Essential Spectra of complete manifolds

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Let $M$ be a compact Riemannian manifold. Then the following are true

1. The spectrum of the Laplacians on $p$-forms are eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$.

2. If $f$ is an eigenform, and if $df \neq 0$, then $df$ is an eigenform also (similar result holds for $\delta$).
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In this talk, we shall concentrate on the essential spectra.
The Weyl Criterion

Theorem (Classical Weyl’s criterion)

A point $\lambda$ belongs to $\sigma(\Delta)$ if, and only if, there exists a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \text{Dom}(\Delta)$ such that

1. $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1,$

2. $(\Delta - \lambda)\psi_n \to 0, \text{ as } n \to \infty \text{ in } \mathcal{H}.$

Moreover, $\lambda$ belongs to $\sigma_{\text{ess}}(\Delta)$ of $H$ if, and only if, in addition to the above properties

3. $\psi_n \to 0 \text{ weakly as } n \to \infty \text{ in } L^2(M).$
Example

The essential spectrum of $\mathbb{R}^n$ is $[0, \infty)$. 
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The proof:

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{n - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} \approx \frac{\partial^2}{\partial r^2}
\]

for \( r \) large. For any \( \lambda > 0 \), \( \sin \sqrt{r} \rho \) is the approximation of the eigenfunction, where \( r \) is the distance function.
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for $r$ large. For any $\lambda > 0$, $\sin \sqrt{r} \rho$ is the approximation of the eigenfunction, where $r$ is the distance function. Can we at least generalize this result to asymptotically flat manifolds?
Fact

On a complete non-compact manifold, the Laplacian of the distance function is locally $L^1$ but not $L^2$ in general.
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Example: let $M = S^1 \times (-\infty, +\infty)$. Let $\rho$ be the distance function to the point $(1, 0)$. Then the cut locus of the function is the set $I = \{(-1, t) \mid t \in \mathbb{R}\}$, and we have

$$\Delta \rho = -\frac{2\pi}{\sqrt{t^2 + \pi^2}} \delta_I + \text{a bounded function}$$
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Thus $\Delta \rho$ is locally in $L^1$ but not in $L^2$. 
Donnelly (1981), Escobar (1986), Escobar-Freire (1992), Chen-L (1992), J. Li (1994), Zhou (1994), Donnelly (1997) essentially assume that the manifold has a pole so that $\Delta \rho \in L^2_{loc}$, or use the Green’s function under very restrictive assumptions of the curvature and the volume growth.
In 1992, Sturm proved the following

**Theorem**

Let $M$ be a complete non-compact manifold whose Ricci curvature has a lower bound. If the volume of $M$ grows uniformly sub-exponentially, then the $L^p$ essential spectra are the same for all $p \in [1, \infty]$. 
In 1997, J-P. Wang proved that, if the Ricci curvature of a manifold $M$ satisfies $\text{Ric} (M) \geq -\delta / r^2$, where $r$ is the distance to a fixed point, and $\delta$ is a positive number depending only on the dimension, then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$. In particular, for a complete non-compact manifold with non-negative Ricci curvature, all $L^p$ spectra are $[0, +\infty)$. 
In 2011, L-Zhou proved the following

**Theorem**

*Let $M$ be a complete non-compact Riemannian manifold. Assume that*

$$\lim_{x \to \infty} \text{Ric}_M(x) \geq 0.$$  

*Then the $L^p$ essential spectrum of $M$ is $[0, +\infty)$ for any $p \in [1, +\infty]$.***
This is the joint work with N. Charalambous.
Theorem (Weyl’s criterion for quadratic forms)

A point \( \lambda \) belongs to \( \sigma(\Delta) \) if, and only if, there exists a sequence \( \{ \psi_n \}_{n \in \mathbb{N}} \subset \text{Dom}(\Delta^{1/2}) \) such that

1. \( \forall n \in \mathbb{N}, \quad \| \psi_n \| = 1 \),

2. \( (\Delta - \lambda)\psi_n \xrightarrow[n \to \infty]{} 0 \) weakly in \( L^2(M) \).

