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Let $(M, h_M)$ be a compact Kähler manifold of dimension $n$. Let $(E, h_E)$ be a holomorphic Hermitian vector bundle on $M$. Let $\Box_{n,q} = (\bar{\partial} + \bar{\partial}^*)^2$ be the Laplacian acting on $A_M^{n,q}(E)$. Let $\zeta_{n,q}(s)$ be the zeta function of $\Box_{n,q}$:

$$
\zeta_{n,q}(s) := \sum_{\lambda \in \sigma(\Box_{n,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \Box_{n,q}),
$$

where $E(\lambda, \Box_{n,q})$ is the eigenspace of $\Box_{n,q}$ with eigenvalue $\lambda$. Let $\omega_M = \det T^* M$ be the canonical bundle of $M$ and write $\omega_M(E) := \omega_M \otimes E$. The holomorphic analytic torsion of $(M, \omega_M(E))$ with respect to the metrics $h_M, h_E$ is the real number

$$
\tau(M, \omega_M(E)) := \exp[-\sum_{q \geq 0} (-1)^q q \zeta'_{n,q}(0)].
$$
Let $X$ be a connected Kähler manifold of dimension $n + 1$.
Let $S \subset \mathbb{C}$ be the unit disc.
Let $\pi: X \to S$ be a surjective holomorphic map with connected fibers.
Let $\Sigma_\pi \subset X$ be the critical locus of $\pi$ and assume $\pi(\Sigma_\pi) = \{0\}$.

$x_0 := \pi^{-1}(0)$ is the only possible singular fiber;
$x_s := \pi^{-1}(s)$ is a compact Kähler $n$-fold for $s \neq 0$.

Let $h_X$ be a Kähler metric on $X$.
Let $(\xi, h_\xi)$ be a holomorphic Hermitian vector bundle on $X$. We set

$h_{X_s} := h_X|_{X_s} (\xi_s, h_{\xi_s}) := (\xi, h_\xi)|_{X_s}$

**Problem**

Determine the behavior of the analytic torsion $\tau(X_s, \omega_{X_s}(\xi_s))$ as $s \to 0$. 
Known results

In the following cases, the behavior of $\tau(X_s, \omega_{X_s}(\xi_s))$ as $s \to 0$ has been known.

- $\pi : X \to S$ is a family of compact hyperbolic Riemann surfaces degenerating to a nodal Riemann surface and $\xi = \mathcal{O}_X$ (Wolpert).
- $\pi : X \to S$ is a family of compact Riemann surfaces degenerating to a nodal Riemann surface, whose fibers are equipped with the metric induced from a metric on $\omega_{X/S}$ or from $h_X$ (Bismut-Bost).
- $X_0 = Y_1 \cup Y_2$ and $Y_1$, $Y_2$ are smooth divisors of $X$ intersecting transversally (Bismut).
- $X_0$ has only isolated rational singularities (Y.).
Goal of talk

When \((\xi, h_{\xi})\) is Nakano semi-positive and \(X\) is an open subset of a projective manifold of the same dimension to which \(\xi\) extends, we prove:

- As \(s \to 0\), \(\log \tau(X_s, \omega_{X_s}(\xi_s))\) has an asymptotic expansion

\[
\log \tau(X_s, \omega_{X_s}(\xi_s)) = \alpha \log |s|^2 + \beta \log(-\log |s|^2) + \gamma + O\left(\frac{1}{\log |s|}\right).
\]

- An explicit value of the logarithmic divergence \(\alpha\) in terms of certain topological data (Gauss map) and algebro-geometric data (semistable reduction) and the integrality \(\beta \in \mathbb{Z}\).

