A VERY SMOOTH DIRAC OPERATOR ON LOOP SPACE

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Joint work in progress with Richard Melrose

In [Wit88] Witten wrote down (strictly formally) a formula for what should be the S^1 equivariant index of a Dirac operator $D_{\mathcal{L}M}$ on the loopspace $\mathcal{L}M$ of a compact spin manifold M, using a formal analog of the Atiyah-Segal formula to localize to the fixed points of S^1 (acting by loop rotation) — which are the constant loops giving a copy of $M \subset \mathcal{L}M$. Valued in formal power series[1] over $\mathbb{Z}$, he observed that for manifolds with $p_1(M) = 0$, the index is valued in modular forms, and computes a so-called *elliptic genus* of M.

This is to be compared with the situation for a finite dimensional manifold M, wherein the $\widetilde{A}$-genus $\widehat{A}(M) \in \mathbb{Q}$ takes values in $\mathbb{Z}$ if M is spin (as it coincides with the index of the spin Dirac operator).

While the results necessary to make sense of Witten's index theorem for $D_{\mathcal{L}M}$ are a long way off, we attempt the modest first step of constructing $D_{\mathcal{L}M}$ in a sufficiently nice way as to (hopefully) allow subsequent analysis. Though attempts have been made at the construction of $D_{\mathcal{L}M}$ (see [Tau89] and [Sta05]), our construction is meant to be *very smooth*. By this we mean that all infinite dimensional manifolds $\mathcal{X}$ are Fréchet, modeled on $C^\infty(X;\mathbb{R}^N)$ for compact X, and that functions (or sections of vector bundles) on such manifolds are *smoothly differentiable*, meaning that the derivative df_p of $f \in C^\infty(\mathcal{X};\mathbb{R})$ — a priori a distribution in $C^{-\infty}(X;\mathbb{R}^N) = (C^\infty(X;\mathbb{R}^N))^* = T^*_p\mathcal{X}$ — is actually a smooth function, or at worst a conormal or Lagrangian distribution. It can be shown that smoothly differentiable partitions of unity exist on such a Fréchet manifold.

The loop space of a compact Riemannian manifold M (we'll take $\dim(M) = 2n$ for simplicity) is defined by $\mathcal{L}M := C^\infty(S^1; M)$. The tangent space at a loop γ consists of smooth sections $C^\infty(S^1;\gamma^*TM \cong \underline{\mathbb{R}^{2n}})$ and such sections with sufficiently bounded length can be exponentiated, giving a smooth Fréchet structure to $\mathcal{L}M$.

Supposing M is spin with associated principal bundle $P_{\mathrm{Spin}} \longrightarrow M$, the looped bundle $\mathcal{L}P_{\mathrm{Spin}}$ forms a principal bundle over $\mathcal{L}M$ with structure group $\mathcal{L}\mathrm{Spin} = C^\infty(S^1;\mathrm{Spin}(2n))$. The ingredients for the Dirac operator are the following: (compare the problem of constructing the spin Dirac operator on a finite dimensional oriented Riemannian manifold)

(a) a universal central extension

$$\mathbb{C}^\times \longrightarrow \widehat{\mathcal{L}\mathrm{Spin}} \longrightarrow \mathcal{L}\mathrm{Spin},$$

(b) the existence of a lift $\widehat{\mathcal{L}}P \longrightarrow \mathcal{L}P_{\mathrm{Spin}}$ over $\mathcal{L}M$ with structure group $\widetilde{\mathcal{L}\mathrm{Spin}}$, (such a lift exists if and only if M admits a so-called *string structure*),

(c) the fundamental representation of $\widehat{\mathcal{L}\mathrm{Spin}}$, from which we may form the associated bundle $\mathbb{S} \longrightarrow \mathcal{L}M$,

[1]Powers of the variable correspond to characters of S^1.

(d) a 'Clifford action' of $T^*\mathcal{L}M$ on $\mathbb{S}$, and
(e) a (smoothly differentiable) connection on $\mathbb{S}$ compatible with the Clifford action.

Our construction of $\widehat{\mathcal{L}G}$ for a compact, simply connected[2] is essentially a 'very smooth' version of that given in [PS88], wherein we make use of Toeplitz pseudodifferential operators. Choosing a representation $G \longrightarrow \mathrm{Aut}(V)$, we observe that $\mathcal{L}G = C^\infty(S^1; G < \mathrm{Aut}(V)) \subset \Psi^0(S^1; V)$. Denoting by Π_H the projection onto the smooth Hardy space $H = \left\{ u \in C^\infty(S^1; V) : u = \sum_{k\geq 0} u_k e^{ik\theta} \right\}$ of functions with vanishing negative Fourier coefficients, we set

$$\mathcal{L}_H G = \left\{ \mu = \Pi_H (l + s) \Pi_H \;\middle|\; l \in \mathcal{L}G, s \in \Psi^{-\infty}, \mu : H \longrightarrow H \text{ invertible.} \right\}$$