Moreover, \( \lambda \) belongs to \( \sigma_{\text{ess}}(\Delta) \) if, and only if, in addition to the above properties

3. \( \psi_n \xrightarrow[w]{n \to \infty} 0 \) in \( L^2(M) \).
Theorem (Charalambous-L)

Let $f$ be a bounded positive continuous function over $[0, \infty)$. A positive real number $\lambda$ belongs to the spectrum $\sigma(\Delta)$ if, and only if, there exists a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset \text{Dom}(\Delta)$ such that

1. $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1$,
2. $(f(\Delta)(\Delta - \lambda)\psi_n, (\Delta - \lambda)\psi_n) \to 0$, as $n \to \infty$ and
3. $((\Delta + 1)^{-1}\psi_n, (\Delta - \lambda)\psi_n) \to 0$, as $n \to \infty$.

Moreover, $\lambda$ belongs to the essential spectrum $\sigma_{\text{ess}}(\Delta)$ if, and only if, in addition to the above properties

4. $\psi_n \to 0$, weakly as $n \to \infty$ in $L^2(M)$. 
Since $\Delta$ is a densely defined self-adjoint operator, the spectral measure $E$ exists and we can write

$$\Delta = \int_0^\infty \lambda \, dE. \quad (1)$$

We pick $\varepsilon > 0$ such that $\lambda > \varepsilon$, and

$$E(\lambda + \varepsilon) - E(\lambda - \varepsilon) = 0.$$  

We write

$$\psi_n = \psi_n^1 + \psi_n^2,$$

where

$$\psi_n^1 = \int_{0}^{\lambda - \varepsilon} \, dE(t) \psi_n,$$

and

$$\psi_n^2 = \psi_n - \psi_n^1.$$
Then we have

\[(f(\Delta)(\Delta - \lambda)\psi_n, (\Delta - \lambda)\psi_n)\]
\[= (f(\Delta)(\Delta - \lambda)\psi_1^n, (\Delta - \lambda)\psi_1^n) + (f(\Delta)(\Delta - \lambda)\psi_2^n, (\Delta - \lambda)\psi_2^n)\]
\[\geq c_1\|\psi_1^n\|^2,\]

where the positive number \(c_1\) is the infimum of the function \(f(t)(t - \lambda)^2\) on \([0, \lambda - \varepsilon]\). Therefore

\[\|\psi_1^n\| \to 0.\]

On the other hand, we similarly get

\[((\Delta + 1)^{-1}\psi_n, (\Delta - \lambda)\psi_n) \geq c_2\|\psi_2^n\|^2 - c_3\|\psi_1^n\|^2,\]

We conclude that both \(\psi_1^n, \psi_2^n\) go to zero. This is a contradiction to \(\|\psi_n\| = 1\).
This is NOT an constructive criterion.
Theorem (Charalambous-L)

Let $M$ be a complete noncompact manifold. Assume that with respect to a fixed point

$$\liminf_{r \to \infty} \text{Ric} \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = 0.$$

Then the essential spectrum of $M$ is $[0, \infty)$. 

Applications

Theorem (Charalambous-L)

*The essential spectrum of a complete shrinking Ricci soliton is \([0, \infty)\).*
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Theorem (Charalambous-L)

The essential spectrum of the Laplacian on 1-forms of a complete Ricci nonnegative manifold is $[0, \infty)$. 
Approximation of the distance function

Proposition

Let $\delta(t)$ be a positive function such that $\delta(t) \to 0$ as $t \to \infty$. Then there exist $C^\infty$ functions $\tilde{\rho}$ and $b$ on $M$ such that

(a). $|\tilde{\rho}(x) - \rho(x)| \leq \delta(\rho(x))$, and

(b). $\|b\|_{L^1(M \setminus B(R))} \leq \delta(R - 1)$ for $R > 2$, and

(c). $\|\nabla \tilde{\rho}^2 - 1\|_{L^1(M \setminus B(R))} \leq \delta(R - 1)$ for $R > 2$, and

(d). $\Delta \tilde{\rho}(x) \leq 2\delta(\rho(x) - 1) + b(x)$ for any $x \in M$ with $\rho(x) > 2$. 
Lemma

