle$

**Example:** The canonical basis $\{e_1, \ldots, e_n\} \subseteq \mathbb{F}^n$ is a basis of $\mathbb{F}^n$.

The space $\mathcal{P}_n(\mathbb{R})$ of polynomials of degree $\leq n$ with real coefficients has as basis $\{1, x, x^2, \ldots, x^n\}$

The kernel of a matrix $A \in \text{Mat}_{m \times n}(\mathbb{F})$ is finite dimensional & it is easy to find a basis: just solve $Ax = 0$.

Eg. $A = \begin{bmatrix} 2 & -1 & 3 & -5 \\ 2 & 0 & 2 & -4 \\ 1 & 1 & 0 & -1 \end{bmatrix}$, $\text{REF}$ of $[A | 0]$ is $\begin{bmatrix} 1 & 0 & 1 & -2 & | 0 \\ 0 & 1 & -1 & 1 & | 0 \\ 0 & 0 & 0 & 0 & | 0 \end{bmatrix}$

So every vector in $\text{Ker} A$ is of the form $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, which is the same as saying $\text{Ker} A = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rangle$

& since these vectors are linearly independent (look at last two entries) they form a basis.
Given a list \( \{v_1, \ldots, v_n\} \) of vectors in \( \mathbb{F}^m \), we've seen that:
1. \( \{v_1, \ldots, v_n\} \) is lin indep \( \iff \) \( \text{RREF of } A = \begin{pmatrix} 1 & \ldots & 1 \\ v_1 & \ldots & v_n \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \end{pmatrix} \) has a pivot \( (n \times m) \) in every col.
2. \( \{v_1, \ldots, v_n\} \) spans \( \mathbb{F}^m \) \( \iff \) \( \text{RREF of } A = \begin{pmatrix} 1 & \ldots & 1 \\ v_1 & \ldots & v_n \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \end{pmatrix} \) has a pivot \( (n \times m) \) in every row.

Hence
\( \{v_1, \ldots, v_n\} \) is a basis of \( \mathbb{F}^m \) \( \iff \) \( \text{RREF of } A = \begin{pmatrix} 1 & \ldots & 1 \\ v_1 & \ldots & v_n \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \end{pmatrix} \) \( \iff \) \( \text{Im} \).

**Theorem**: A list of vectors \( \{v_1, \ldots, v_n\} \) is \( \text{a basis of } V \) if and only if every vector we \( \in V \) can be written in a unique way as a linear combination of \( \{v_1, \ldots, v_n\} \).

**Proof**

\( \Rightarrow \) Suppose \( \{v_1, \ldots, v_n\} \) is a basis of \( V \) and \( \alpha \in V \).

We need to show that \( \alpha \) can be written in a unique way as a linear combination of \( \{v_1, \ldots, v_n\} \).

We know \( \{v_1, \ldots, v_n\} \) spans \( V \), so we can find \( a_1, \ldots, a_n \in \mathbb{F} \) s.t.
\[ \alpha = \sum_{j=1}^{n} a_j v_j \]

If we also have \( \alpha = \sum_{j=1}^{n} a'_j v_j \), then substituting we find \( 0 = \alpha - \alpha = \sum_{j=1}^{n} (a_j - a'_j)v_j \);

so linear independence \( \Rightarrow \) \( a_j - a'_j = 0 \) \( \forall j \) \( \Rightarrow \) \( a_j = a'_j \) \( \forall j \).

\( \Leftarrow \) If every vector \( \alpha \in V \) can be written in a unique way as a linear combination of \( \{v_1, \ldots, v_n\} \),
then clearly \( \text{span} \{v_1, \ldots, v_n\} = V \).

Moreover, if we have \( 0 = \sum_{j=1}^{n} a_j v_j \), then since we also have \( 0 = \sum_{j=1}^{n} 0 \cdot v_j \), uniqueness \( \Rightarrow \) \( a_j = 0 \) \( \forall j \).

So \( \{v_1, \ldots, v_n\} \) is linearly indep.
Suppose $V$ is a nonzero vector space. If we can find a list $\{v_1, \ldots, v_n\}$ that spans $V$, then we can find a sublist $S_B \subseteq \{v_1, \ldots, v_n\}$ that forms a basis of $V$. We’ll do this in $n$ steps:

**Step 1:** If $v_1 \neq 0$ then add it to $S_B$.

So if $S_B \neq \emptyset$ then $\langle v_1 \rangle \subseteq \langle S_B \rangle$.

**Step 2:** If $v_2 \notin \langle v_1 \rangle$ then add it to $S_B$.

Note if $S_B \neq \emptyset$ then $\langle v_1, v_2 \rangle \subseteq \langle S_B \rangle$.

**Step j:** If $v_j \notin \langle v_1, \ldots, v_{j-1} \rangle$ then add it to $S_B$.

Note if $S_B \neq \emptyset$ then $\langle v_1, \ldots, v_j \rangle \subseteq \langle S_B \rangle$.

After the $n$th step we’ve constructed $S_B$ s.t. $\langle v_1, \ldots, v_n \rangle = \langle S_B \rangle$ but also s.t. $S_B$ is linearly independent by the linear dependence lemma, hence $S_B$ is a basis.

Notice that hence every finite-dimensional space has a basis.

If our vector space is $\mathbb{F}^m$ there is a better algorithm than the $n$ steps above.

If $\{v_1, \ldots, v_n\} \subseteq \mathbb{F}^m$ we can find a basis of $\langle v_1, \ldots, v_n \rangle$, $S_B$, by putting the matrix $A = \left( \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \right)$ in $RREF$ and then including $v_j \in S_B$ if and only if the $j$th column has a pivot.
Indeed, if $A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is already in RREF then our $n$-step algorithm would pick out the columns that have pivots because if $v_j$ does not have a pivot then $v_j = \sum_{k=1}^{j-1} c_k v_k$.

This relation is equivalent to knowing that $\begin{bmatrix} v_1 & \cdots & v_{j-1} & v_j \end{bmatrix}$ is a solution to the system $\begin{bmatrix} c_1 & \cdots & c_{j-1} \end{bmatrix}$.

As this is unchanged by row operations the two algorithms pick the same subset of $\{v_1, \ldots, v_n\}$.  

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