ANALYSIS ON NON-COMPACT MANIFOLDS NOTES FOR 18.158, SPRING 2008

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Contents

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Sources

The notes are for the course 18.158 Spring 2008. They are culled from many sources, including:

- [Carron, 'L^2 harmonic forms on non-compact manifolds']
- [Carron, 'Un théorème de l'indice relatif']
- [de Rham, 'Differentiable Manifolds']
- [Kato, 'Perturbation theory for linear operators']
- [Mazzeo, 'Analysis and geometry on asymptotically hyperbolic spaces', Lecture notes for course offered at ETHZ, Summer 2003]
- [Mazzeo, Elliptic theory of differential edge operators]
- [Melrose, 'Lectures on pseudodifferential operators', Lecture notes for 18.157 offered at MIT, Fall 2005]
- [Melrose, 'The Atiyah-Patodi-Singer Index Theorem']
- [Taylor, 'Pseudodifferential operators']
- [Taylor, 'Partial differential equations']

1. Pseudo-differential operators (in construction)

For the sake of completeness we recall the definition of pseudo-differential operators.

An integral operator P is one whose action on functions is given by integration against a measurable function, $\mathcal{K}_P$, known as the Schwartz kernel of P, so that

$$Pf(\zeta) = \int_M \mathcal{K}_P(\zeta,\zeta')f(\zeta')\,d\zeta'.$$

A similar statement is true for the identity map but the integral kernel is not a function but a distribution (namely the Dirac delta function of the diagonal, $\mathcal{K}_{\mathrm{Id}}(\zeta,\zeta') = \delta(\zeta-\zeta')$). In fact the *Schwartz kernel theorem* says that if M is a smooth manifold and P is a linear map

$$P : \mathcal{C}^\infty_c(M) \to \mathcal{C}^{-\infty}(M),$$

then P has a distributional kernel $\mathcal{K}_P \in \mathcal{C}^{-\infty}(M^2)$ so that

$$\langle Pf, h\rangle_M = \langle \mathcal{K}_P, h\otimes f\rangle_{M^2}, \quad \text{for every } f,h\in\mathcal{C}^\infty_c(M). \tag{1.1}$$

Strikingly, this theorem puts integral and differential operators on the same formal footing.

Even if one's aim is to study elliptic differential operators it is very useful to enlarge the class of operators studied to pseudo-differential operators. We define this class by restricting the singularities that can occur in their distributional kernels. The usual motivation for this larger class starts with considering a constant coefficient differential operator on $\mathbb{R}^m$, say,

$$P = \sum_{|\alpha|\le k} a_\alpha(\zeta)D^\alpha,$$

with $D_j = \frac{1}{i}\partial_{\zeta_j}$. We can realize (1.1) by applying first the Fourier transform and then the inverse Fourier transform; thus, for any smooth compactly supported function, f, we have

$$Pf(\zeta) = \frac{1}{(2\pi i)^{m/2}}\int e^{i\xi\cdot(\zeta-\zeta')}\left(\sum a_\alpha(\zeta)\xi^\alpha\right)f(\zeta')\,d\zeta' d\xi.$$

On $\mathbb{R}^m$ a (properly supported) pseudo-differential operator of order $s\in\mathbb{R}$ is an operator on $\mathcal{C}^\infty_c(\mathbb{R}^m)$ whose action is given by

$$Pf(\zeta) = \frac{1}{(2\pi i)^{m/2}}\int e^{i\xi\cdot(\zeta-\zeta')}a_P(\zeta,\zeta',\xi)f(\zeta')\,d\zeta' d\xi,$$

where a_P is no longer assumed to be a polynomial but instead to be a smooth function satisfying

$$\left|D^\alpha_\zeta D^\beta_{\zeta'} D^\gamma_\xi a_P(\zeta,\zeta',\xi)\right| \le C_{\alpha\beta\gamma}(1+|\xi|^2)^{\frac{1}{2}(s-|\gamma|)}. \tag{1.2}$$

The function a_P is said to be **the symbol** of P, and P is said to be a *quantization* of a_P.

2. Symbolic Calculus and elliptic theory

The aim of the course is to study the generalization of part of the theory of elliptic operators from closed manifolds to non-compact manifolds. We start by recalling the definition of an elliptic operator and the symbolic calculus of pseudo-differential operators. Familiarity with pseudo-differential operators is assumed, though we will review their definition later on, in the process of extending it.

A differential operator of order ℓ on M is an operator that in local coordinates looks like

$$D = \sum_{|I|\leq k} a_I(\zeta) \frac{\partial^{i_1}}{\partial x_1} \cdots \frac{\partial^{i_m}}{\partial x_m}.$$

Formally D is an element of the enveloping algebra of the vector fields on M. The principal symbol of D is given by

$$\sigma(D) = \sum_{|I|=k} a_I(\zeta)(\xi^1)^{i_1} \cdots (\xi^m)^{i_m},$$

in local coordinates, and patches together to define a function on T^*M.

More generally, if E is a vector bundle over M, then the differential operators on sections of E are elements of the algebra generated by $\hom(E)$ and ∇_X where ∇ is any connection on E and X any vector field on M. This algebra is filtered by the order of the operator, e.g.,

$$\mathrm{Diff}^k(M, E) = \underset{\Gamma(\hom(E))}{\mathrm{span}} \left\{ \nabla_{X_1} \cdots \nabla_{X_j} : j \leq k \right\},$$

and endowed with a principal symbol map, now valued in $\mathcal{C}^\infty(T^*M, \pi^* \hom(E))$. To define this map, notice that if $f \in \mathcal{C}^\infty(M)$ satisfies $df(\zeta) = \xi$, then

$$e^{-itf} \circ [a(\zeta)\nabla_{X_1} \cdots \nabla_{X_k}] \circ e^{itf} = (it)^k a(\zeta)\xi(X_1)(\zeta) \cdots \xi(X_k)(\zeta) + \mathcal{O}(t^{k-1})$$

and so we can define the symbol on $\mathrm{Diff}^k(M, E)$ by

$$\sigma_k(D)(\zeta, \xi) = \lim_{t\to\infty} \frac{e^{-itf} \circ D \circ e^{itf}}{t^k}.$$

It is easy to see that this is well-defined independently of the choice of f. Also it follows that the symbol is equal to

$$\sigma_k(D)(\zeta, \xi) = \frac{(-i)^k}{k!} (\mathrm{ad} f)^k D.$$

The right hand side is a zero-th order differential operator and hence an element of $\hom(E)_\zeta$.

The principal symbol σ_k vanishes precisely on $\mathrm{Diff}^{k-1}(M, E)$ and so we get a short exact sequence

$$0 \to \mathrm{Diff}^{k-1}(M, E) \to \mathrm{Diff}^k(M, E) \xrightarrow{\sigma} \mathcal{P}_k(S^*M, \pi^* \hom(E)) \to 0 \tag{2.1}$$

where the space on the right is the space of sections of $\pi^* \hom(E)$ over T^*M that are homogeneous polynomials of degree k along the fibers, restricted to the cosphere bundle S^*M. (From now on, unless indicated otherwise, we will think of symbols as being defined on the cosphere bundle.) A differential operator is called **elliptic** if its principal symbol is invertible.

2.1. Parametrices of elliptic operators.

The inverse of the symbol of an elliptic operator is not itself the symbol of a differential operator, but it is the symbol of a *pseudo-differential* operator. For the moment we will take a 'black-box' approach to pseudo-differential operators, though later we will review their definition in order to generalize it. The space of pseudo-differential operators on M acting on sections of E

$$\Psi^s(M,E) \ni A : \mathcal{C}^\infty(M,E) \to \mathcal{C}^\infty(M,E)$$

form an algebra filtered by the 'degree' $s \in \mathbb{R}$,

$$\Psi^s(M,E) \subseteq \Psi^t(M,E) \iff s \le t, \quad \Psi^s(M,E) \circ \Psi^t(M,E) \subseteq \Psi^{s+t}(M,E)$$

with associated graded algebra isomorphic to the symbols of order s so that

$$0 \to \Psi^{s-1}(M,E) \to \Psi^s(M,E) \xrightarrow{\sigma} \mathcal{C}^\infty(S^*M, N_s \otimes \pi^* \hom(E)) \to 0, \tag{2.2}$$

where N_s is a trivial line bundle over S^*M that 'carries the homogeneity' of σ, is a short exact sequence of algebras.

Pseudo-differential operators are *asymptotically complete* in the sense that if, for every $i \in \mathbb{N}_0$ we have an operator $P_i \in \Psi^{k-i}(M,E)$, then there is an operator $P \in \Psi^k(M,E)$ such that

$$P - \sum_{i \le j} P_i \in \Psi^{k-j-1}(M,E), \text{ for every } j \in \mathbb{N}_0.$$

Finally, the 'residual' space of pseudo-differential operators

$$\Psi^{-\infty}(M,E) = \cap_{s\in\mathbb{R}} \Psi^s(M,E)$$

coincides with the space of smoothing operators, i.e.,

$$\Psi^{-\infty}(M,E) \ni A : \mathcal{C}^{-\infty}(M,E) \to \mathcal{C}^\infty(M,E),$$

where $\mathcal{C}^{-\infty}(M,E)$ denotes the space of distributional sections dual to $\mathcal{C}^\infty$ on M[1].

If $Q \in \Psi^*(M;F,E)$ and $P \in \Psi^*(M;E,F)$ are both pseudo-differential operators, we say that Q is a **parametrix** for P if

$$PQ - \mathrm{Id} \in \Psi^{-\infty}(M,F), \quad QP - \mathrm{Id} \in \Psi^{-\infty}(M,E).$$

We say that Q is a **right parametrix** for P if the first equation holds and a **left parametrix** if the second equation holds.

Proposition 2.1. *If $P \in \Psi^k(M;E,F)$ is elliptic then P has a parametrix $Q \in \Psi^{-k}(M;F,E)$.*

[1] $\mathcal{C}^{-\infty}(M,E) = (\mathcal{C}^\infty(M, E^* \otimes \Omega_M))'$

Proof. We carry out the 'Levi parametrix procedure' to construct Q iteratively. To start, since P is elliptic and (2.2) is exact, we can find $Q_0 \in \Psi^{-k}(M;F,E)$ such that

$$\sigma(Q_0) = \sigma(P)^{-1}.$$

It follows, using that σ is an algebra homomorphism and (2.2) is exact, that

$$PQ_0 = \mathrm{Id} - R_0, \quad R_0 \in \Psi^{-1}(M;F).$$

For every $j \in \mathbb{N}_0$, define

$$Q_j = Q_0(\mathrm{Id} + R_0 + R_0^2 + \ldots + R_0^j) \in \Psi^{-k}(M;F),$$

so that

$$PQ_j = (\mathrm{Id} - R_0)(\mathrm{Id} + R_0 + R_0^2 + \ldots + R_0^j) = \mathrm{Id} - R_0^{j+1},$$

with $R_0^{j+1} \in \Psi^{-1-j}(M;F)$. Using asymptotic completeness we can find a $Q \in \Psi^{-k}(M;F,E)$ such that

$$Q - Q_j \in \Psi^{-k-j-1}(M;F,E), \text{ for every } j \in \mathbb{N}_0$$

and thus $PQ = \mathrm{Id} - R$ with $R \in \Psi^{-\infty}(M;F)$ so Q is a right parametrix for P.

