Last Time:
1. Given any $A \in \mathbb{M}_{m,n}(\mathbb{F})$ there is $B \in \mathbb{M}_{m,n}(\mathbb{F})$ in RREF
   & elementary matrices $E_1, ..., E_k \in \mathbb{M}_{m,n}(\mathbb{F})$ s.t. $A = E_k \cdots E_1 B$
2. A invertible $\iff$ its RREF = I
   In that case the RREF of $[A | \text{Im}]$ is $[\text{Im} | A^{-1}]$

This time: The LU decomposition
Here $L$ stands for lower triangular & $U$ for upper triangular
A matrix $A \in \mathbb{M}_{m,n}(\mathbb{F})$ is lower triangular if $a_{ij} = 0$ whenever $i > j$
A matrix $A \in \mathbb{M}_{m,n}(\mathbb{F})$ is upper triangular if $a_{ij} = 0$ whenever $i < j$

eg. $P_{c,i,j}$ is lower triangular if $i > j$ & upper triangular if $i < j$

eg. $P_{5,3,1}$ in $\mathbb{M}_3(\mathbb{R})$ is

Given a matrix $A \in \mathbb{M}_{m,n}(\mathbb{F})$
Suppose we can use row operation $R_1$ (add a multiple $c$ of row $j$ to row $i$)
repeatedly, but only adding a row to lower rows
& ends up with an upper triangular matrix $U$

That is, in terms of elementary matrices, $P_{c_1,i,j_1} \cdots P_{c_k,i,j_k} A = U$ if $i_k > j_k$

Thus, solving for $A$, we find

$$A = (P_{c_k,i_k,j_k} \cdots P_{c_1,i_1,j_1})^{-1} U = P_{c_1,i_1,j_1}^{-1} \cdots P_{c_k,i_k,j_k}^{-1} U$$

$$= P_{-c_1,i_1,j_1} \cdots P_{-c_k,i_k,j_k} U$$

$$= L$$
Algorithm to find the LU decomposition of A:
1) Try to reduce A to an upper triangular matrix using only row operations R1 & only ever adding a multiple of row j to row i if i ≠ j.
   If this succeeds, call the result U.
   If it fails, A does not have an LU decomposition.
2) If the previous step succeeded, define L to be the lower triangular matrix with 1's on the diagonal & -c in the (i,j) entry if during the elimination process c times row j was added to row i.

\[ A = LU \]

\[ \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ -9 & -1 & 14 & -5 & -1 \\ 15 & 9 & -23 & 12 & 2 \\ 3 & 9 & -9 & 7 & 11 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ 0 & 2 & -1 & 1 & -4 \\ 0 & 4 & 2 & 2 & +2 \\ 0 & 8 & -4 & 5 & +2 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -5 & 2 & 0 \\ 0 & 2 & -1 & 1 & -4 \\ 0 & 0 & 4 & 0 & +4 \\ 0 & 0 & 0 & 1 & +5 \end{bmatrix} \]

\[ U = \begin{bmatrix} -4/3, 2 & -3 & 1 & -5/3, 1 & 1/3, -5, 2, 0 \end{bmatrix} \]

\[ A = P_{3,2,1} P_{5,3,1} P_{4,4,1} P_{4,4,1} P_{4,4,1} P_{3,3,2} P_{3,3,2} P_{4,4,1} U \]

\[ U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \]

Given LU decomposition, we solve \( Ax = b \) \( \Rightarrow \) \( LUx = b \)
by setting \( y = Ux \) so that \( Ly = b \).

It is easy to solve \( Ly = b \) & once we know \( y \), it is easy to solve \( Ux = y \).
This is usually the preferred method for solving linear systems.
**Example:** With $A$ as above, solve $Ax = \begin{bmatrix} 4 \\ -10 \\ 20 \\ 13 \end{bmatrix}$

First we solve $Ly = b$:

- $y_1 = 4$
- $3y_1 + y_2 = -10$  $\Rightarrow$  $y_2 = 2$
- $5y_1 + 2y_2 + y_3 = 20$  $\Rightarrow$  $y_3 = -4$
- $y_1 + 4y_2 + y_4 = 13$  $\Rightarrow$  $y_4 = 1$

(Unique solution)

Next we solve $Ux = y$ (with $x_5$ as free variable):

- $3x_1 + x_2 - 5x_3 + 2x_4 = 4$  $\Rightarrow$  $x_1 = \frac{3}{5}x_2 - \frac{1}{2} + \frac{5}{2}x_5$
- $2x_2 - x_3 + x_4 - x_5 = 2$  $\Rightarrow$  $x_2 = \frac{5}{2}x_3$
- $4x_3 + 4x_5 = -4$  $\Rightarrow$  $x_3 = -1 - x_5$
- $x_4 + 5x_5 = 1$  $\Rightarrow$  $x_4 = 1 - 5x_5$

So the set of solutions is:

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{5}{6} \\ \frac{1}{2} \\ -1 \\ -5 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Sometimes a matrix does not have an LU decomposition.  
Example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$u_1l_1 = 0$  $u_2l_1 = 1$

If we allow exchange of rows, then it is always possible.

**Theorem:** For every $A \in M_{m \times n}(\mathbb{F})$, there is a matrix $P$ which is a product of elementary matrices of the form $R_{ij}$, an upper triangular matrix $U$, and a lower triangular matrix $L$, such that $PA = LU$.

This is the LU decomposition of $A$, or $LU$ by pivoting.
PF Any A can be put into upper triangular form by using row operations of type 1 & 3.

Note that $R_{i,j}$, $P_{c,k,l}$, $R_{i,j}$, if \( \{i,j\} \not\subseteq \{k,l\} \)
\[
R_{i,j} = \begin{cases} 
P_{c,k,l} & \text{if } i = k, j \neq l \\
P_{c,j,i} & \text{if } i = l, j \neq k \\
P_{c,k,l} & \text{if } i = k, j = l 
\end{cases}
\]

So it is possible to carry out all of the row exchanges first & then use the row operations of type \( R_{i,j} \).

**Range** \( T \in \mathbb{L}(V, W) \)

\[
\text{Range}(T) = T(V) = \{ T(v) \mid v \in V \} \subseteq W.
\]

Note that Range(\( T \)) is a subspace of \( W \).

1) \( \forall v \in \text{Range}(T), ~ w, w' \in \text{Range}(T) \Rightarrow w, w' \in \text{Range}(T), ~ \text{weRange}(T) = \text{cwwe}(T) \)

If \( T \in \mathbb{L}(F^n, F^m) \) is represented by \( A \in M_{m \times n}(F) \)

Then the range of \( T \) is exactly the span of the columns of \( A \).

PF

\[
A = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{pmatrix}, \quad a_i = T(e_i) \in \text{Range}(T) \text{ & since Range}(T) \text{ is a subspace}
\]

\[
\Rightarrow \langle a_1, \ldots, a_n \rangle \subseteq \text{Range}(T).
\]

On the other hand, \( w \in \text{Range}(T) \)

\[
\text{Then } w = Tv \text{ for some } v \in F^n \text{ & hence } w = \sum_{i=1}^n v; a_i \in \langle a_1, \ldots, a_n \rangle
\]

\[
\Rightarrow \text{Range}(T) = \langle a_1, \ldots, a_n \rangle \Rightarrow \text{Range}(T) = \langle a_1, \ldots, a_n \rangle
\]

The span of the columns of \( A \)

is known as the column space of \( A \) & denoted \( \text{C}(A) \).