Strategy

Analytic torsion is the ratio of Quillen and L2 metrics on the determinant of cohomology \(\det H(X_s, \omega_{X_s}(\xi_s)) = \bigotimes_{q \geq 0} (\det H^q(X_s, \omega_{X_s}(\xi_s)))^{-1})^q\)

\[
\| \cdot \|_Q^2 = \tau(X_s, \omega_{X_s}(\xi_s)) \times \| \cdot \|_{L^2}^2.
\]

We determine the behavior of \(\| \cdot \|_Q\) and \(\| \cdot \|_{L^2}\) as \(s \to 0\).
Singularity of Quillen metrics (Strategy)

- Let $\sigma$ be a frame of $\det R_{\pi^*}\omega_{X/S}(\xi) := \bigotimes_{q \geq 0} (\det R^q_{\pi^*}\omega_{X/S}(\xi))^{(-1)^q}$ as a holomorphic line bundle on $S$.
- We apply the Bismut-Lebeau embedding formula for Quillen metrics to the family of embeddings $X_s \hookrightarrow X$ ($s \neq 0$).
- As a result, we get an expression of $\log \|\sigma\|_Q^2(s)$ as a function on $S$

$$
\log \|\sigma\|_Q^2(s) \equiv - \int_{X_s} \tilde{Td}(TX_s, TX|_{X_s}, N_{X_s/X}; h_X|_{X_s}, h_X, h_{N_{X_s/X}}) \text{ch}(\xi, h_{\xi})
$$

mod $C^0(S)$, where $h_{N_{X_s/X}}$ is a Hermitian metric with $h_{N_{X_s/X}}^{-1}(d\pi, d\pi) = 1$.
- The integrand can be regarded as the pull-back of a certain universal Bott-Chern form on the projective bundle $\mathbb{P}(TX)^\vee$ via the Gauss map.
- The singularity of $\log \|\sigma\|_Q^2(s)$ can be understood through the resolution of the indeterminacy of the Gauss map.
The Gauss map

Let \( \mathbb{P}(TX)^\vee \) be the projective bundle such that \( \mathbb{P}(T_xX)^\vee \) is the set of hyperplanes of \( T_xX \). The Gauss map \( \gamma : X \setminus \Sigma_\pi \rightarrow \mathbb{P}(TX)^\vee \) is the section

\[
\gamma(x) := [T_xX_{\pi(x)}] \in \mathbb{P}(T_xX)^\vee, \quad x \in X \setminus \Sigma_\pi.
\]

Fact (Hironaka: resolution of indeterminacy)

There is a resolution \( q : (\tilde{X}, E) \rightarrow (X, \Sigma_\pi) \) of the indeterminacy of \( \gamma \) with:

- \( q|_{\tilde{X}\setminus E} : \tilde{X} \setminus E \cong X \setminus \Sigma_\pi \) is an isomorphism.
- \( \tilde{\gamma} := \gamma \circ q \) extends to a holomorphic map from \( \tilde{X} \) to \( \mathbb{P}(TX)^\vee \).

Let \( \mathcal{H} = \mathcal{O}_{\mathbb{P}(TX)^\vee}(1) \) be the tautological quotient bundle on \( \mathbb{P}(TX)^\vee \), i.e.,

\[
0 \rightarrow \mathcal{U} \rightarrow \Pi^*TX \rightarrow \mathcal{H} \rightarrow 0 \quad \text{(exact),}
\]

where \( \Pi : \mathbb{P}(TX)^\vee \rightarrow X \) is the projection and \( \mathcal{U} \rightarrow \mathbb{P}(TX)^\vee \) is the universal bundle parametrizing the hyperplanes.
Theorem (Y.)

Recall \( \det R_{\pi_*}\omega \vert_{X/S}(\xi) = O_S\sigma \). Then, as functions on \( S \), one has

\[
\log \tau(X_s, \omega_{X_s}(\xi)) + \log \|\sigma(s)\|_{L^2}^2 \equiv \alpha(\gamma) \log |s|^2 \pmod{C^0(S)},
\]

where \( \alpha(\gamma) \) is the topological constant given by the following formula:

\[
\alpha(\gamma) := \int_{(\pi \circ q)^{-1}(0) \cap E} \tilde{\gamma}^* \left\{ \frac{Td(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)} \right\} \ q^* \{ \text{Td} (TX) \text{ch}(\xi) \} \in \mathbb{Q}
\]