In the same way, we can find a left parametrix for P, say Q', such that $Q'P = \mathrm{Id} - R'$ with $R' \in \Psi^{-\infty}(M;E)$. We can compare Q and Q' by writing $Q'PQ$ in two different ways

$$Q'PQ = (\mathrm{Id} - R')Q = Q - R'Q$$
$$Q'PQ = Q'(\mathrm{Id} - R) = Q' - Q'R$$

and hence $Q - Q' = R'Q - Q'R \in \Psi^{-\infty}(M;F,E)$. It follows that Q and Q' are both parametrices of P. □

By taking a closer look at smoothing operators we can deduce mapping properties of pseudo-differential operators and improve the parametrix construction for elliptic operators.

A choice of a Riemannian metric g on M, gives us a density dvol_g which we use to define an inner product on $\mathcal{C}^\infty(X)$, by

$$\langle f, h \rangle = \int_M f(\zeta)\overline{h(\zeta)} \, \mathrm{dvol}_g .$$

If we also choose a Hermitian metric on E, then we can define an inner product on $\mathcal{C}^\infty(M,E)$ by

$$\langle u, v \rangle = \int_M (u(\zeta), v(\zeta))_E \, \mathrm{dvol}_g .$$

The completion of $\mathcal{C}^\infty(M,E)$ with respect to the induced norm

$$\|s\|_{L^2} = \sqrt{< s, s >}$$

is denoted $L^2(M,E)$ and is independent of the choices.

By the Schwartz kernel theorem[2], if P is a pseudo-differential operator, then the bilinear form

$$\mathcal{C}^\infty(M,E)\times\mathcal{C}^\infty(M,E)\ni(u,v)\mapsto\langle Pu,v\rangle\in\mathbb{C}$$

is given by pairing with a distribution $\mathcal{K}_P\in\mathcal{C}^{-\infty}(M^2,\mathrm{Hom}(E))$,

$$\langle Pu,v\rangle=(\mathcal{K}_P,\pi_L^*v\otimes\pi_R^*u),$$

where π_L and π_R are the left and right projections $M^2\to M$ and $\mathrm{Hom}(E)=\pi_L^*E\otimes\pi_R^*E'$. We often represent this symbolically by

$$Pu(\zeta)=\int\mathcal{K}_P(\zeta,\zeta')u(\zeta')\,d\zeta', \tag{2.3}$$

however if $P\in\Psi^{-\infty}(M,E)$ then this representation is correct and $\mathcal{K}_P\in\mathcal{C}^\infty(M^2,\mathrm{Hom}(E))$ (and conversely). Since M is compact it follows that $P\in\Psi^{-\infty}(M,E)$ defines a bounded linear operator on $L^2(M,E)$. Furthermore it maps $L^2(M,E)$ continuously into $\mathcal{C}^\infty(M,E)$ and hence, by factoring P

$$\begin{array}{ccc} L^2(M,E) & \dashrightarrow & L^2(M,F) \\ \downarrow{\scriptstyle P} & & \uparrow \\ \mathcal{C}^\infty(M,F) & \hookrightarrow & \mathcal{C}^0(M,F) \end{array}$$

and applying the Arzela-Ascoli theorem, we see that it is a compact operator[3].

Recall that an operator between two topological vector spaces is Fredholm if its null space is finite dimensional, its range is closed, and its range has a finite dimensional complement. The index of a Fredholm operator is defined to be

$$\mathrm{ind}(P)=\dim\ker(P)-\dim\mathrm{coker}(P).$$

Lemma 2.2. *If S is a smoothing operator, then $\mathrm{Id}-S$ is a Fredholm operator on either $\mathcal{C}^\infty(M,E)$ or $L^2(M,E)$.*

Proof. First we prove that $\mathrm{Id}-S$ is Fredholm on $L^2(M,E)$. Notice that if u is an element of the null space of $\mathrm{Id}-S$ as an operator on $L^2(M,E)$ (which we denote $\mathrm{null}_{L^2}(\mathrm{Id}-S)$), then $u=Su$, so S acts as the identity on $\mathrm{null}_{L^2}(\mathrm{Id}-S)$. This implies that elements of this space are smooth and, since S is compact, that this space is finite dimensional. It also follows that the null space of $\mathrm{Id}-S$ as an operator on $\mathcal{C}^\infty(M,E)$ coincides with its L^2-null space.

To check that the range is closed, let (u_k) be a sequence of elements in $L^2(M,E)$, orthogonal to $\mathrm{null}_{L^2}(\mathrm{Id}-S)$, such that $(\mathrm{Id}-S)u_k$ converges, say to $v\in L^2$. If u_k has a bounded subsequence, then a subsequence of Su_k converges and hence so does a subsequence of $(\mathrm{Id}-S)u_k+Su_k=u_k$. If the limit of this subsequence is u, then continuity of $\mathrm{Id}-S$ implies $(\mathrm{Id}-S)u=$

[2] See [Taylor vol.1, §4.6]

[3] That is, it maps bounded sets to pre-compact sets

v. If instead $\|u_k\| \to \infty$, then the sequence $\overline{u}_k = u_k/\|u_k\|$ is bounded and $(\mathrm{Id} - S)\overline{u}_k$ converges (to zero). By the previous argument, there is a subsequence of $\overline{u}_k$ converging to an element of $\mathrm{null}_{L^2}(\mathrm{Id} - S)$ with unit norm, but this can not happen since we assumed that all u_k are orthogonal to $\mathrm{null}_{L^2}(\mathrm{Id} - S)$.

Since the range of $\mathrm{Id} - S$ is closed, its orthocomplement can be identified with the null space of its adjoint, $\mathrm{Id} - S^*$. Since S^* is also smoothing, the previous argument shows that

$$\mathrm{null}_{L^2}(\mathrm{Id} - S^*) = \mathrm{null}_{\mathcal{C}^\infty}(\mathrm{Id} - S^*) \text{ is finite dimensional.}$$

Hence the range of $\mathrm{Id} - S$ as an operator on $L^2(M, E)$ has a finite dimensional complement, and $\mathrm{Id} - S$ is Fredholm.

To check that the range is closed as an operator on $\mathcal{C}^\infty(M, E)$, let (u_k) be a sequence of elements in $\mathcal{C}^\infty(M, E)$ such that $(\mathrm{Id} - S)u_k$ converges in $\mathcal{C}^\infty(M, E)$ to v. Then by the above argument a subsequence of u_k converges in L^2, and since $u_k = (\mathrm{Id} - S)u_k + Su_k$ it actually converges in $\mathcal{C}^\infty(M, E)$, and so the range is closed. Finally we claim that

$$(\mathrm{Id} - S)\left(\mathcal{C}^\infty(M, E)\right) + \mathrm{null}(\mathrm{Id} - S^*) = \mathcal{C}^\infty(M, E),$$

Notice that the left hand side is certainly a closed subspace of $\mathcal{C}^\infty(M, E)$, so it suffices to show that if $v \in \mathcal{C}^{-\infty}(M; E)$ pairs to zero with both summands, then $v = 0$. And indeed, if v is such that

$$\begin{gathered} (v, (\mathrm{Id} - S)u) = 0 \text{ for every } u \in \mathcal{C}^\infty(M, E), \\ \text{and } (v, w) = 0 \text{ for every } w \in \mathrm{null}(\mathrm{Id} - S^*), \end{gathered}$$

then using the first condition we see that $v = S^*v$ hence $v \in \mathcal{C}^\infty(M, E)$ and v is in $\mathrm{null}(\mathrm{Id} - S^*)$, but then the second condition shows that $v = 0$. □

A generalized inverse (or pseudo-inverse) of an operator A between Hilbert spaces with closed range is an operator B such that

$$BA = \mathrm{Id} - \Pi_{\mathrm{null}(A)}, \quad AB = \mathrm{Id} - \Pi_{\mathrm{null}(A^*)}$$

with Π_* equal to the orthogonal projection to each space.

Proposition 2.3. *If $P \in \Psi^k(M; E, F)$ is an elliptic operator then it defines a Fredholm operator acting on smooth sections or L^2 sections, with smooth kernel and cokernel. Furthermore, its generalized inverse as an operator between L^2 sections of E and F is a pseudo-differential parametrix.*

Proof. Because P is elliptic we know that it has left and right parametrices, Q and Q', such that

$$PQ = \mathrm{Id} - R, \quad Q'P = \mathrm{Id} - R'$$

with R and R' smoothing operators.

Notice that the range of P contains the range of $PQ = \mathrm{Id} - R$ which is closed and complemented by a finite dimensional space of smooth sections. It follows that the range of P is closed and has a finite dimensional complement consisting of smooth sections. Similarly the null space of P is contained in the null space of $Q'P = \mathrm{Id} - R$ which is a finite dimensional space of smooth sections. It follows that the null space of P is finite dimensional and smooth.

The same reasoning shows that the range of P^* is closed, and hence we can decompose the sections of E or F into

$$\mathcal{C}^\infty(M,E) = \mathrm{null}(P) \oplus \mathrm{Ran}(P^*), \quad \mathcal{C}^\infty(M,F) = \mathrm{null}(P^*) \oplus \mathrm{Ran}(P),$$

and P induces an invertible map

$$\widetilde{P} : \mathrm{Ran}(P^*) \to \mathrm{Ran}(P).$$

The generalized inverse of P is defined in terms of this decomposition by

$$G = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{P}^{-1} \end{pmatrix},$$

and clearly satisfies

$$GP = \mathrm{Id} - \Pi_{\mathrm{null}(P)}, \quad PG = \mathrm{Id} - \Pi_{\mathrm{null}(P^*)}.$$

Let $\{s_1, \ldots, s_\ell\}$ be a basis of null(P) consisting of smooth sections, then for any $u \in L^2(M,E)$ we have

$$\Pi_{\mathrm{null}(P)}(u) = \sum \langle s_i, u \rangle s_i \in \mathcal{C}^\infty(M,E)$$

and since $\Pi_{\mathrm{null}(P)}$ is self-adjoint this implies $\Pi_{\mathrm{null}(P)} \in \Psi^{-\infty}(M,E)$. Similarly $\Pi_{\mathrm{null}(P^*)} \in \Psi^{-\infty}(M,F)$. To show that G is a pseudo-differential operator, we write $Q'PG$ in two ways to conclude

$$G = R'G + Q' - Q'\Pi_{\mathrm{null}(P^*)},$$

we write GPQ in two ways to conclude

$$G = GR + Q - \Pi_{\mathrm{null}(P)}Q,$$

and substitute the second expression in the first to find

$$G = Q' + R'Q - R'\Pi_{\mathrm{null}(P)}Q - Q'\Pi_{\mathrm{null}(P^*)} + R'GR. \tag{2.4}$$

The last term on the right hand side is smoothing, because both it and its adjoint map L^2 sections to smooth sections, hence all of the terms on the right hand side are pseudo-differential and we get $G \in \Psi^{-k}(M;F,E)$. □

The symbolic calculus and the behavior of smoothing operators is enough to analyze the mapping properties of pseudo-differential operators.