The symbol $\sigma(\mu) = l$ gives a homomorphism to $\mathcal{L}G$ which is surjective in light of the Toeplitz index formula[3]. The kernel of the symbol map is a group $\mathbb{G}_H^{-\infty}$ of smoothing perturbations of the identity on H, on which the determinant $\det_H : \mathbb{G}_H^{-\infty} \longrightarrow \mathbb{C}^\times$ is defined, and taking the quotient everywhere by the subgroup by $\det_H^{-1}(1)$ yields a central extension

$$\mathbb{C}^\times \longrightarrow \widehat{\mathcal{L}G} = \mathcal{L}_H G / \det_H^{-1}(1) \longrightarrow \mathcal{L}G$$

A basic[4] connection on $\widetilde{\mathcal{L}G}$ is given by $\overline{\mathrm{Tr}}(g^{-1}dg)$, where $g^{-1}dg$ is the Maurer-Cartan form[5] on $\mathcal{L}_H G$, and $\overline{\mathrm{Tr}}(\,\cdot\,) = \mathrm{f.p.}_{z\to 0} \mathrm{Tr}((1 + D_\theta)^{-z}\,\cdot\,)$ is the regularized trace of Wodzicki and Guillemin. The Lie algebra cocycle of the central extension (which classifies it) can be computed by evaluating $d\overline{\mathrm{Tr}}(g^{-1}dg) = \mathrm{Tr}_R([g^{-1}dg, g^{-1}dg])/2$ at $T_{\mathrm{Id}}\mathcal{L}G = \mathcal{L}\mathfrak{g}$ using the trace-defect formula, giving

$$\mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \ni (\eta(\theta), \xi(\theta)) \longmapsto \frac{1}{4\pi i} \int_{S^1} \mathrm{tr}_V \left(\eta'(\theta)\, \xi(\theta) \right) d\theta.$$

For the correct choice of representation $G \longrightarrow \mathrm{Aut}(V)$, the universal central extension may be obtained.

The problem of lifting the principal bundle on $\mathcal{L}M$ to one with structure group $\widehat{\mathcal{L}\mathrm{Spin}}$ is purely topological in nature. Indeed, $\mathcal{L}P_{\mathrm{Spin}}$ along with the central extension $\widehat{\mathcal{L}\mathrm{Spin}}$ give rise to a so-called *lifting bundle gerbe* on $\mathcal{L}M$ with corresponding cohomology class $\sigma \in H^3(\mathcal{L}M; \mathbb{Z})$, the nonvanishing of which obstructs the existence of a lift. From a result of McLaughlin [McL92], σ is the transgression[6] of the class $p_1(M)/2 \in H^4(M; \mathbb{Z})$ (since M is assumed to be spin, $p_1(M)$ is even). Thus M is called a *string manifold* provided it is oriented, spin, and satisfies $p_1(M)/2 = 0$.

Provided M is string, the principal $\widehat{\mathcal{L}\mathrm{Spin}}$ bundle $\widehat{\mathcal{L}P} \longrightarrow \mathcal{L}M$ exists, and is seen to be a smooth Fréchet manifold in light of our construction of $\widehat{\mathcal{L}\mathrm{Spin}}$. Moreover, using a smoothly differentiable partition of unity subordinate to a trivializing cover of $\widehat{\mathcal{L}P}$ and the 1-form $\overline{\mathrm{Tr}}(g^{-1}dg)$, the Levi-Civita connection on $\mathcal{L}P_{\mathrm{Spin}}$ may be lifted to a smoothly differentiable connection on $\widehat{\mathcal{L}P}$.

[2] The assumption $\pi_1(G) = 0$ simplifies matters, but is not strictly necessary.

[3] $\Pi_H l \Pi_H$ is Fredholm with index $0 = -\mathrm{wn}(l)$ and so has an invertible perturbation by smoothing operators

[4] meaning it is compatible with multiplication in $\widehat{\mathcal{L}G}$

[5] upon taking the trace, the form passes to the quotient by $\det_H^{-1}(1)$.

[6] via the map $\int_{S^1} \mathrm{ev}^* : H^k(M; \mathbb{Z}) \longrightarrow H^{k-1}(\mathcal{L}M; \mathbb{Z})$, $\mathrm{ev} : S^1 \times \mathcal{L}M \longrightarrow M$ denoting loop evaluation.

Though the appropriate representation[7] of $\widehat{\mathcal{L}\mathrm{Spin}}$ and the Clifford action are the subjects of ongoing work, it should be pointed out that a critical result of our construction will be that the 'Clifford contraction' (i.e. the composition of the connection with the Clifford action) defining the Dirac operator will be modeled point-wise by the *smooth pairing* of functions, rather than pairing with distributions or passing to L^2 completions.

References

[McL92] D.A. McLaughlin, *Orientation and string structures on loop space*, Pacific J. Math **155** (1992), no. 1, 143–156.

[PS88] A. Pressley and G. Segal, *Loop groups*, Clarendon press, 1988.

[Sta05] A. Stacey, *The geometry of the loop space and a construction of a dirac operator*, Arxiv preprint math/0505077 (2005).

[Tau89] C.H. Taubes, *S 1 actions and elliptic genera*, Communications in mathematical physics **122** (1989), no. 3, 455–526.

[Wit88] E. Witten, *The index of the dirac operator in loop space*, Elliptic curves and modular forms in algebraic topology (1988), 161–181.

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[7] Or more correctly, a very smooth version thereof.