To understand the behavior of \( \tau(X_s, \omega_{X_s}(\xi_s)) \) as \( s \to 0 \), we study the behavior of the \( L^2 \)-metric \( \|\sigma(s)\|_{L^2}^2 \) as \( s \to 0 \). For this, we must assume certain positivity of \( (\xi, h_{\xi}) \), the Nakano semi-positivity.
Definition

Let $R^\xi = (\nabla^\xi)^2$ be the curvature of $(\xi, h_\xi)$, where $\nabla^\xi$ is the Chern connection, i.e., holomorphic Hermitian connection. Write

$$h_\xi(\sqrt{-1} R^\xi(\cdot), \cdot) = \sum_{i,j,\alpha,\beta} R_{\alpha\bar{\beta} ij} (e^\vee_\alpha \otimes \bar{e}^\vee_\beta) \otimes (\theta_i \wedge \bar{\theta}_j),$$

where $\{e^\vee_\alpha\}$ (resp. $\{\theta_i\}$) is a local unitary frame of $\xi^\vee$ (resp. $\Omega^1_X$).

$(\xi, h_\xi)$ is said to be Nakano semi-positive $\iff$

$$\sum_{i,j,\alpha,\beta} R_{\alpha\bar{\beta} ij} \zeta_i^\alpha \bar{\zeta}_j^\beta \geq 0 \quad (\forall (\zeta_i^\alpha) \in \mathbb{C}^r(n+1)).$$
Fact (Takegoshi)

If \((\xi, h_\xi)\) is Nakano semi-positive on \(X\), then \(\dim H^q(X_s, \omega_{X_s}(\xi_s))\) is constant on \(S\). As a result, the \(q\)-th direct image

\[
R^q \pi_* \omega_{X/S}(\xi) := \bigcup_{s \in S} H^q(X_s, \omega_{X_s}(\xi_s))
\]

has the structure of a holomorphic vector bundle over \(S\) for all \(q \geq 0\). Set

\[
\ell_q := \text{rk } R^q \pi_* \omega_{X/S}(\xi) = \dim H^q(X_s, \omega_{X_s}(\xi_s)).
\]

Definition (L2 metric)

By Hodge theory, each fiber \(H^q(X_s, \omega_{X_s}(\xi_s)), s \in S \setminus \{0\}\), is endowed with a Hermitian structure via the identification with the harmonic forms. This Hermitian metric on \(R^q \pi_* \omega_{X/S}(\xi)\) is called the \(L^2\)-metric and is denoted by \(h_{L^2}\) or \(h_{R^q \pi_* \omega_{X/S}(\xi)}\).
Nakano semi-positivity of direct images

Fact (Berndtsson, Berndtsson-Paun, Mourougane-Takayama)

Assume that \((\xi, h_\xi)\) is Nakano semi-positive and let \(R(s) \, ds \wedge d\bar{s}\) be the curvature form of \((R^q\pi_*\omega_{X/S}(\xi), h_{L^2})\) w.r.t. the Chern connection. Then the direct image \((R^q\pi_*\omega_{X/S}(\xi), h_{L^2})\) is again Nakano semi-positive

\[
\sqrt{-1}R(s) \, ds \wedge d\bar{s} \geq 0,
\]

i.e., \(R(s)\) is a semi-positive Hermitian endomorphism on \(S \setminus \{0\}\).

After this, it is natural to ask:

Problem (Behavior of L2 metric)

Determine the behaviors of \(h_{L^2}\) and the curvature of \((R^q\pi_*\omega_{X/S}(\xi), h_{L^2})\) as \(s \to 0\). When \(\xi \cong \mathcal{O}_X\), the Nilpotent orbit theorem of Schmid in VHS gives an answer. By using semi-stable reduction, we get a similar result.
To understand the singularity of $L^2$ metrics, we recall the semi-stable reduction theorem, which gives a canonical model of degenerations.

**Theorem (Mumford et al: Semistable reduction theorem)**

Let $(T,0)$ be another pointed unit disc of $\mathbb{C}$. Then there exist $\nu \in \mathbb{Z}$ and a family $f : (Y, Y_0) \to (T,0)$ with the following properties.