Proposition 2.4. *If $P \in \Psi^0(M;E,F)$ then P defines a bounded linear operator between $L^2(M,E)$ and $L^2(M,F)$.*

Proof. We follow Hörmander and reduce to the smoothing case as follows. Assume we can find $Q \in \Psi^0(M,E)$, $C > 0$, and $R \in \Psi^{-\infty}(M,E)$ such that

$$P^*P + Q^*Q = C\,\mathrm{Id} + R$$

where Q^* is defined by requiring $(Qu, v) = (u, Qv)$ for every $u, v \in L^2(M, E)$ and P^* is defined by requiring $\langle Pu, v\rangle_F = \langle u, P^* v\rangle_E$ for every $u \in L^2(M, E)$ and $v \in L^2(M, F)$. Then we have

$$\begin{aligned}\|Pu\|^2_{L^2(M,F)} &= \langle Pu, Pu\rangle_F = \langle P^*Pu, u\rangle_E \\ &\leq \langle P^*Pu, u\rangle + \langle Q^*Qu, u\rangle = C\langle u, u\rangle + \langle Ru, u\rangle \leq (C + \|\mathcal{K}_R\|)\langle u, u\rangle,\end{aligned}$$

so the proposition is proved once we construct Q, C, and R.

We proceed iteratively. Choose $C > 0$ large enough so that $C - \sigma(P)^*\sigma(P)$ is, at each point of S^*M, a positive element of $\hom(E)$, then choose Q_0' so that its symbol is a positive square root of $C - \sigma^*(P)\sigma(P)$, and let $Q_0 = \frac{1}{2}(Q_0' + (Q_0')^*)$. Note that Q_0 is a self-adjoint elliptic operator with the same symbol as Q_0'.

Suppose we have found self-adjoint operators $Q_0, \dots, Q_N$ satisfying $Q_i \in \Psi^{-i}(M; E)$ and

$$P^*P + \left(\sum_{i=0}^N Q_i\right)^2 = C\,\mathrm{Id} - R_N, \quad R_N \in \Psi^{-1-N}(M; E).$$

Then, if $Q'_{N+1} \in \Psi^{-N-1}(M, E)$, we have

$$\begin{aligned}&P^*P + \left(\sum_{i=0}^N Q_i + Q'_{N+1}\right)^2 - C\,\mathrm{Id} \\ &= -R_N + Q'_{N+1}\left(\sum_{i=0}^N Q_i\right) + \left(\sum_{i=0}^N Q_i\right)Q'_{N+1},\end{aligned}$$

so in order for this to be in $\Psi^{-2-N}(M; E)$, Q'_{N+1} must satisfy

$$\sigma(Q'_{N+1}Q_0 + Q_0Q'_{N+1}) = \sigma(R_N).$$

We can always solve this equation because $\sigma(Q_0)$ is self-adjoint and positive as shown in the following lemma.

Lemma 2.5. *Let A be a positive self-adjoint $k \times k$ matrix, and let*

$$\Phi_A(B) = AB + BA,$$

then Φ_A is a bijection from the space of self-adjoint matrices to itself.

Proof. We can find a matrix Q such that $Q^* = Q^{-1}$ and $Q^*AQ = D$ is diagonal. Define $\Psi(B) = Q^*BQ$ and note that $\Psi^{-1}\Phi_A\Psi = \Phi_{\Psi(A)}$ so we may replace A with $D = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$. Let E_{st} denote the matrix with entries

$$(E_{st})_{ij} = \begin{cases} 1 \text{ if } s = i, t = j \\ 0 \text{ otherwise}\end{cases}$$

and note that

$$\Phi_D(E_{st} + E_{ts}) = (\lambda_s + \lambda_t)(E_{st} + E_{ts}).$$

Hence the matrices $\{E_{st} + E_{ts}\}$ with $s \leq t$ form an eigenbasis of Φ_D on the space of self-adjoint $k \times k$ matrices. Since, by assumption, each $\lambda_s > 0$ it follows that Φ_D is bijective. □

So using the lemma we can find $\sigma(Q'_{N+1})$ among self-adjoint matrices and then define $Q_{N+1} = \frac{1}{2}(Q'_{N+1} + (Q'_{N+1})^*)$, finishing the iterative step. Finally, we define Q by asymptotically summing the Q_i and the result follows. □

Corollary 2.6. *Let $A \in \Psi^s(M; E, F)$, $s \leq 0$, and let*

$$|\sigma| = \sup_{S^*M} |\sigma(A)(\zeta, \xi)|,$$

then there is a self-adjoint operator $R \in \Psi^{-\infty}(M, E)$ such that, for any $u \in L^2(M, E)$

$$\|Au\|^2_{L^2(M,F)} \leq |\sigma|^2 \|u\|_{L^2(M,E)} + \langle Ru, u\rangle_E.$$

In particular, if $|\sigma| = 0$, then A is compact as an operator on L^2.

Remark 2.7. The closure of $\Psi^0(M)$ as a subalgebra of bounded operators on $L^2(M)$ is a C^*-algebra, $\mathcal{A}$, with the space of compact operators $\mathcal{K}$ as a closed ideal. The symbol map is a $*$-homomorphism and hence extends to a continuous homomorphism fitting into a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{A} \xrightarrow{\sigma} \mathcal{C}^0(S^*M) \to 0.$$

The quotient algebra, $\mathcal{A}/\mathcal{K}$ is a commutative C^*-algebra and hence by the Gelfand-Naimark theorem isometrically isomorphic to the space of continuous functions on a compact Hausdorff space. The exact sequence shows that σ is an isomorphism of the quotient with $\mathcal{C}^0(S^*M)$ and the corollary can be used to show that it is an isometry.

Boundedness on L^2 spaces also gives us boundedness on Sobolev spaces. We can define Sobolev spaces on M, for any $s \in \mathbb{R}$, by

$$H^s(M, E) = \left\{u \in \mathcal{C}^{-\infty}(M, E) : Pu \in L^2(M, E) \text{ for every } P \in \Psi^s(M, E)\right\}.$$

Then we put a norm on these spaces using the L^2-norm on $L^2(M, E)$ and a fixed family of invertible pseudo-differential operators of order s, D_s,

$$\|u\|_{H^s} = \|D_s u\|_{L^2}.$$

For instance, we can take

$$D_s = (\operatorname{Id} + \Delta)^{s/2},$$

which yields the usual Sobolev norm on $\mathbb{T}^m$. A distributional section is smooth precisely when it is in the intersection of the Sobolev spaces of all orders.

Corollary 2.8. *If $P \in \Psi^k(M; E, F)$ then P defines a bounded linear operator*

$$P : H^s(M, E) \to H^{s-k}(M, F), \text{ for any } s \in \mathbb{R}.$$

If P is elliptic all of these operators are Fredholm and have the same index. Furthermore,

$$H^s(M,E) = \\ \left\{ u \in \mathcal{C}^{-\infty}(M,E) : \exists P \in \Psi^s(M,E) \text{ elliptic, s.t. } Pu \in L^2(M,E) \right\}.$$

Proof. That P defines a linear map from $H^s(M,E)$ to $H^{s-k}(M,E)$ is automatic. If we use a family of invertible pseudo-differential operators, D_s^E and D_s^F, on sections of E and F respectively to identify the Sobolev space with L^2, then P as a map between H^s and H^{s-k} is equal to $D_{s-k}^F P (D_s^E)^{-1}$ as a map between $L^2(M,E)$ and $L^2(M,F)$, and this map is bounded by Proposition 2.4. Similarly all three operators in this composition are Fredholm, hence so is P. The kernel and cokernel of P acting on different Sobolev spaces is the same under the image of D_*^E and D_*^F so their dimensions do not change.

If $s > 0$ then $P \in \Psi^s(M,E)$ acts as an ubounded linear operator on $L^2(M,E)$. Recall that the maximal domain of such an operator is

$$\mathcal{D}_{\max}(P) = \left\{ u \in L^2(M,E) : Pu \in L^2(M,E) \right\}.$$

It is always true that $H^s(M,E) \subseteq \mathcal{D}_{\max}(P)$, and if P is elliptic then the converse is also true. Indeed, let Q be a parametrix for P, so that $QP = \mathrm{Id} - R$ with $R \in \Psi^{-\infty}(M,E)$, then whenever $u \in \mathcal{D}_{max}(P)$ we have $u = QPu + Ru$ with $QPu \in H^s(M,E)$ and $Ru \in \mathcal{C}^\infty(M,E)$. It follows that $\mathcal{D}_{\max}(P) = H^s(M,E)$, which proves the characterization above for $s > 0$ and for $s \leq 0$ by duality. □

The same argument shows elliptic regularity, i.e., if $P \in \Psi^s(M;E,F)$ is elliptic and u is a distributional section of E then

$$Pu \in H^k(M,F) \iff u \in H^{k+s}(M,E).$$

It also yields the basic elliptic estimate. Indeed, if Q is a parametrix for P we have

$$\|u\|_{H^{k+s}(M,E)} = \|QPu + Ru\|_{H^{k+s}(M,E)} \\ \leq \|QPu\|_{H^{k+s}(M,E)} + \|Ru\|_{H^{k+s}(M,E)} \leq C(\|Pu\|_{H^k(M,F)} + \|u\|_{H^k(M,E)}).$$

The last norm $\|u\|_{H^k(M,E)}$ can be taken in $H^r(M,E)$ for any $r < k+s$.

The basic elliptic estimate extends to arbitrary manifolds locally, in the following way:

> If M is a smooth manifold (not necessarily compact), $P \in \mathrm{Diff}^s(M;E,F)$ is an elliptic operator, and $\mathcal{U} \subseteq\subseteq \Omega \subseteq\subseteq M$, then, for every $u \in \mathcal{C}_c^\infty(\Omega;E)$, we have
>
> $$\|u\|_{H^{k+s}(\mathcal{U},E)} \leq C(\|Pu\|_{H^k(\Omega,F)} + \|u\|_{H^k(\Omega,E)}).$$

In particular, if we define $H^s_{\mathrm{loc}}(M,E)$ of a manifold to be

$$H^s_{\mathrm{loc}}(M,E) = \{u : \mathcal{C}_c^\infty(M)u \in H^s_c(M,E)\},$$

then we see that, if P is elliptic of order s and $Pu \in H^k_{\mathrm{loc}}(M,F)$, then $u \in H^{k+s}_{\mathrm{loc}}(M,E)$.

2.2. **Essentially self-adjoint operators.**
We briefly recall some concepts from the theory of unbounded operators on a Hilbert space. If $\mathcal{H}$ is a Hilbert space, the graph of a linear operator $T : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(T)$ is the set $\{(x, Tx) \in \mathcal{H} \times \mathcal{H} \ : x \in \mathcal{D}(T)\}$. An operator is **closed** if its graph is a closed subset of $\mathcal{H} \times \mathcal{H}$ (i.e., if $(x_n) \subseteq \mathcal{D}(T)$, $x_n \to x$, and $Tx_n \to y$, then $Tx = y$). The closed graph theorem says that an everywhere defined operator with a closed graph is automatically bounded, so when dealing with unbounded operators we need to also keep track of their domain.

An extension of T is an operator T' such that $\mathcal{D}(T) \subset \mathcal{D}(T')$ and $Tx = T'x$ for every $x \in \mathcal{D}(T)$. An operator is *closable* if the closure of its graph is the graph of a linear operator.

If $\mathcal{D}(T)$ is dense in $\mathcal{H}$, then we can define an adjoint of T, T^*. Its domain is

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} : x \mapsto (Tx, y) \text{ is continuous on } \mathcal{D}(T)\},$$

and if $y \in \mathcal{D}(T^*)$ then T^*y is defined by the relation

$$(Tx, y) = (x, T^*y) \text{ for every } x \in \mathcal{D}(T).$$

This definition makes T^* a closed operator. If T is closed, then T^* is densely defined and $T^{**} = T$.

