Last time: Gaussian elimination: using row operations we can put any matrix into Reduced Row Echelon Form.

This time: A geometric interpretation of a linear system.

An $n$-dimensional vector over $\mathbb{R}$ is a column \( \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) with \( \forall i \in \mathbb{R} \).

The set of $n$-dimensional vectors over $\mathbb{R}$ is denoted $\mathbb{R}^n$.

We have two algebraic operations on $\mathbb{R}^n$:

1) Scalar multiplication:

   if $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then $cv \in \mathbb{R}^n$ is the vector
   \[ cv = c \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}. \]

2) Addition:

   if $v, w \in \mathbb{R}^n$, then $v + w \in \mathbb{R}^n$ is the vector
   \[ v + w = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}. \]

A linear combination of $v_1, \ldots, v_k \in \mathbb{R}^n$ is a vector of the form \( \sum_{i=1}^{k} c_i v_i = c_1 v_1 + \cdots + c_k v_k \) where $c_i \in \mathbb{R}$.

The span of $v_1, \ldots, v_k$ is the set $\langle v_1, \ldots, v_k \rangle$ of all linear combinations of $v_1, \ldots, v_k$: $\langle v_1, \ldots, v_k \rangle = \left\{ \sum_{i=1}^{k} c_i v_i : c_i \in \mathbb{R} \right\}$. 
E.g., Is \( \mathbf{w} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix} \) a linear combination of \( \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \)?

\( \iff \) Can we find \( c_1, c_2, c_3 \in \mathbb{R} \) s.t. \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{w} \)?

\( \iff \) Can we solve \( 3c_1 + 2c_2 + c_3 = 39 \) ? Yes, this is \( 2c_1 + 3c_2 + c_3 = 34 \) from the Chinese ex.

\( \iff \) Can we solve \( c_1 + 2c_2 + 3c_3 = 26 \) from last class

E.g., Any \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) can be written \( \mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n \)

We call \( (\mathbf{e}_1, \ldots, \mathbf{e}_n) \) the standard basis of \( \mathbb{R}^n \)

E.g., \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \)

A vector \( \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \) is in \( \text{span} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \iff w_3 = 0 \)

Indeed, \( \iff \) is clear & to see that \( \iff \) is true, note that we can write \( \mathbf{w} = w_1 \mathbf{v}_1 + (w_1 - w_2) \mathbf{v}_2 = w_1 \mathbf{v}_1 + (w_1 - w_2) \mathbf{v}_1 \)

Alternately, we can write \( \mathbf{w} = w_2 \mathbf{v}_2 + (w_1 - 2w_2) \mathbf{v}_2 = w_2 \mathbf{v}_2 + (w_1 - 2w_1) \mathbf{v}_2 \)

So \( \mathbf{w} \) is a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) in more than one way.
We have an equivalence of problems:
Solve a linear system of equations
\( \Leftrightarrow \) Write a vector as a linear combination of given vectors

Geometrically, \( v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{array}{c} x \\ y \end{array}, \quad w = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow \begin{array}{c} x \\ y \end{array} \)

The two operations above have geometric interpretations:

- \( cv \) has length \( \|cv\| = |c|\|v\| \)
- A point is in the same direction as \( v \) if \( c > 0 \)
- opposite \( c < 0 \)
- \( v + w \) is the vector obtained by moving one of the vectors from the origin to the tip of the other

The span of a non-zero vector \( v \), \( \langle v \rangle = \{cv : c \in \mathbb{R}\} \), is the line through the origin in the direction of \( v \)

If \( w \) is a fixed vector then \( \{w + cv : c \in \mathbb{R}\} = w + \langle v \rangle \) is a line parallel to \( \langle v \rangle \) passing through the end of \( w \)
E.g., last time we saw that \( x + \frac{1}{2} y = 10 \) has solutions \( x = 10 + \frac{1}{2} w \) and \( y = 10 - \frac{3}{2} w \).

As vectors, the solutions are:

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix} = \begin{bmatrix}
  10 + \frac{1}{2} w \\
  10 - \frac{3}{2} w \\
  0 \\
  w
\end{bmatrix} + w \begin{bmatrix}
  \frac{1}{2} \\
  -\frac{3}{2} \\
  0 \\
  1
\end{bmatrix}
\]

Geometrically, this is a line in \( \mathbb{R}^4 \):

It is the line through \( \begin{bmatrix}
  10 \\
  10 \\
  0 \\
  0
\end{bmatrix} \) in the direction \( \begin{bmatrix}
  \frac{1}{2} \\
  -\frac{3}{2} \\
  0 \\
  1
\end{bmatrix} \).

E.g., the system \( x_1 + 2x_2 + 5x_4 = -3 \) has solutions:

\[
\begin{align*}
-x_1 - 2x_2 + x_3 - 6x_4 + x_5 &= 2 \\
-2x_1 + 4x_2 - 10x_4 + x_5 &= 8
\end{align*}
\]

\[
x_1 = 3 - 2x_2 - 5x_4 \\
x_3 = 1 + x_4 \\
x_5 = -2
\]

\( x_2, x_4 \in \mathbb{R} \)

This is a translate of the plane \( \langle \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}, \begin{bmatrix}
  -5 \\
  0 \\
  1
\end{bmatrix} \rangle \) in \( \mathbb{R}^3 \).

We can visualize this in \( \mathbb{R}^3 \):

If \( v_1, v_2, w \in \mathbb{R}^3 \) are non-zero vectors not all on the same plane, then...