- $f : Y \setminus Y_0 \to T \setminus \{0\}$ is the pull-back of $\pi : X \setminus X_0 \to S \setminus \{0\}$ by the map $\mu : T \to S$, $\mu(t) = t^\nu$.
- $Y$ is smooth and $Y_0$ is a reduced, normal crossing divisor, i.e., locally,

$$Y_t = \{z_0 \cdots z_k = t\}, \quad 0 \leq \exists k \leq n.$$  

- There is a map $F : Y \to X$ sending the fibers of $f$ to fibers of $\pi$ such that

$$\pi \circ F = \mu \circ f.$$
This situation can be summarized as the commutative diagram:

\[
F : (Y, Y_0) \to (X, X_0)
\]

\[
f \downarrow \quad \downarrow \pi
\]

\[
\mu : (T, 0) \to (S, 0)
\]

with \( s = t^\nu \), and \( Y_0 \) reduced and normal crossing divisor, and \( Y \) smooth.

**Remark**

Since \((F^*\xi, h_{F^*\xi} := F^*h_\xi)\) is Nakano semi-positive, \( R^q f_* \omega_{Y/T}(F^*\xi) \) is again a holomorphic vector bundle over \( T \).

**Definition (degenerate Kähler metric on \( Y \))**

Define the **degenerate** Kähler form \( h_Y \) on \( Y \) by

\[
h_Y := F^*h_X + f^*(dt \otimes d\bar{t}).
\]

In what follows, \( Y \) is equipped with the degenerate metric \( h_Y \).
Comparison of direct images between $Y/T$ and $X/S$

Fact (Mourougane-Takayama)

There is a natural injective homomorphism of holomorphic vector bundles

$$\varphi_q : R^q f_* \omega_{Y/T}(F^* \xi) \hookrightarrow \mu^* R^q \pi_* \omega_{X/S}(\xi),$$

which is an isomorphism on $T \setminus \{0\}$ and preserves the $L^2$-metrics, i.e.,

$$\varphi^*_q \mu^* h_{R^q \pi_* \omega_{X/S}(\xi)} = h_{R^q f_* \omega_{Y/T}(F^* \xi)}.$$

Notation

Regard $R^q f_* \omega_{Y/T}(F^* \xi) \subset \mu^* R^q \pi_* \omega_{X/S}(\xi)$ via $\varphi_q \in M(\ell_q, \mathbb{C}\{t\})$. Write

$$\left( \frac{\mu^* R^q \pi_* \omega_{X/S}(\xi)}{R^q f_* \omega_{Y/T}(F^* \xi)} \right)_0 \cong \bigoplus_{1 \leq \alpha \leq \ell_q} \mathbb{C}\{t\}/(t^{e^{(q)}_\alpha}), \quad \exists e^{(q)}_\alpha \in \mathbb{Z}_{\geq 0}.$$

$\{e^{(q)}_\alpha\}$ measures the difference of direct images between $X/S$ and $Y/T$. 

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Singularities and Analytic Torsion  
July 30, 2012
Theorem (A structure theorem for the singularity of L2 metric)

- By a suitable choice of a basis $\{\psi_{\alpha}^{(q)}\}$ of $R^q \pi_* \omega X / S(\xi)$, the Hermitian matrix $G(s) := \left(h_{L^2}(\psi_{\alpha}^{(q)}(s), \psi_{\beta}^{(q)}(s))\right)$ admits the expression

$$G(t^\nu) = D(t) \cdot H(t) \cdot \overline{D(t)}, \quad D(t) = \text{diag}(t^{-e_1^{(q)}}, \ldots, t^{-e_{\ell q}^{(q)}}).$$

- One has the expression on $T \setminus \{0\}$

$$H(t) = \sum_{m=0}^{n} A_m(t) (\log |t|^2)^m, \quad A_m(t) \in C^\infty(S, \text{Herm}(\ell_q)).$$

- Defining the real-valued functions $a_m(t) \in C^\infty(S), 0 \leq m \leq n\ell_q$, by

$$\det H(t) = \sum_{m=0}^{n\ell_q} a_m(t) (\log |t|^2)^m,$$

one has $a_m(0) \neq 0$ for $0 \leq \exists m \leq n\ell_q$. Set $\rho_q := \max\{m; a_m(0) \neq 0\} \in \mathbb{Z}$. 
Singularity of the curvature of the $L^2$-metric