An operator is symmetric if $(Tx, y) = (x, Ty)$ whenever $x, y \in \mathcal{D}(T)$, and self-adjoint if moreover $\mathcal{D}(T) = \mathcal{D}(T^*)$. A symmetric operator is always closable since its adjoint is a closed extension. An operator is essentially self-adjoint if it has a unique closed, self-adjoint extension.

A pseudo-differential operator $P \in \Psi^s(M,E)$ is formally self-adjoint if is it symmetric on $\mathcal{C}^\infty_c(M,E)$. If $s > 0$, then P defines an unbounded linear operator on $L^2(M,E)$, with two canonical closed extensions from $\mathcal{C}^\infty_c(M,E)$. The minimal extension of P is defined by taking the closure of the graph of P,

$$\mathcal{D}(P_{\min}) = \\ \left\{ \omega \in L^2(M,E) : \exists (\omega_j) \subseteq \mathcal{C}^\infty_0(M,E), \omega_j \to \omega, \text{ and } P\omega_j \text{ converges} \right\},$$

and the maximal closure is defined by letting P act distributionally,

$$\mathcal{D}(P_{\max}) = \left\{ \omega \in L^2(M,E) : P\omega \in L^2(M,E) \right\}.$$

Using the L^2 pairing on smooth functions we can define P' by $\langle Pu, v\rangle = \langle u, Pv\rangle$ (this is not the adjoint of P because P is not a closed operator on $\mathcal{C}^\infty_c(M,E)$). We point out that $\mathcal{D}(P'_{\min}) = \mathcal{D}(P_{\max})$ and $\mathcal{D}(P'_{\max}) = \mathcal{D}(P_{\min})$, so a formally self-adjoint pseudo-differential operator is essentially self-adjoint precisely when $\mathcal{D}(P_{\min}) = \mathcal{D}(P_{\max})$. We will generally abuse notation and refer to the domains of the minimal and maximal extensions by $\mathcal{D}_{\min}(P)$ and $\mathcal{D}_{\max}(P)$ respectively.

On a closed manifold we clearly have $H^s(M,E) \subseteq \mathcal{D}_{\min}(P)$ for any $P \in \Psi^s(M,E)$, and, from the proof of Corollary 2.8, $\mathcal{D}_{\max}(P) = H^s(M,E)$ whenever P is elliptic. In view of the above, this has the following interpretation.

Corollary 2.9. *If $s > 0$ and $P \in \Psi^s(M,E)$ is elliptic, then P has a unique closed extension from $\mathcal{C}^\infty(M,E)$ as a linear operator on $L^2(M,E)$. The domain of this extension is $H^s(M,E)$. If P is formally self-adjoint, this extension is self-adjoint.*

Example 2.10. If $P = \Delta$ is the Laplacian on a compact manifold with boundary, it is formally self-adjoint with core domain $\mathcal{C}_c^\infty(M)$, but Corollary 2.9 does not hold. Indeed, the minimal extension of Δ has domain

$$\mathcal{D}_{\min}(\Delta) = H_0^2(M) = \text{closure of } \mathcal{C}_c^\infty(M) \text{ in } H^2(M)$$

while $\Delta u \in L^2(M) \iff u \in H^2(M)$, so $\mathcal{D}_{\max}(\Delta) = H^2(M)$.

Thus there is no unique self-adjoint extension of Δ. From Green's formula

$$(\Delta u, v) - (u, \Delta v) = \int_{\partial M} \frac{\partial u}{\partial \nu}\overline{v} - u\frac{\partial \overline{v}}{\partial \nu}\, dS$$

we see that the Laplacian with either of the domains

$$\mathcal{D}_{\text{Dir}} = \left\{ u \in H^2(M) : i^*u = 0 \right\}, \quad \mathcal{D}_{\text{Neum}} = \left\{ u \in H^2(M) : i^*\frac{\partial u}{\partial \nu} = 0 \right\}$$

is self-adjoint.

Example 2.11. On the other hand, if M is not compact but is complete, then Δ is essentially self-adjoint with core domain $\mathcal{C}_c^\infty(M)$. To prove this we recall the following characterization of complete manifolds.

Lemma 2.12 (Gordon, de Rham, Borel)**.** *A smooth Riemannian manifold M is complete if and only if there is a smooth proper function $M \xrightarrow{\mu} [0,\infty)$ whose gradient is uniformly bounded.*

Proof. If M is complete and $p \in M$, note that the distance to p is a proper function, differentiable almost everywhere and with gradient of length one. We obtain μ by smoothing out $d(p,\cdot)$. Another approach with more machinery is to use the Nash embedding theorem to find an isometric embedding $J : M \to \mathbb{R}^N$. Then completeness of M is equivalent to properness of J. Define $F : \mathbb{R}^N \to \mathbb{R}$ by $F(\zeta) = \log(1 + |\zeta|^2)$ so that F is proper, smooth, and has gradient of length bounded by one, and note that the same is true for $\mu = F \circ J$.

Conversely, given such a function, let $\gamma : \mathcal{J} \subseteq \mathbb{R} \to M$ be a geodesic segment parametrized by arclength. If the length of γ is finite then, since $|\dot\gamma| = 1$, the variation of $\mu \circ \gamma$ on $\mathcal{J}$ is bounded. Because μ is proper this implies that the range of γ is contained in a compact subset of M and hence can be extended at both ends. Hence M is complete. $\square$

Theorem 2.13 (Gaffney). *Let M be a complete Riemannian manifold, and let $D \in \mathrm{Diff}^1(M; E, F)$ satisfy $|\sigma(D)(\zeta, \xi)| \leq C(1 + |\xi|)$ uniformly on M, then*

a) $\mathcal{D}_{\min}(D) = \mathcal{D}_{\max}(D)$,
b) $\mathcal{D}_{\min}(D^*D) = \mathcal{D}_{\max}(D^*D)$, *hence D^*D is essentially self-adjoint.*

Operators satisfying the hypothesis of the theorem include the exterior derivative d, its adjoint δ, any covariant derivative ∇, and more generally any Dirac-type operator $\eth$. Applying part (*b*) to the operator $D = d + \delta$ shows that the Hodge Laplacian is essentially self-adjoint.

Proof.
[**a**)] We need to show that $\mathcal{D}_{\max}(D) \subseteq \mathcal{D}_{\min}(D)$, so we fix $u \in \mathcal{D}_{\max}(D) \subseteq H^1_{\text{loc}}(M)$ and we will find a sequence $(u_j) \subseteq \mathcal{D}(D)$ such that $u_j \to u$ and $Du_j \to Du$ in L^2. Fix $\chi \in \mathcal{C}^\infty(\mathbb{R}; [0, 1])$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \end{cases}$$

and define $\chi_j(q) = \chi(\mu(q) - j)$, so that

$$\chi_j(q) = \begin{cases} 1 & \text{if } \mu(q) \leq j \\ 0 & \text{if } \mu(q) \geq j + 1 \end{cases}$$

Notice that we can find $C > 0$ such that $|\chi_j| \leq C$ independently of j. Define $u_j = \chi_j u$ and note that $u_j \to u$ and

$$Du_j = D(\chi_j u) = \chi_j Du + [D, \chi_j]u = \chi_j Du + \sigma(D)(\cdot, d\chi_j)u$$

Clearly the first term on the right converges to Du. The second term converges pointwise to zero, since $[D, \chi_j]$ is supported in $\mu^{-1}([j, j + 1])$, and its pointwise norm is bounded above by $C|u|_E$, hence by Lebesgue dominated convergence it converges to zero in $L^2(M, F)$.

[**b**)] Our strategy is to find an expression for $\mathcal{D}.(D^*D)$ in terms of the domains of D and D^* (both of which satisfy (*a*)). Let $u \in L^2(M, E)$ be an element of $\mathcal{D}_{\max}(D^*D)$ and denote $\chi_{j+1}u$ by u_{j+1}, note that

$$\langle \chi_j^2 u, D^*Du \rangle = \langle \chi_j^2 u_{j+1}, D^*D(u_{j+1}) \rangle$$

since χ_{j+1} is identically equal to one on the support of χ_j. For any $\phi \in \mathcal{C}^\infty_c(M)$ we know that ϕu is a compactly supported element of $H^2_{\text{loc}}(M; E)$ and hence is in the domain of both D^* and D. It follows that

$$\begin{aligned} \langle \chi_j^2 u_{j+1}, D^*D(u_{j+1}) \rangle &= \langle D(\chi_j^2 u_{j+1}), D(u_{j+1}) \rangle \\ &= \langle \chi_j^2 D(u_{j+1}) + 2\chi_j \sigma(D)(d\chi_j)u_{j+1}, D(u_{j+1}) \rangle \\ &= \langle \chi_j D(u_{j+1}), \chi_j D(u_{j+1}) \rangle + \langle 2\chi_j \sigma(D)(d\chi_j)u_{j+1}, D(u_{j+1}) \rangle. \end{aligned}$$

However, it is easy to see that

$$\langle 2\chi_j\sigma(D)u_{j+1}, D(u_{j+1})\rangle \geq -\|\chi_j D(u_{j+1})\| \, \|2\sigma(D)u_{j+1}\|$$
$$= \frac{1}{2}\left[(\|\chi_j D(u_{j+1})\| - \|2\sigma(D)u_{j+1}\|)^2 - \|\chi_j D(u_{j+1})\|^2 - \|2\sigma(D)u_{j+1}\|^2 \right]$$
$$\geq -\frac{1}{2}\|\chi_j D(u_{j+1})\|^2 - C'\|u_{j+1}\|^2,$$

for some $C' > 0$, and we can conclude that

$$\frac{1}{2}\|\chi_j D(u_{j+1})\|^2 \leq C'\|u_{j+1}\|^2 + \langle \chi_j^2 u_{j+1}, D^*D(u_{j+1})\rangle$$
$$\leq C'\|u\|^2 + \langle u, D^*Du\rangle.$$

It follows that $Du \in L^2(M,F)$ and $D^*(Du) \in L^2(M,E)$, i.e.,

$$\mathcal{D}_{\max}(D^*D) = \left\{ u \in L^2(M,E) : u \in \mathcal{D}_{\max}(D) \text{ and } Du \in \mathcal{D}_{\max}(D^*) \right\}.$$

Now consider $u \in \mathcal{D}_{\min}(D^*D)$. Choose $u_j \in \mathcal{C}_c^\infty(M,E)$ such that both u_j and D^*Du_j converge in $L^2(M,E)$. Notice that

$$\|Du_j - Du_k\|^2 = \langle D^*D(u_j - u_k), u_j - u_k\rangle \to 0,$$

i.e., Du_j converges in $L^2(M,F)$ and hence $u \in \mathcal{D}_{\min}(D)$. Convergence of $D^*(Du_j)$ shows that $Du \in \mathcal{D}_{\min}(D^*)$ and hence

$$\mathcal{D}_{\min}(D^*D) = \left\{ u \in L^2(M,E) : u \in \mathcal{D}_{\min}(D) \text{ and } Du \in \mathcal{D}_{\min}(D^*) \right\}.$$

By part (a), $\mathcal{D}_{\min}(D^*D) = \mathcal{D}_{\max}(D^*D)$.

