Last Time: \( \mathcal{L}(V,W) \) is a vector space & other properties of linear transformations

This Time: Matrix multiplication

If \( S \in \mathcal{L}(F^n, F^m) \), \( T \in \mathcal{L}(F^p, F^n) \) then \( SoT \in \mathcal{L}(F^p, F^m) \)

On the other hand, every linear map between these spaces is multiplication by some matrix.

If \( A \) is the matrix associated to \( S \)

\( B \) is the matrix associated to \( T \)

then we define the matrix associated to \( SoT \)

to be the product of \( A \) with \( B \)

& we denote it \( AB \)

That is, \( AB \) is the matrix s.t. \( (AB)v = A(Bv) \) \( \forall v \in F^p \)

Note that \( A \in M_{mn}, B \in M_{np} \) & \( AB \in M_{mp} \)

There are many ways of thinking about \( AB \)

i) The \( j^{th} \) column of \( AB \) is \( (AB)e_j = A(\overline{\text{column } j \text{ of } B}) \)

So if we write \( B = \begin{pmatrix} b_1 & b_2 & \cdots & b_r \\ b_1' & b_2' & \cdots & b_r' \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{(r)} & b_2^{(r)} & \cdots & b_r^{(r)} \end{pmatrix} \)

then we have \( AB = \begin{pmatrix} A b_1 & A b_2 & \cdots & A b_r \\ A b_1' & A b_2' & \cdots & A b_r' \\ \vdots & \vdots & \ddots & \vdots \\ A b_1^{(r)} & A b_2^{(r)} & \cdots & A b_r^{(r)} \end{pmatrix} \)

In particular, every column of \( AB \) is a linear combination of the columns of \( A \).
ii) This expression gives us a formula for the entries of $AB$

$$[AB]_{ij} = \left(A(b_j)\right)_i = \sum_{k=1}^{n} a_{ik} b_{kj}$$

A row vector is a matrix of size $1 \times m$ for some $m \in \mathbb{N}$ (but like a column vector of size $n \times 1$ is a matrix of size $n \times 1$).

If $v \in M_{1,m}(F) \& w \in M_{m,1}(F)$

then $vw$ makes sense when $n=m$

& then is a matrix of size $1 \times 1$, which is just a #

$$vw = [v_1 \cdots v_m][w_1 \cdots w_n] = \sum_{k=1}^{n} v_k w_k$$

Comparing we see that $[AB]_{ij} = \text{matrix product of}$

the $i^{th}$ row of $A$ \& $j^{th}$ column of $B$.

Thus if $A = \begin{bmatrix} -a_1 & - & - \\ - & \ddots & - \\ - & - & -a_m \end{bmatrix}$, $B = \begin{bmatrix} b_1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & b_p \end{bmatrix}$ then $AB = \begin{bmatrix} a_1 b_1 \cdots a_1 b_p \\ \vdots & \ddots & \vdots \\ a_m b_1 \cdots a_m b_p \end{bmatrix}$

iii) Finally, this can be interpreted as $AB = \begin{bmatrix} -a_1 b \cdots -a_1 b \\ \vdots & \ddots & \vdots \\ -a_m b \cdots -a_m b \end{bmatrix}$

That is, the $j^{th}$ row of $AB$

is the product of the $j^{th}$ row of $A$ \& $B$. 
Example: \[ A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix} \]

\[ AB = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -2 & 12 \end{bmatrix} \] note that \( BA \) is not defined

Example: \[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \]

If \( A \in M_{m \times n}(\mathbb{F}) \), \( A = [a_{ij}]_{1 \leq i \leq m}^{1 \leq j \leq n} \)

the transpose of \( A \), \( A^T \in M_{n \times m}(\mathbb{F}) \),

has entries \( [A^T]_{ij} = a_{ji} \)

So \( A = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{bmatrix} \) then \( A^T = \begin{bmatrix} 1 & \cdots & 1 \\ a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \)

Note: \((A^T)^T = A \) & \((AB)^T = B^T A^T \)

If \( \mathbb{F} = \mathbb{C} \), \( B^* \) is the matrix obtained from \( B \)

by transposing & then taking complex conjugates

\[ B^* = \text{conjugate transpose of } B \]
Def \( A \in M_n(\mathbb{F}) \). We say that \( B \in M_n(\mathbb{F}) \) is an inverse of \( A \) if \( AB = BA = I_n \).

If \( A \) has an inverse, we say that \( A \) is invertible.

If \( B \) is an inverse of \( A \), then it is unique since multiplying by \( B \) yields \( x = Bb \).

If a matrix \( A \in M_n(\mathbb{F}) \) has an inverse, then it is unique.

Indeed, if \( B, C \) are both inverses of \( A \), then
\[ B = B(AC) = (BA)C = C. \]

If \( A, B \in M_n(\mathbb{F}) \) are both invertible, then \( (AB)^{-1} = B^{-1}A^{-1} \).

Also, note that \( (A^{-1})^{-1} = A \).

Example: Suppose \( a, b, c, d \in \mathbb{F} \) and \( ad \neq bc \),
then \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is invertible and \( \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \) is its inverse.

Example: \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is not invertible (over any field).

Indeed, \( A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \), so if \( A \) were invertible, then \( A^2 \) would be too.