As an immediate application of the previous structure theorem for the singularity of $h_{L^2}$, we get:

**Theorem (Y.)**

- The curvature form $\mathcal{R}(s)\, ds \wedge d\bar{s}$ of $(R^q \pi_* \omega_{X/S}(\xi), h_{L^2})$ has Poincaré growth near $0 \in S$. Namely, there is a constant $C > 0$ such that

$$0 \leq \mathcal{R}(s) \leq \frac{C}{|s|^2(\log |s|)^2} \operatorname{Id}_{R^q \pi_* \omega_{X/S}(\xi)}$$
on S \setminus \{0\}.

- The Chern form $c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2})$ has the asymptotic as $s \to 0$:

$$c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = \left\{ \frac{\rho q}{|s|^2(\log |s|)^2} + O \left( \frac{1}{|s|^2(\log |s|)^3} \right) \right\} \sqrt{-1} \, ds \wedge d\bar{s}$$
Main Theorem — Singularity of analytic torsion

Theorem (Y.)

If \((\xi, h_\xi)\) is Nakano semi-positive on \(X\) and if there is a projective manifold containing \(X\) as an open subset to which \(\xi\) extends, then

\[
\log \tau(X_s, \omega_{X_s}(\xi_s)) = \left\{ \alpha(\gamma) + \frac{1}{\deg \mu} \right\} \chi \left( \frac{\mu^* R\pi_* \omega_X/S(\xi)}{Rf_* \omega_Y/T(F^* \xi)} \right) \log |s|^2 \\
+ \rho \log(-\log |s|^2) + c + O(1/\log |s|),
\]

where \(\rho := \sum_{q \geq 0} (-1)^q \rho_q \in \mathbb{Z}\), \(c \in \mathbb{R}\) and

\[
\alpha(\gamma) = \int_{(\pi \circ q)^{-1}(0) \cap E} \tilde{\gamma}^* \left\{ \frac{Td(H^\vee)^{-1} - 1}{c_1(H^\vee)} \right\} q^* \{ Td(TX)ch(\xi) \} \in \mathbb{Q},
\]

\[
\chi \left( \frac{\mu^* R\pi_* \omega_X/S(\xi)}{Rf_* \omega_Y/T(F^* \xi)} \right) := \sum_{q \geq 0} (-1)^q \dim \mathcal{O}_T/m_0 \left( \frac{\mu^* R^q \pi_* \omega_X/S(\xi)}{R^q f_* \omega_Y/T(F^* \xi)} \right)_0 \in \mathbb{Z}.
\]
A topological interpretation of the logarithmic divergence

The coefficient of the logarithmic divergence in the asymptotic of 
\[ \log \tau(X_s, \omega_X(\xi_s)) \]

\[ C_0 := \alpha(\gamma) + \frac{1}{\deg \mu} \chi \left( \frac{\mu^* R\pi_* \omega_X S(\xi)}{Rf_* \omega_Y / T(F^* \xi)} \right) \]

admits another topological expression.

Define a coherent sheaf \( Q \) on \( Y \) supported on \( Y_0 \) by the exact sequence

\[ 0 \longrightarrow \mathcal{O}_Y(TY) \longrightarrow \mathcal{O}_Y(F^* TX) \longrightarrow Q \longrightarrow 0. \]

Then we define

\[ \text{Td}(Q) = \frac{\text{Td}(F^* TX)}{\text{Td}(TY)} \]

\[ \det Q = \det(F^* TX) \otimes (\det TY)^\vee = \omega_Y \otimes (F^* \omega_X)^\vee \]
Define

\[ H_{Y_0}(Y, \mathbb{R}) := \ker \{ H_{cpt}(Y, \mathbb{R}) \to H(Y \setminus Y_0, \mathbb{R}) \}. \]

We have

\[ \frac{Td(Q)}{\text{ch}(\det Q)} \text{ch}(\omega_{T/S}) - 1 \in H_{Y_0}(Y, \mathbb{R}). \]