□

If D is an operator covered by Gaffney's theorem, and $u, v \in \mathcal{D}(D^*D)$, then

$$\langle D^*Du, v\rangle = \langle Du, Dv\rangle.$$

In particular, for the Hodge Laplacian on k-forms on a complete manifold, this implies that

$$\langle \Delta\omega, \eta\rangle = \langle (d+\delta)\omega, (d+\delta)\omega\rangle = \langle d\omega, d\eta\rangle + \langle \delta\omega, \delta\eta\rangle.$$

In particular harmonic forms coincide with forms that are closed and co-closed, i.e.,

$$\operatorname{null}(\Delta) = \operatorname{null}(d) \cap \operatorname{null}(\delta).$$

2.3. **The spectrum of an elliptic operator.**

Different closed extensions of the same operator can have different spectral sets. Let $P \in \Psi^s(M,E)$ be a formally self-adjoint pseudo-differential operator with $s > 0$ on a closed manifold M. Since P is essentially self-adjoint, we can unambiguously define its spectrum as the spectrum of its closure, $\overline{P}$, as an unbounded operator on $L^2(M,E)$. Recall that this means that $\lambda \in \mathbb{C}$ is *not* in the spectrum if

$$\lambda \in \mathbb{C} \setminus \operatorname{Spec}(P) \iff (\overline{P} - \lambda)^{-1} \text{ exists as a bounded operator in } L^2(M,E).$$

We can divide the spectrum into three disjoint sets according to how $\overline{P} - \lambda$ fails to be invertible,

$$\lambda \in \operatorname{Spec}_p(P) \iff \operatorname{null}_{L^2}(\overline{P} - \lambda) \neq \emptyset,$$

and, if $\lambda \notin \operatorname{Spec}_p(P)$, then

$$\lambda \in \operatorname{Spec}_c(P) \iff \operatorname{Ran}(\overline{P} - \lambda) \text{ open and dense in } L^2(M, E)$$
$$\lambda \in \operatorname{Spec}_r(P) \iff \operatorname{Ran}(\overline{P} - \lambda) \text{ is not dense in } L^2(M, E).$$

It is easy to see that $\operatorname{Spec}(P)$ is a closed subset of $\mathbb{C}$ and that the resolvent of P, $(P - \lambda)^{-1}$, is holomorphic as a function from $\mathbb{C} \setminus \operatorname{Spec}(P)$ to the set of bounded operators on $L^2(M, E)$. Also notice that self-adjoint operators do not have residual spectrum (since $\operatorname{Ran}(P - \lambda)^{\perp} = \operatorname{null}(P - \overline{\lambda})$).

If λ is in the spectrum of P then $P - \lambda$ is not invertible, but it might still be Fredholm (in fact it will be). This motivates a second partition of the spectrum of P into the essential and disrete spectra:

$$\lambda \in \operatorname{Spec}_{\text{ess}}(P) \iff \overline{P} - \lambda \text{ is not Fredholm}$$
$$\operatorname{Spec}_d(P) = \operatorname{Spec}(P) \setminus \operatorname{Spec}_{\text{ess}}(P).$$

Lemma 2.14. *(Weyl's criterion) A complex number λ is in the essential spectrum of a self-adjoint operator L if and only if there is a sequence of unit length vectors u_i converging weakly to zero such that $(L-\lambda)u_i$ converges strongly to zero. Hence*

$$\operatorname{Spec}_{\text{ess}}(L) = \operatorname{Spec}_c(L) \bigcup \{\lambda \in \operatorname{Spec}_p(L) : \dim \operatorname{null}_{L^2}(L - \lambda) = \infty\}$$
$$\operatorname{Spec}_d(L) = \{\lambda \in \operatorname{Spec}_p(L) : \lambda \textit{ is isolated, and } \dim \operatorname{null}_{L^2}(L - \lambda) < \infty\}.$$

Proof. Assume λ is in the essential spectrum. Since L is self-adjoint this means that either L does not have closed range or it has infinite dimensional null space. In the latter case we can take (u_i) to be an orthonormal basis of the null space. If the null space of $L - \lambda$ is finite dimensional it has a closed complement $V = (\operatorname{null}(L - \lambda))^{\perp}$ on which $L - \lambda$ has an unbounded inverse. This allows us to find a sequence of unit-length vectors (v_i) in V such that $(L - \lambda)v_i \to 0$. Furthermore, since $L - \lambda$ is injective on V, v_i can not have a convergent subsequence and the span of $\{v_i\}$ must be infinite dimensional so we can extract a subsequence that converges weakly to zero.

Conversely, assume that (u_i) is a sequence of unit vectors that converges weakly to zero such that $(L - \lambda)u_i \to 0$. We need to show that if the null space of $L - \lambda$ is finite dimensional then the range of $L - \lambda$ is not closed. With this assumption, the orthogonal projection $\mathcal{P}$ onto $\operatorname{null}(L - \lambda)$ is a compact operator and hence

$$\|(\operatorname{Id} - \mathcal{P})u_i\| \to 1.$$

After dropping any elements of (u_i) that may be in $\operatorname{null}(L - \lambda)$ we can replace each u_i by its orthogonal projection off of $\mathcal{U}$, normalized to have norm one, say $\overline{u}_i$. It is easy to see that $(\overline{u}_i)$ converges weakly to zero and

$(L-\lambda)\overline{u}_i$ converges strongly to zero - hence the inverse of $L-\lambda$ restricted to $\mathrm{null}(L-\lambda)^{\perp}$ is unbounded. By the closed graph theorem this implies that the range of $L-\lambda$ is not closed and thus $L-\lambda$ is not Fredholm. □

Since on a closed manifold any elliptic operator of positive order is Fredholm, this implies the following theorem.

Theorem 2.15. *If $P \in \Psi^s(M,E)$ with $s>0$ is elliptic and self-adjoint and M is a closed manifold then* $\mathrm{Spec}(P)$ *consists of a countable number of real eigenvalues with no finite accumulation point. The eigenspaces are finite dimensional and the eigensections are smooth.*

For the Laplacian we can also say something about how quickly the eigenvalues are growing. The key observation is that the any Riemannian metric is locally close to the Euclidean metric and so any geometric operator, such as the Laplacian, should be 'locally close' to the corresponding operator on Euclidean space (or the flat torus). For this particular purpose, it is useful to involve an integral transform of the Laplacian such as the heat or wave kernel. The heat kernel of an operator P is the fundamental solution to the initial value problem

$$\begin{cases} (\partial_t + P)u(t,\zeta) = 0 \\ u(0,\zeta) = f(\zeta) \end{cases}$$

its existence - as a bounded operator on $L^2(M,E)$ - is guaranteed for any P elliptic and self-adjoint by the functional calculus, and can be expressed by

$$e^{-tP} = \int_\Gamma e^{-t\mu}(P-\mu)^{-1}\,d\mu = \sum_{\lambda\in\mathrm{Spec}(P)} e^{-t\lambda_i}\Pi_i.$$

Choose an orthonormal basis of eigenfunctions of P, ϕ_i, and note that

$$\begin{aligned}(e^{-tP}u)(\zeta) &= \sum e^{-t\lambda_i}\langle u,\phi_i\rangle\phi_i = \sum e^{-t\lambda_i}\left[\int u(\zeta')\phi_i(\zeta')\,d\zeta'\right]\phi_i(\zeta) \\ &= \int\left[\sum e^{-t\lambda_i}\phi_i(\zeta)\phi_i(\zeta')\right]u(\zeta')\,d\zeta' = \int \mathcal{K}_{e^{-tP}}(t,\zeta,\zeta')u(\zeta')\,d\zeta'.\end{aligned}$$

For comparison, recall the the heat equation on Euclidean space is solved by

$$f \mapsto u(t,\zeta) = \frac{1}{(4\pi t)^{m/2}}\int \exp\left(\frac{|\zeta-\zeta'|^2}{4t}\right) f(\zeta')\,d\zeta'$$

and also that, in normal coordinates, the metric has the form

$$g_{ij} = \delta_{ij} + \mathcal{O}(\|\zeta\|^2).$$

This suggests looking for an expression for $\mathcal{K}_{e^{-t\Delta}}(t,\zeta,\zeta')$ of the form

$$\frac{1}{(4\pi t)^{m/2}}\exp\left(\frac{|\zeta-\zeta'|^2}{4t}\right)\left[\sum_{i=0}^{\infty}\mathcal{K}_i(\zeta,\zeta')t^i\right]. \tag{2.5}$$

It is possible to find a (unique) formal solution of this form and show that it is asymptotic to $\mathcal{K}_{e^{-t\Delta}}$ as $t \to 0$. The leading coefficient $\mathcal{K}_0(\zeta, \zeta')$ is given by parallel translation from ζ to ζ', so in particular, $\mathcal{K}_0(\zeta, \zeta) = 1$. This is all we need to prove Weyl's theorem.

Theorem 2.16 (Weyl)**.** *Let λ_i denote the eigenvalues of the Laplacian, counted with multiplicity, then as $t \to 0$,*

$$\sum e^{-t\lambda_i} = (4\pi t)^{-m/2} \operatorname{Vol}(M) + \mathcal{O}(t^{-m/2+1}),$$

and, as $\ell \to \infty$,

$$N(\ell) = \#\{\lambda_i < \ell\} \sim \frac{\operatorname{Vol}(M)}{(4\pi)^{m/2}\Gamma(m/2+1)} \ell^{m/2}. \tag{2.6}$$

Proof. The first statement follows from

$$\sum e^{-t\lambda_i} = \sum e^{-t\lambda_i} \|\phi_i\|^2 = \int \sum e^{-t\lambda_i} \phi_i(\zeta)\phi_i(\zeta)\, d\zeta$$
$$= \int \mathcal{K}_{e^{-t\Delta}}(t, \zeta, \zeta)\, d\zeta \sim \frac{1}{(4\pi t)^{m/2}} \operatorname{Vol}(M).$$

The second statement follows from the first by means of a Tauberian theorem.

Lemma 2.17 (Karamata)**.** *If μ is a positive measure on $[0, \infty)$, $\alpha \in [0, \infty)$, then*

$$\int_0^\infty e^{-t\lambda}\, d\mu(\lambda) \sim a t^{-\alpha} \text{ as } t \to 0$$

implies

$$\int_0^\ell d\mu(\lambda) \sim \left[\frac{a}{\Gamma(\alpha+1)}\right] \ell^\alpha \text{ as } \ell \to \infty.$$

Proof. We start by pointing out that

$$t^\alpha \int e^{-t\lambda}\, d\mu(\lambda) = \int e^{-\lambda}\, d\mu_t(\lambda)$$

with $\mu_t(\lambda)(A) = t^\alpha \mu(t^{-1}A)$, so we can rewrite the hypothesis as

$$\lim_{t\to 0} \int e^{-\lambda}\, d\mu_t(\lambda) = a = \left[\frac{a}{\Gamma(\alpha)}\right] \int_0^\infty \lambda^{\alpha-1} e^{-\lambda}\, d\lambda$$

and the conclusion as

$$\lim_{t\to 0} \int \chi(\lambda)\, d\mu_t(\lambda) = \left[\frac{a}{\Gamma(\alpha)}\right] \int_0^\infty \lambda^{\alpha-1} \chi(\lambda)\, d\lambda$$

with $\chi(\lambda)$ equal to the indicator function for $[0, \ell]$. But this last equality is true for $\chi(\lambda) = e^{s\lambda}$ for any $s \in \mathbb{R}^+$, hence by density for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and hence for Borel measurable functions, in particular for the indicator function of $[0, \ell]$. ∎

We get (2.6) by applying the lemma with $d\mu(\lambda) = \sum \delta(\lambda - \lambda_i)$.