For all $\varepsilon > 0$ there exists an $R_1 > 0$ such that for $r > R_1$ one of the following holds

(a) If $\text{vol}(M)$ is infinite

$$\int_{B_p(r) \setminus B_p(R_1)} |\Delta \tilde{\rho}| \leq 2\varepsilon V(r) + 2\text{vol}(\partial B_p(R_1))$$

(b) If $\text{vol}(M)$ is finite

$$\int_{M \setminus B_p(r)} |\Delta \tilde{\rho}| \leq 2\varepsilon \text{vol}(M) + 2\text{vol}(\partial B_p(r))$$
The trick here is that

$$|\Delta \tilde{\rho}(x)| \leq \Delta \tilde{\rho}(x) + 4\delta(\rho(x) - 1) + 2b(x).$$
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Reason: if $a \leq b$ and $b \geq 0$, then $|a| \leq 2b - a$. 

Let $x, y, R$ be large positive numbers such that $x > 2R > 2\mu + 4$ and $y > x + 2R$. We take the cut-off function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$, smooth with support on $[x/R - 1, y/R + 1]$ and such that $\psi = 1$ on $[x/R, y/R]$ and $|\psi'|, |\psi''|$ bounded. We let

$$\varphi(\tilde{r}) = \psi\left(\frac{\tilde{\rho}}{R}\right) e^{i\sqrt{\lambda}\tilde{\rho}}$$

(2)

$$|\Delta \varphi + \lambda \varphi| = |e^{i\sqrt{\lambda}\tilde{\rho}} \left[ \frac{1}{R^2} \psi'' |\nabla \tilde{\rho}|^2 
+ 2i\sqrt{\lambda} \frac{1}{R} \psi' |\nabla \tilde{\rho}|^2 
+ (\frac{1}{R} \psi' + \psi i\sqrt{\lambda}) \Delta \tilde{\rho} 
+ \lambda \varphi (1 - |\nabla \tilde{\rho}|^2) \right] 
\leq (\frac{C'}{R} + C\delta^2 (x - R)) + C |\Delta \tilde{\rho}|.$$
When the volume of $M$ is infinite, if we choose $R, x$ large enough, then

$$\int_M |\Delta \varphi + \lambda \varphi| \leq \varepsilon V(y + R) + 2C \text{vol} (\partial B_p(x - R))$$

we can prove

$$\int_M (\varphi, \Delta \varphi + \lambda \varphi) \leq 2\varepsilon V(y + R).$$
By the curvature assumption, the volume of the manifold grows subexponentially, which implies that there exists a sequence of $y_k \to \infty$ such that $V(y_k + R) \leq 2V(y_k)$. 
Lemma

The operator

\[ (-\Delta + 1)^{-N} : L^\infty \to L^\infty \]

is a bounded operator.
We have

\[ |((-\Delta + 1)^{-N} \varphi, \Delta \varphi + \lambda \varphi)| \leq ||\Delta \varphi + \lambda \varphi||_{L^1} \leq C\varepsilon V(y_k + R) \]
We have

\[ \left| \left( ( - \Delta + 1 )^{-N} \varphi, \Delta \varphi + \lambda \varphi \right) \right| \leq \| \Delta \varphi + \lambda \varphi \|_{L^1} \leq C \varepsilon V( y_k + R ) \]

Since

\[ V( y_k + R ) \leq 2 V( y_k ) \leq C \| \varphi \|_{L^2}^2 \]

The proof is complete.
Now we turn to the essential spectrum of $k$ forms.
Theorem

The essential spectrum of the Laplacian on functions is contained in that of 1-forms.
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Corollary
Assume the Ricci curvature of $M$ is nonnegative, then the essential spectrum on 1-forms is $[0, \infty)$.