A topological formula for the coefficient of the logarithmic divergence

Let \( F \) be a general fiber of \( f : Y \to T \). Then

\[
\deg \mu \cdot C_0 = \alpha(Y_0, \omega_{Y/T}(F^*\xi)) - \frac{\deg \mu - 1}{2} \chi(F, \omega_F(\xi)) \\
- \int_Y \left\{ \frac{Td(Q)}{\text{ch}(\det Q)} \text{ch}(\omega_{T/S}) - 1 \right\} Td(TY)\text{ch}(\omega_{Y/T}(F^*\xi))
\]

It is very likely that the right hand side is expressed in terms of the characteristic classes of \( Y_{i_1} \cap \cdots \cap Y_{i_p} \), its normal bundles and \( F^*\xi \).
Some implications of the Main Theorem

**Theorem (Y.)**

Assume that \((\xi, h_\xi)\) is Nakano semi-positive on \(X\).

- If \(X_0\) is reduced, normal and has only canonical (equivalently rational) singularities, then there exist \(c \in \mathbb{R}\), \(r \in \mathbb{Q}_{>0}\), \(\nu \in \mathbb{Z}_{\geq 0}\) such that

\[
\log \tau(X_s, \omega_{X_s}(\xi_s)) = \alpha(\gamma) \log |s|^2 + c + O(|s|^r(\log |s|)\nu).
\]

- The complex Hessian of \(\log \tau(X_s, \omega_{X_s}(\xi_s))\) has Poincaré growth at \(s = 0\)

\[
\partial_{s\bar{s}} \log \tau(X_s, \omega_{X_s}(\xi_s)) = \frac{\rho}{|s|^2(\log |s|)^2} + O\left(\frac{1}{|s|^2(\log |s|)^3}\right),
\]

where \(\rho \in \mathbb{Z}\) is the same integer as in the Main Theorem.
Singularity of $L^2$-metrics

Let $\{\Theta_1(q), \ldots, \Theta_{\ell_q}(q)\}$ be frame fields of the holomorphic vector bundle $R^q f_* \omega_Y/_{T(F^*\xi)}$ on $T$. Since

$$R^q f_* \omega_Y/_{T(F^*\xi)} \subset \mu^* R^q \pi_* \omega_X/_{S(\xi)},$$

we may assume by an appropriate choice of the frames $\{\psi_1^{(q)}, \ldots, \psi_{\ell_q}^{(q)}\}$ and $\{\Theta_1^{(q)}, \ldots, \Theta_{\ell_q}^{(q)}\}$ that

$$\mu^* (\psi_{\alpha}^{(q)}) = t^{-e_{\alpha}^{(q)}} \Theta_{\alpha}^{(q)}, \quad e_{\alpha}^{(q)} \in \mathbb{Z}_{\geq 0}.$$

This is because $\mu^* R^q \pi_* \omega_X/_{S(\xi)}$ and $R^q f_* \omega_Y/_{T(F^*\xi)}$ are free modules of the same rank $\ell_q$ over the ring of convergent power series $\mathbb{C}\{t\}$. 
Theorem (Structure theorem for the singularity of L2 metric)

- The Hermitian matrix $G(s) := \left( h_{L^2}(\Psi^{(q)}_{\alpha}(s), \Psi^{(q)}_{\beta}(s)) \right)$ has the expression

$$G(t') = D(t) \cdot H(t) \cdot \overline{D(t)}, \quad D(t) = \text{diag} \left( t^{-e_{1}^{(q)}}, \ldots, t^{-e_{\ell_{q}}^{(q)}} \right).$$

- Here

$$H(t) = \sum_{m=0}^{n} A_{m}(t) (\log |t|^{2})^{m}, \quad A_{m}(t) \in C^{\infty}(T, \text{Herm}(\ell_{q})), \quad 0 \leq m \leq n.$$

- Defining the real-valued functions $a_{m}(t) \in C^{\infty}(T), 0 \leq m \leq n\ell_{q}$ by

$$\det H(t) = \sum_{m=0}^{n\ell_{q}} a_{m}(t) (\log |t|^{2})^{m},$$

one has $a_{m}(0) \neq 0$ for some $0 \leq m \leq n\ell_{q}.$
Proof of the structure theorem of the $L^2$ metric

Let $\kappa_Y$ be the Kähler form of the degenerate metric $h_Y$.