∎

2.4. **The Hodge theorem and Kodaira decomposition.**
Now consider the complex of forms on M,

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m M,$$

or more generally any sequence of vector bundles $E_i \to M$ and pseudo-differential operators $P_i \in \Psi^k(M; E_i, E_{i+1})$ such that

$$\mathcal{C}^\infty(M, E_1) \xrightarrow{P_1} \mathcal{C}^\infty(M, E_2) \xrightarrow{P_2} \ldots \xrightarrow{P_{n-1}} \mathcal{C}^\infty(M, E_n) \tag{2.7}$$

forms a complex, i.e., $P_{i+1}P_i = 0$. We say that such a complex is elliptic if, for any $(\zeta, \xi) \in S^*M$, the corresponding complex of symbols

$$0 \to (E_1)_\zeta \xrightarrow{\sigma(P_1)(\zeta,\xi)} (E_2)_\zeta \xrightarrow{\sigma(P_2)(\zeta,\xi)} \ldots \xrightarrow{\sigma(P_{n-1})(\zeta,\xi)} (E_n)_\zeta \to 0$$

is exact.

For the de Rham complex, note that $e^{-ift} d(e^{ift}\omega) = itdf \wedge \omega + d\omega$ so the symbol of d is given by exterior product,

$$\sigma(d)(\zeta, \xi) = i\mathfrak{e}_\xi.$$

The symbolic complex

$$0 \to \Lambda^0(T^*_\zeta M) \xrightarrow{i\mathfrak{e}_\xi} \Lambda^1(T^*_\zeta M) \xrightarrow{i\mathfrak{e}_\xi} \ldots \xrightarrow{i\mathfrak{e}_\xi} \Lambda^n(T^*_\zeta M) \to 0$$

is exact, hence the de Rham complex is elliptic.

Given an elliptic complex, we choose inner products on all of the bundles involved and an smooth measure on M in order to define the adjoint complex,

$$\mathcal{C}^\infty(M, E_1) \xleftarrow{P_1^*} \mathcal{C}^\infty(M, E_2) \xleftarrow{P_2^*} \ldots \xleftarrow{P_{n-1}^*} \mathcal{C}^\infty(M, E_n) \tag{2.8}$$

which is again elliptic. The advantage is that the operators

$$\Delta_i = P_i^* P_i + P_{i-1} P_{i-1}^*$$

are elliptic, self-adjoint, and map $\mathcal{C}^\infty(M, E_i)$ to itself.

Indeed, we can check that they are elliptic by noting that, for $\omega \in (E_i)_\zeta$,

$$(\sigma(\Delta_i)\omega, \omega) = (\sigma(P_i)\omega, \sigma(P_i)\omega) + (\sigma(P_{i-1}^*)\omega, \sigma(P_{i-1}^*)\omega)$$

hence

$$\omega \in \mathrm{null}[\sigma(\Delta_i)(\zeta, \xi)] \iff \omega \in \mathrm{null}[\sigma(P_i)(\zeta, \xi)] \cap \mathrm{null}[\sigma(P_{i-1}^*)(\zeta, \xi)].$$

But, if $\xi \neq 0$, then ellipticity of the original complex means that

$$\mathrm{null}[\sigma(P_i)(\zeta, \xi)] = \mathrm{Ran}[\sigma(P_{i-1})(\zeta, \xi)] \subseteq \left(\mathrm{null}[\sigma(P_{i-1}^*)(\zeta, \xi)]\right)^\perp,$$

so $\omega = 0$.

The same reasoning applied to Δ_i instead of $\sigma(\Delta_i)$, and integration by parts, shows that

$$\mathrm{null}(\Delta_i) = \mathrm{null}(P_i) \cap \mathrm{null}(P_{i-1}^*).$$

In the next theorem we compare this space to the cohomology of the complex (2.7), (thought of as vector spaces)

$$H^k(\{E_i, P_i\}) = \frac{\ker P_i}{\operatorname{Image} P_{i-1}} = \frac{\{\omega \in \mathcal{C}^\infty(M, E_i) : P_i\omega = 0\}}{\{P_{i-1}\eta \in \mathcal{C}^\infty(M, E_i) : \eta \in \mathcal{C}^\infty(M, E_{i-1})\}}.$$

Before proving the theorem, we note that the same data allows us to define complexes from sections with less regularity. On one extreme, we can define the complex of distributional sections (using P_i^* above to define the action of P_i on $\mathcal{C}^{-\infty}(M, E_i)$)

$$\mathcal{C}^{-\infty}(M, E_1) \xrightarrow{P_1} \mathcal{C}^{-\infty}(M, E_2) \xrightarrow{P_2} \dots \xrightarrow{P_{n-1}} \mathcal{C}^{-\infty}(M, E_n). \tag{2.9}$$

In between these extremes, we have the complex of L^2 forms, of which we will have much to say for general manifolds. In this case, though we write

$$L^2(M, E_1) \xrightarrow{P_1} L^2(M, E_2) \xrightarrow{P_2} \dots \xrightarrow{P_{n-1}} L^2(M, E_n), \tag{2.10}$$

this is an abuse of notation, and at each step we should really have $\mathcal{D}_{\max}(P_i)$ so as to have an actual complex. As is well-known (and shown in the next theorem), these complexes have the same cohomology as (2.7).

Theorem 2.18. *If the complex* (2.7) *is elliptic, then every element of* $\mathcal{C}^\infty(M, E_i)$ *(or* $\mathcal{C}^{-\infty}(M, E_i)$ *or* $L^2(M, E_i)$*) can be written in a unique way as the sum of an element in* $\operatorname{Ran}(P_{i-1})$*, an element in* $\operatorname{Ran}(P_i^*)$*, and an element in the null space of* Δ_i*. Furthermore, there is a canonical vector space isomorphism*

$$\mathcal{H}^k(\{E_i, P_i\}) \cong \operatorname{null}(\Delta_k). \tag{2.11}$$

Proof. Let $G_i \in \Psi^{-2k}(M, E_i)$ be the generalized inverse of Δ_i as an operator on $L^2(M, E_i)$, so that

$$\Delta_i G_i = G_i \Delta_i = \operatorname{Id} - \Pi_i$$

with $\Pi_i \in \Psi^{-\infty}(M, E_i)$ the orthogonal projection onto the null space of Δ_i. Then, in any of the three complexes above, existence of the decomposition is a consequence of

$$\operatorname{Id} = P_i^*(P_i G_i) + P_{i-1}(P_{i-1}^* G_i) + \Pi_i.$$

To prove uniqueness of the decomposition, it suffices to consider the case of distributions. Let $T \in \mathcal{C}^{-\infty}(M, E_i)$ be given by $T_1 + T_2 + T_3$ with $T_1 = P_i^* T_i'$, $T_2 = P_{i-1} T_2'$, and $T_3 \in \operatorname{null}(\Delta_i)$. Let ϕ be any test function and note that

$$\begin{aligned}\langle \phi, T_1 \rangle &= \langle \left(P_i^* P_i G_i + P_{i-1} P_{i-1}^* G_i + \Pi_i \right) \phi, T_1 \rangle \\ &= \langle P_i^* P_i G_i \phi, T_1 \rangle = \langle P_i^* P_i G_i \phi, T \rangle = \langle \phi, G_i P_i^* P_i T \rangle = \langle \phi, P_i^* P_i G_i T \rangle,\end{aligned}$$

and since ϕ was arbitrary, we conclude $T_1 = P_i^* P_i G_i T$. Similarly we identify T_2 with $P_{i-1} P_{i-1}^* G_i T$ and T_3 with $\Pi_i T$, thus proving uniqueness.

Next, since elliptic regularity implies $\mathrm{null}(\Delta_k) \subseteq \mathcal{C}^\infty(M, E_k)$, it suffices to prove (2.11) for smooth sections. Using $\mathrm{null}(\Delta_i) \subseteq \mathrm{null}(P_i)$, we have a well-defined vector space homomorphism

$$\Phi : \mathrm{null}(\Delta_i) \to \mathcal{H}^k(\{E_i, P_i\})$$

defined by assigning to each $u \in \mathrm{null}(\Delta_i)$ its class in the quotient, $[u]$. We claim that Φ is bijective. Indeed, if $\Phi(u) = 0$ then $u \in \mathrm{null}(\Delta_i) \cap \mathrm{Ran}(P_{i-1})$ so, as explained above, $u = 0$. Similarly given any $v \in \mathrm{null}(P_i)$, then

$$v = \Pi v + G\Delta_i v = \Pi v + P_{i-1}(P_{i-1}^* G v)$$

shows that $[v] = \Phi(\Pi v)$, and hence Φ is surjective. □

In particular, we can apply this to the de Rham complex of forms.

Corollary 2.19 (The compact Hodge Theorem)**.** *The Hodge cohomology of a closed manifold M is canonically isomorphic to the de Rham cohomology of M.*

Hence by the de Rham theorem, the null spaces of the Hodge Laplacian can be identified with the topological cohomology spaces of M.

3. L^2 cohomology on non-compact manifolds

On any smooth manifold, compact or not, the de Rham cohomology groups

$$H^k_{dR}(M) = \frac{\{\omega \in \mathcal{C}^\infty(\Lambda^k T^*M) : d\omega = 0\}}{d\mathcal{C}^\infty(\Lambda^{k-1}T^*M)}$$

are isomorphic to the real cohomology groups $H^k(M, \mathbb{R})$, and the compactly supported de Rham cohomology groups

$$H^k_c(M) = \frac{\{\omega \in \mathcal{C}^\infty_c(\Lambda^k T^*M) : d\omega = 0\}}{d\mathcal{C}^\infty_c(\Lambda^{k-1}T^*M)}$$

are isomorphic to the real cohomology groups with compact support, $H^k_c(M, \mathbb{R})$. If M is the interior of a manifold with boundary, $\overline{M}$, then

$$H^k_c(M) \cong H^k(\overline{M}, \partial\overline{M}) = \frac{\{\omega \in \mathcal{C}^\infty(\Lambda^k T^*\overline{M}) : d\omega = 0, i^*\omega = 0\}}{\{d\beta : \beta \in \mathcal{C}^\infty(\Lambda^{k-1}T^*\overline{M}), i^*\beta = 0\}}$$

where $i : \partial\overline{M} \to \overline{M}$ is the inclusion map.

If M is oriented, the bilinear map

$$H^k(M) \times H^{m-k}_c(M) \ni ([\alpha], [\beta]) \mapsto I([\alpha], [\beta]) = \int_M \alpha \wedge \beta \in \mathbb{R}$$

is well-defined and non-degenerate. Thus, if α is a closed form,

$$I([\alpha], [\beta]) = 0 \text{ for every } \beta \iff \alpha = d\gamma \text{ for some smooth form } \gamma.$$

In particular if both spaces are finite dimensional then we have Poincaré Duality

$$H^k(M) \cong (H^{n-k}_c(M))^*, \text{ and } (H^k(M))^* \cong H^{n-k}_c(M).$$

(If they are not finite dimensional, the first equality still holds though the second need not.)