This generalizes a result of Charalambous, where the existence of a pole and curvature nonnegativity is assumed.
Moreover, we have

**Theorem**

*Suppose that $\lambda$ belongs to the essential spectrum of the Laplacian on $k$-forms, $\sigma_{\text{ess}}(k, \Delta)$. Then one of the following holds:*

(a) $\lambda \in \sigma_{\text{ess}}(k - 1, \Delta)$, or  
(b) $\lambda \in \sigma_{\text{ess}}(k + 1, \Delta)$. 
Let $\lambda > 0$ and $\lambda \in \sigma_{\text{ess}}(k, \Delta)$. For any $\varepsilon > 0$, have a $k$-form $\omega_\varepsilon$ such that $\|\omega_\varepsilon\|_{L^2} = 1$,

$$\|(\Delta - \lambda)\omega_\varepsilon\|_{L^2} \leq \varepsilon\|\omega_\varepsilon\|_{L^2}$$

(3)

and $\omega_\varepsilon \to 0$ weakly as $\varepsilon \to 0$.

If we choose $\varepsilon < \frac{1}{2}\lambda$, then we obtain

$$\|\Delta\omega_\varepsilon\|_{L^2} \geq \frac{\lambda}{2}\|\omega_\varepsilon\|_{L^2}.$$  

(4)
WLOG, we assume that $\omega_\varepsilon$ are smooth with compact support. We have

$$\int_M \langle \omega, \Delta \omega \rangle \geq \frac{1}{2\lambda} \| \Delta \omega \|_{L^2}^2 + \frac{1}{2\lambda} (\lambda^2 - \varepsilon^2) \| \omega \|_{L^2}^2 \geq c \| \omega \|_{L^2}^2$$

where $c > 0$ is a constant. Integration by parts yields

$$\int_M \langle \omega, \Delta \omega \rangle = \| d\omega \|_{L^2}^2 + \| \delta \omega \|_{L^2}^2 \geq c \| \omega \|_{L^2}^2. \tag{5}$$

Since $d$ and $\delta$ commute with the Laplacian $\Delta$, then they also commute with $(\Delta + 1)^{-m}$ for any integer $m$. For $m = 1, 2$, we compute

$$((\Delta + 1)^{-m} d\omega, (\Delta - \lambda) d\omega) + ((\Delta + 1)^{-m} \delta \omega, (\Delta - \lambda) \delta \omega)$$

$$= ((\Delta + 1)^{-m} \Delta \omega, (\Delta - \lambda) \omega)$$

after integration by parts.
Furthermore,
\[
((\Delta + 1)^{-m} \Delta \omega, (\Delta - \lambda)\omega) = ((\Delta + 1)^{-m+1} \omega, (\Delta - \lambda)\omega) - (\lambda + 1)((\Delta + 1)^{-m} \omega, (\Delta - \lambda)\omega).
\]

Since \((\Delta + 1)^{-m}\) is a bounded operator for \(m = 0, 1, 2\), we obtain
\[
((\Delta+1)^{-m} d\omega, (\Delta-\lambda)d\omega) + ((\Delta+1)^{-m} \delta \omega, (\Delta-\lambda)\delta \omega) \leq C\varepsilon \|\omega\|^2_{L^2}
\]
for some constant \(C\).
Estimate (5) gives

\[(\Delta + 1)^{-m} d\omega, (\Delta - \lambda) d\omega) + ((\Delta + 1)^{-m} \delta\omega, (\Delta - \lambda) \delta\omega) \leq C \varepsilon (\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2)\]

for a possibly larger constant $C$. Thus there exists a subsequence of $\omega_{\varepsilon_n}$ with $\varepsilon_n \to 0$ (which we also denote as $\omega_\varepsilon$) such that either

\[((\Delta + 1)^{-m} d\omega_\varepsilon, (\Delta - \lambda) d\omega_\varepsilon) \leq \frac{1}{2} C \varepsilon \|d\omega_\varepsilon\|_{L^2}^2, or\]

\[((\Delta + 1)^{-m} \delta\omega_\varepsilon, (\Delta - \lambda) \delta\omega_\varepsilon) \leq \frac{1}{2} C \varepsilon \|\delta\omega_\varepsilon\|_{L^2}^2\]

is valid.
Now we consider the essential spectrum on $p$-forms.
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**Theorem**