### Representation of cohomology by harmonic forms

- **Lefschetz theorem** w.r.t. the degenerate Kähler form $\kappa_Y$ (Takegoshi, Mourougane-Takayama, Y.)

  \[ \exists \text{ relative holomorphic forms } \psi_\alpha \text{ with } \]

  \[ \Theta^{(q)}_\alpha = [\psi_\alpha \wedge \kappa^q_Y], \quad (f^* dt) \wedge \psi_\alpha \in H^0(Y, \Omega^{n+1-q}_Y(\xi)). \]

(Here the Nakano semi-positivity is used in the essential manner.)

Since $\psi_\alpha$ is holomorphic and hence $\psi_\alpha \wedge \kappa^q_Y|_{Y_t}$ is the harmonic representative of $\Theta^{(q)}_\alpha|_{Y_t}$, we get

\[ h_{L^2}(\Theta^{(q)}_\alpha(t), \Theta^{(q)}_\beta(t)) = h_{L^2}(\psi_\alpha \wedge \kappa^q_Y, \psi_\beta \wedge \kappa^q_Y) \]

\[ = \int_{Y_t} (\sqrt{-1})^{(n-q)^2} F^* h_\xi(\psi_\alpha \wedge \overline{\psi_\beta}) \wedge \kappa^{n-q}_Y. \]

We must study the behavior of this fiber integral as $t \to 0$. 
Asymptotic expansion of fiber integral — Barlet’s theorem

Since $Y_0$ is a reduced normal crossing divisor of $Y$, we get by Barlet

$$\int_{Y_t} (\sqrt{-1})^{(n-q)^2} F^* h_\xi (\psi_\alpha \wedge \overline{\psi}_\beta) \wedge \kappa_{Y}^{n-q} = \sum_{k=0}^{n} c_{\alpha\bar{\beta},k}^{(q)} (- \log |t|)^k + O(|t|^r),$$

where $c_{\alpha\bar{\beta},k}^{(q)}$ are constants and $r > 0$. (Here we use that $Y_0$ is a reduced, normal crossing divisor in the essential manner.) In particular,

$$\det \left( (\Theta^{(q)}_{\alpha}(t), \Theta^{(q)}_{\beta}(t))_{L^2} \right) = \sum_{k=0}^{n\ell_q} c_{k}^{(q)} (- \log |t|)^k + O \left( |t|^r (- \log |t|)^{n\ell_q} \right),$$

where $c_{k}^{(q)}$ are real constants.

To finish the proof, we must show $c_{k}^{(q)} \neq 0$ for some $0 \leq k \leq n\ell_q$. 
The determinant does not decrease rapidly

- **Fujita type estimate** (Mourougane-Takayama)

There exists $\epsilon > 0$ such that

$$\det \left( h_{L^2}(\Theta^{(q)}_\alpha(t), \Theta^{(q)}_\alpha(t)) \right) \geq \epsilon > 0$$

for all $t \in T \setminus \{0\}$. In particular, $c_{k}^{(q)} \neq 0$ for some $0 \leq k \leq n\ell_{q}$ in the previous asymptotic expansion

$$\det \left( (\Theta^{(q)}_\alpha(t), \Theta^{(q)}_\beta(t))_{L^2} \right) = \sum_{k=0}^{n\ell_{q}} c_{k}^{(q)} (-\log |t|)^k + O \left( |t|^r (-\log |t|)^{n\ell_{q}} \right).$$

Hence, by setting $\rho_{q} = \max\{k; c_{k}^{(q)} \neq 0\}$, one has as $t \to 0$,

$$\det \left( (\Theta^{(q)}_\alpha(t), \Theta^{(q)}_\beta(t))_{L^2} \right) = (-\log |t|)^{\rho_q} \left\{ c_{\rho}^{(q)} + O \left( \frac{1}{\log |t|} \right) \right\}.$$