Given a Riemannian metric g on M forms in $L^2\Omega^k M$ form a Hilbert space with inner product

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \, \mathrm{dvol}_g \, .$$

We define the formal adjoint of d,

$$\mathcal{C}^\infty_c \Omega^{k+1} M \xrightarrow{\delta} \mathcal{C}^\infty_c \Omega^k M,$$

by the formula

$$\langle \delta\alpha, \beta \rangle = \langle \alpha, d\beta \rangle, \quad \text{for every } \alpha \in \mathcal{C}^\infty_c \Omega^{k+1} M, \beta \in \mathcal{C}^\infty_c \Omega^k M.$$

3.1. **Kodaira decomposition.**
If $\alpha \in \mathcal{C}^{-\infty}\Omega^k M$ and $\gamma \in \mathcal{C}^{-\infty}\Omega^{k+1} M$, we say that $d\alpha = \gamma$ weakly if

$$\langle d\alpha, \beta \rangle = \langle \gamma, \delta\beta \rangle, \quad \text{for every } \beta \in \mathcal{C}^\infty_c \Omega^{k+1} M.$$

The space of closed L^2 forms is defined by

$$Z^k_2(M) = \left\{ \omega \in L^2\Omega^k M : d\omega = 0 \text{ weakly} \right\}.$$

Thus we can identify

$$Z^k_2(M) = \left(\delta \mathcal{C}^\infty_c \Omega^{k+1} M \right)^\perp$$

which shows that $Z^k_2(M)$ is a closed subspace of $L^2\Omega^k M$. Similarly the space of co-closed forms is equal to

$$\left\{ \omega \in L^2\Omega^k M : \delta\omega \text{ weakly} \right\} = \left(d\mathcal{C}^\infty_c \Omega^{k-1} M \right)^\perp,$$

and the space of closed an co-closed forms (often called the space of L^2 harmonic fields) is

$$\begin{aligned} L^2\mathcal{H}^k(M,g) &= \left\{ \alpha \in L^2\Omega^k(M) : d\alpha = 0, \delta\alpha = 0 \right\} \\ &= \left(\delta\mathcal{C}^\infty_c \Omega^{k+1} M \right)^\perp \cap \left(d\mathcal{C}^\infty_c \Omega^{k-1} M \right)^\perp. \end{aligned}$$

Note that, by applying elliptic regularity to $d + \delta$,

$$L^2\mathcal{H}^k(M,g) \subseteq \mathcal{C}^\infty\mathcal{H}^k(M,g).$$

In general $L^2\mathcal{H}^k(M,g)$ is not equal to $\mathrm{null}(\Delta_i)$.

The following theorem follows easily from the fact that, whenever $\alpha, \beta \in \mathcal{C}^\infty_c \Omega^* M$,

$$\langle d\alpha, \delta\beta \rangle = \langle d^2\alpha, \beta \rangle = 0,$$

and hence

$$\delta\mathcal{C}^\infty_c \Omega^{k+1} M \perp d\mathcal{C}^\infty_c \Omega^{k-1} M.$$

Theorem 3.1 (Kodaira)**.** *On an arbitrary manifold M (not necessarily complete), there is an orthogonal decomposition*

$$L^2\Omega^k(M,g) = L^2\mathcal{H}^k(M,g) \oplus \overline{d\mathcal{C}^\infty_c \Omega^{k-1}} \oplus \overline{\delta\mathcal{C}^\infty_c \Omega^{k+1}}.$$

3.2. **Reduced L^2 cohomology.**
Recall that the exterior derivative has two canonical closed extensions from its core domain $\mathcal{C}^\infty_c \Omega^k$. Let $\mathcal{D}^k(d)$ denote the maximal domain of d (Gaffney's theorem shows that $\mathcal{D}(d)$ is also the minimal domain of d if g is complete). The range of the exterior derivative, as a map $\mathcal{D}^k(d) \to L^2\Omega^{k+1}$, need not be closed. We define two L^2 cohomology spaces: the $k^{\mathbf{th}}$ **reduced** L^2 **cohomology space of** M is

$$\overline{H}^k_2(M) = Z^k_2(M) / \overline{d\mathcal{D}^{k-1}(d)}$$

and is automatically a Hilbert space; the k^{th} **un-reduced L^2 cohomology space of M** is

$$H_2^k(M) = Z_2^k(M)/d\mathcal{D}^{k-1}(d)$$

and need not even be Hausdorff. This latter space is infinite dimensional if the range of d is not closed, and coincides with $\overline{H}_2^k(M)$ if the range of d is closed (e.g., on a closed manifold).

Proposition 3.2. *If (M, g) is a complete manifold*

$$\overline{H}_2^k(M) \cong L^2\mathcal{H}^k(M) \cong \mathrm{null}(\Delta_k), \tag{3.1}$$

and, if M is also orientable, then we have Poincaré duality

$$\overline{H}_2^k(M) \cong \overline{H}_2^{m-k}(M).$$

Proof. On a complete manifold we know that $\overline{d\mathcal{D}^{k-1}(d)} = \overline{d\mathcal{C}_c^\infty\Omega^{k-1}(d)}$, and the Kodaira decomposition theorem implies that $Z_2^k(M) = L^2\mathcal{H}^k(M) \oplus \overline{d\mathcal{D}^{k-1}(d)}$, which together imply $\overline{H}_2^k(M) \cong L^2\mathcal{H}^k(M)$. We also know from Gaffney's theorem that $L^2\mathcal{H}^k(M) = \mathrm{null}(\Delta_k)$. Finally, orientability of M gives an isomorphism

$$* : L^2\Omega^k \to L^2\Omega^{m-k}$$

which intertwines Δ_k and Δ_{n-k}, in particular it establishes $\mathrm{null}(\Delta_k) \cong \mathrm{null}(\Delta_{m-k})$ and hence $\overline{H}_2^k(M) = \overline{H}_2^{m-k}(M)$. □

An interesting consequence of this proposition is that the Hodge cohomology $L^2\mathcal{H}^k(M)$ is an invariant under quasi-isometric changes of a (complete) metric. We say that two metrics g and g' are quasi-isometric if there is a positive constant C, such that

$$Cg \leq g' \leq \frac{1}{C}g$$

as quadratic forms. Quasi-isometric metrics define the same L^2 spaces and hence define isomorphic L^2 cohomology groups, $\overline{H}_2^k(M)$, so, if the metrics are complete, isomorphic $L^2\mathcal{H}^k(M)$. This is surprising because quasi-isometry involves no derivatives of the metric while defining $d+\delta$ requires derivatives of the metric.

Corollary 3.3.
a) On any complete Riemannian manifold,

$$\dim H_2^0(M) = \#\ \textit{of connected components of } M \textit{ with finite volume.}$$

b) For $\mathbb{R}^n$ with its flat metric, $\overline{H}_2^k(\mathbb{R}^n) = \{0\}$ for every k.

Proof.
a) For functions on a complete manifold, $\mathrm{null}(\Delta) = \mathrm{null}(d)$, hence harmonic functions are locally constant.
b) On $\mathbb{R}^n$, any form can be written

$$\alpha = \sum \alpha_I dx^I,$$

and will be harmonic if and only if its coefficient functions are all harmonic, hence locally constant. The norm of a form on Euclidean space is equivalent to the sum of the L^2-norms of the coefficient functions, hence the fact that $\mathbb{R}^n$ has infinite volume implies that there are no harmonic L^2 forms on $\mathbb{R}^n$. □

Remark 3.4. If M is not complete, the behavior of these vector spaces is very different. For instance, for functions on the unit interval we have $\mathrm{null}(d) = \{\text{constants}\}$ and $\mathrm{null}\,\Delta = \{\text{affine functions}\}$, so $\mathrm{null}(\Delta) \neq \mathrm{null}(d) \cap \mathrm{null}(\delta)$. For a Riemannian manifold with boundary we always have, on functions, $\mathrm{null}(d) = \{\text{constants}\}$, but $\mathrm{null}(\Delta)$ is infinite dimensional whenever $m > 1$.

For $0 < k < \dim M$, the space of closed and co-closed k forms on a manifold with boundary is always infinite dimensional. Indeed, a simple construction is possible using the following unique continuation principle:

> Let N is a connected Riemannian manifold, $\mathcal{U} \subseteq N$ a non-empty open subset of N, and α, β smooth closed and co-closed forms that coincide on $\mathcal{U}$, then $\alpha = \beta$.

Proposition 3.5. *Assume that (M, g) is a smooth, compact, Riemannian manifold with boundary, and $0 < k < m$, then*

$$\dim L^2\mathcal{H}^k(M) = \infty.$$

Proof. Let $2M = M \#_{\partial M} M$ be the double of M and for any $\ell \in \mathbb{N}$ let

$$N_\ell = 2M \# \left(\#_\ell \mathbb{T}^n \right).$$

Choose any extension of g to a smooth Riemannian metric on N_ℓ, and note that, by the unique continuation principle, the restriction map

$$L^2\mathcal{H}^k(N_\ell) \to L^2\mathcal{H}^k(M)$$

is injective. On the other hand, for k as above, we have

$$L^2\mathcal{H}^k(N_\ell) \cong H^k(N_\ell) \cong H^k(2M) \oplus \Big(\oplus_\ell H^k(\mathbb{T}^m) \Big)$$

and hence

$$\dim L^2\mathcal{H}^k(M) \geq \dim H^k(2M) + \ell \binom{m}{k}.$$

Since ℓ is arbitrary, the result follows. □

Next we want to show that, on a complete manifold, the L^2 cohomology is between $H^*_c(M)$ and $H^*(M)$, at least in the sense that the natural map between $H^k_c(M)$ and $H^k(M)$ factors through the reduced L^2 cohomology,

$$\begin{array}{ccc} H^k_c(M) & \longrightarrow & H^k(M) \\ & \searrow \quad \nearrow & \\ & \overline{H}^k_2(M) & \end{array} \qquad (3.2)$$

The first map $H^k_c(M) \to \overline{H}^k_2(M)$ is easily seen to be well-defined, however for the second map $\overline{H}^k_2(M) \to H^k(M)$, we need the following result of de Rham.

Lemma 3.6 (de Rham)**.** *Let α be a smooth form in Z^k_2 and suppose that $[\alpha]$ is zero as an element of reduced L^2 cohomology. Then there is a smooth $k-1$ form β such that $\alpha = d\beta$.*

In general we can not say anything about the behavior of β at infinity. Notice that the hypothesis does not say that we can choose β in L^2, but rather that there is a sequence of smooth compactly supported $k-1$ forms, β_ℓ, such that $d\beta_\ell$ converges to α in $L^2(M)$.

Proof. By Poincaré duality it suffices to show that if $\phi \in \mathcal{C}^\infty_c\Omega^{n-k}$ is closed then

$$\int \alpha \wedge \phi = 0,$$

and this follows from

$$\int \alpha \wedge \phi = \lim \int d\beta_\ell \wedge \phi = \lim \int d(\beta_\ell \wedge \phi) = 0.$$

□

Lemma 3.6 shows that the second map in (3.2) is well-defined. In general neither map need be injective or surjective, though we do have the following result.

Proposition 3.7 (Anderson)**.** *There is a natural injective map*

$$\operatorname{Im}\left(H^k_c(M) \to H^k(M)\right) \to \overline{H}^k_2(M).$$

Proof. Using (3.2) we have

$$\operatorname{Im}\left(H^k_c(M) \to H^k(M)\right) \hookrightarrow \operatorname{Im}\left(\overline{H}^k_2(M) \to H^k(M)\right) \hookrightarrow \overline{H}^k_2(M),$$

where we use the Hilbert space structure of $\overline{H}^k_2(M)$ to define the final map.
□

This sometimes gives an easy lower bound for the reduced L^2 cohomology.