Let $M$ be a complete noncompact manifold. Assume that the curvature of $M$ goes to zero at infinity. Then the essential spectra of $M$ on $k$-forms are $[0, \infty)$. 
Definition
A nilpotent structure on a manifold $\mathcal{M}$ is a sheaf $\mathcal{N}$ of vector fields such that for each $p$ there exists a neighborhood $U_p$ and an action of group $G_p$ on a Galois covering $\tilde{U}_p$ of $U_p$ with the following properties

1. The connected component $N_p$ of $G_p$ is nilpotent;
2. The deck transformation group $\Gamma_p$ of the covering: $\tilde{U}_p \to U_p$ is contained in $G_p$;
3. $G_p$ is generated by $N_p$ and $\Gamma_p$;
4. $[\Gamma_p : \Gamma_p \cap N_p] = [G_p : N_p]$ is finite;
5. The lift of $\mathcal{N}$ to $\tilde{U}_p$ is generated by the action of $N_p$. 
Theorem (Cheeger-Fukaya-Gromov)

There exists \( \varepsilon_n > 0 \) and \( k_n \) with the property that, for each \( M \in \mathcal{M}_n \) there exists a nilpotent structure \( N \) such that the following holds in addition

1. \( U_p \) contains \( B_{\varepsilon_n}(p, M) \);
2. The injectivity radius of \( \tilde{U}_p \) is larger than \( \varepsilon_n \);
3. \( [G_p : N_p] < k_n \).
Definition
A Riemannian metric $g$ on $M$ is said to be $(\varepsilon, k)$-round, if there exists a nilpotent structure $\mathcal{N}$ on $M$ such that it satisfies (1)-(3) in addition and that the section of $\mathcal{N}$ is a killing vector field of the metric $g$.

Theorem (Cheeger-Fukaya-Gromov)
For each $\delta$ there exist $\varepsilon = \varepsilon(\delta, n)$ and $k = k(\delta, n)$ such that for each Riemannian manifold $(M, g)$ in the class $\mathcal{M}_n$ we can find a metric $g_\delta$ on $M$ with the following properties:

1. $(M, g_\delta)$ is $(\varepsilon, k)$-round;
2. $|g - g_\delta| < \delta$. 

Let $M$ be a complete noncompact manifold whose curvature goes to zero at infinity. for any $\sigma > 0$ small, we can rescale the metric of $M$ so that at infinity, the curvature still tends to zero. By the above theorems, there exists a neighbor $U_p$ such that

1. The injectivity radius of $\tilde{U}_p$ is greater than $\varepsilon_n$;
2. $U_p = A \times N$, where $A$ is a Euclidean ball and $N$ is an infra-nilpotent manifold.
3. The metric on $U_p$ can be approximated by $(\varepsilon, k)$-round metric.
Rescaling back, we obtain the following result

**Theorem**

There exists a nonnegative integer $0 \leq k \leq n - 1$ such that for any $R > 0$, there is any embedding

$$B(R) \times N \rightarrow M$$

where $B(R)$ is the ball of radius $R$ in the Euclidean space and $N$ is an infranilmanifold. That is, $N$ is the quotient of a Nilpotent Lie group by a discrete group. Moreover, the embedding is almost isometric in the sense that

$$\frac{1}{2} g_0 \leq f^*(g) \leq 2 g_0,$$

where $g_0, g$ are the Riemannian metrics of $B(R) \times N$ and $M$, respectively.
A Proposition essentially due to Lott.

**Proposition**

Let $N$ be a compact nilpotent manifold and let $B(\sigma)$ be the ball of radius $\delta$. Then there exists a content depending only on $\sigma$ such that the first eigenvalue with respect to the Dirichlet Laplacian is bounded.
Essential spectrum of the Laplacian on $p$ forms on hyperbolic manifolds.
Using Davies-Simons, Melrose-Taylor, we obtain

**Theorem**

The $L^p$ essential spectrum of the Laplacian $\Delta$ on $k$-forms of hyperbolic space, is exactly the set of points to the right of the parabola

$$Q_p = \left\{ -\left( \frac{N}{p} + is - k \right)\left( \frac{N}{p} + is + k - N \right) \mid s \in \mathbb{R} \right\}.$$
Question: let $k(t, x, y)$ be the heat kernel on $k$ forms, estimate $|k(t, x, y)|$ from below.
Thank you!