Corollary 3.8. *If S is a complete Riemann surface,*

$$\operatorname{genus}(S) \leq \dim \overline{H}^1_2(M).$$

Proof. We show that if a connected Riemann surface S is obtained from another connected Riemann surface S' by attaching a handle, then

$$\dim \operatorname{Im}\left(H^k_c(S') \to H^k(S')\right) + 1 \leq \dim \operatorname{Im}\left(H^k_c(S) \to H^k(S)\right).$$

We represent the handle by an emebedding f of $\mathbb{S}^1 \times [-1,1]$ into S such that $S \setminus \mathrm{Im}(f)$ is connected. Let $\chi : [-1,1] \to [0,1]$ be any smooth function such that

$$\chi(t) = \begin{cases} 1 & \text{if } t > \frac{1}{2} \\ 0 & \text{if } t < -\frac{1}{2} \end{cases}$$

and let $\pi : \mathbb{S}^1 \times [-1,1] \to [-1,1]$ be the projection onto the second factor. The form $\alpha = (f^{-1})^* \pi^* d\chi$ is a smooth compactly supported closed form on S with support in $\mathrm{Im}(f)$. Choose a path $\gamma : [0,1] \to S \setminus \mathrm{Im}(f)$ connecting $f(1,1)$ with $f(1,-1)$ and define

$$\widetilde{\gamma}(t) = \begin{cases} f(1,t) & \text{if } t \in [-1,1] \\ \gamma(t-1) & \text{if } t \in [1,2] \end{cases}$$

Then we have

$$\int_{\widetilde{\gamma}} \alpha = \int_{-1}^{1} d\chi = 1,$$

which shows that α is not zero as an element of $H^1(S)$. □

For a Riemann surface of finite topology, we can say more about the L^2 cohomology. The key fact about the degree one reduced L^2 cohomology of a surface is its conformal invariance. Indeed, more generally we have conformal invariance of the L^2 spaces of middle degree forms.

Proposition 3.9 (Conformal invariance of middle-degree cohomology)**.** *Let M be an even dimensional manifold and let g and g' be conformally related metrics on M (not necessarily complete), then, if $m = \dim M$, we have*

$$L^2\mathcal{H}^{m/2}(M,g) = L^2\mathcal{H}^{m/2}(M,g').$$

Proof. Let ω be a smooth function such that $g' = e^{2\omega} g$, a simple computation shows that

$$*_k(g) = e^{(m-2k)\omega} *_k (g'),$$

so the Hilbert space inner products on middle degree forms, and hence the Hilbert spaces $L^2(M,g)$ and $L^2(M,g')$, coincide. A form in $L^2\mathcal{H}^{m/2}(M,g)$ is smooth, closed, and orthogonal to the image of $d\mathcal{C}_c^\infty\Omega^{k-1}$ hence an element of $L^2\mathcal{H}^{m.2}(M,g')$ and vice-versa. □

A surface S has finite topology if it has finite genus and a finite number of ends. The uniformization theorem for surfaces shows that there is a compact Riemann surface $\overline{S}$ so that a complete metric on S is conformally equivalent to the hyperbolic metric on $\overline{S}$ minus a finite number of points and disks. Each finite volume end (known as a cusp) corresponds to one of the 'missing' points, and each end of infinite volume (known as a funnel) corresponds to one of the 'missing' disks. The following lemma will imply that the reduced L^2 cohomology does not notice the missing points.

Lemma 3.10. *Let $\mathbb{D}$ be the unit disk with its flat metric, and $\alpha \in L^2\mathcal{H}^1(\mathbb{D}\setminus\{0\})$, then α extends smoothly across the origin; hence we have*

$$L^2\mathcal{H}^1(\mathbb{D}\setminus\{0\}) = L^2\mathcal{H}^1(\mathbb{D}).$$

Proof. Any α as above is in the null space of $d+\delta$ in $\mathbb{D}\setminus\{0\}$, we show that it satisfies $(d+\delta)\alpha = 0$ weakly in $\mathbb{D}$. Note that we can choose cut-off functions

$$\chi_n(r,\theta) = \begin{cases} 0 & \text{if } r \leq \frac{1}{2n} \\ 2nr - 1 & \text{if } \frac{1}{2n} \leq r \leq \frac{1}{n} \\ 1 & \text{if } r \geq \frac{1}{n} \end{cases}$$

that satisfy $\|d\chi_n\|_{L^2}^2 \to 0$. Then, for any $\beta \in \mathcal{C}_c^\infty\Omega^*\mathbb{D}$, we have $\chi_n\beta \in \mathcal{C}_c^\infty\Omega^*(\mathbb{D}\setminus\{0\})$ hence

$$\langle \alpha, (d+\delta)(\chi_n\beta)\rangle = 0.$$

Hence we have

$$\begin{aligned}|\langle \alpha, (d+\delta)\beta\rangle| &= \lim_n |\langle \alpha, \chi_n(d+\delta)\beta\rangle| = \lim_n |\langle \alpha, [\chi_n, (d+\delta)]\beta\rangle| \\ &= \lim_n |\langle \alpha, \sigma(d+\delta)(d\chi_n)\beta\rangle| \leq \lim_n \|\alpha\|_{L^2}\|\beta\|_{L^\infty}\|d\chi_n\|_{L^2}^2 = 0.\end{aligned}$$

Elliptic regularity implies that $(d+\delta)\alpha = 0$ holds classically on $\mathbb{D}$. □

On the other hand, as we see in the next theorem, the presence of funnels will cause the reduced L^2 cohomology to blow-up.

Theorem 3.11. *Let (S,g) be a complete connected Riemann surface with finite topology. If $\dim L^2\mathcal{H}^1(S,g) < \infty$, then S is conformally equivalent to $\overline{S}\setminus\{p_1,\dots,p_n\}$, a compact Riemann surface with a finite number of points removed, and*

$$L^2\mathcal{H}^1 S \cong \mathrm{Im}(H_c^1(S) \to H^1(S)) \cong H^1(\overline{S}).$$

Proof. Let $\overline{S}$ be a compact Riemann surface so that

$$S \cong \overline{S}\setminus \mathbb{D}_1 \cup \dots \mathbb{D}_\ell \cup \{p_1,\dots,p_n\}.$$

We know from the lemma that an L^2 harmonic form on S extends to an L^2 harmonic form on $\overline{S}$ minus the disks. So we only need to show that if there are any disks removed then there are infinitely many harmonic forms. Notice that, given any $f \in \mathcal{C}^\infty(\partial\mathbb{D}_1)$, we can find a harmonic extension of f to a function u on $\overline{S}\setminus\mathbb{D}_1$. The form du is then in $L^2\mathcal{H}^1(\overline{S}\setminus\mathbb{D}_1)$ and restricts to an L^2 harmonic form on S. Thus we have a linear map $\mathcal{C}^\infty(\partial\mathbb{D}_1) \to L^2\mathcal{H}^1(S)$ whose null space consists of constant functions. Thus finite dimensional reduced L^2 cohomology implies that S is conformally equivalent to $\overline{S}\setminus\{p_1,\dots,p_n\}$. □

3.3. A long exact sequence in L^2 cohomology.

Because the L^2 cohomology of M is invariant under quasi-isometries we expect it to only depend on the asymptotic behavior of the ends of M. A natural approach to studying it is hence to use a long exact sequence in cohomology to try to separate out the contribution from the ends. In particular we will see a result of John Lott that whether or not a metric has infinite dimensional reduced L^2 cohomology only depends on its geometry at infinity.

Let M be a complete Riemannian manifold, K be a compact subset of M with smooth boundary, and let $\mathcal{U} = M \setminus K$. The long exact sequences in de Rham cohomology induces by including K or $\mathcal{U}$ into M have analogous sequences in *unreduced* L^2 cohomology:

$$\ldots \to H^k_2(K, \partial K) \to H^k_2(M) \to H^k_2(\mathcal{U}) \to \ldots \tag{3.3}$$

$$\ldots \to H^k_2(\mathcal{U}, \partial \mathcal{U}) \to H^k_2(M) \to H^k_2(K) \to \ldots \tag{3.4}$$

In general these sequences are not exact for *reduced* L^2 cohomology, though they are if the range of d is closed. However parts of the sequence are always exact.

To make this precise, we need to define reduced L^2 cohomology spaces for manifolds with boundary. Thus define

$$\begin{aligned} Z^k_2(\mathcal{U}) &= \left\{ \omega \in L^2\Omega^k\mathcal{U} : (\omega, \delta\phi) = 0 \text{ for all } \phi \in \mathcal{C}^\infty_c \Omega^{k+1}\mathcal{U}^\circ \right\} \\ &= (\delta \mathcal{C}^\infty_c \Omega^{k+1}\mathcal{U}^\circ)^\perp, \\ Z^k_2(\mathcal{U}, \partial\mathcal{U}) &= \left\{ \omega \in L^2\Omega^k\mathcal{U} : (\omega, \delta\phi) = 0 \text{ for all } \phi \in \mathcal{C}^\infty_c \Omega^{k+1}\mathcal{U} \right\} \\ &= (\delta \mathcal{C}^\infty_c \Omega^{k+1}\mathcal{U})^\perp, \end{aligned}$$

and, as before, these spaces are closed. Note that Green's formula shows that, if $\omega \in L^2\Omega^k(M)$ is smooth, $i : \partial\mathcal{U} \to \mathcal{U}$ is the inclusion, and ν is a normal vector field to the boundary, then

$$\omega \in Z^k_2(\mathcal{U}) \iff d\omega = 0, \quad \omega \in Z^k_2(\mathcal{U}, \partial\mathcal{U}) \iff d\omega = 0, i^*\omega = 0.$$

The reduced L^2 cohomology spaces are defined by

$$\begin{aligned} \overline{H}^k_2(\mathcal{U}) &= Z^k_2(\mathcal{U}) / \overline{d\mathcal{C}^\infty_c \Omega^{k-1}\mathcal{U}}, \\ \overline{H}^k_2(\mathcal{U}, \partial\mathcal{U}) &= Z^k_2(\mathcal{U}, \partial\mathcal{U}) / \overline{d\mathcal{C}^\infty_c \Omega^{k-1}\mathcal{U}^\circ}, \end{aligned}$$

and have an interpretation in terms of harmonic forms:

$$\begin{aligned} L^2\mathcal{H}^k_2(\mathcal{U}) &= \left\{ \omega \in L^2\Omega : d\omega = \delta\omega = \mathrm{i}_\nu \omega = 0 \right\} \\ L^2\mathcal{H}^k_2(\mathcal{U}, \partial\mathcal{U}) &= \left\{ \omega \in L^2\Omega : d\omega = \delta\omega = i^*\omega = 0 \right\}. \end{aligned}$$

Lemma 3.12. *The following sequence is exact:*

$$\overline{H}^k_2(M) \to \overline{H}^k_2(\mathcal{U}) \to H^{k+1}(K, \partial K), \tag{3.5}$$

and so is

$$\overline{H}^k_2(\mathcal{U}, \partial\mathcal{U}) \to \overline{H}^k_2(M) \to H^k(K). \tag{3.6}$$

Proof. We start by recalling the maps involved in (3.5). The first map is induced by the inclusion $j_{\mathcal{U}} : \mathcal{U} \to M$; the second map is the co-boundary map $b : \overline{H}^k_2(\mathcal{U}) \to H^k(K, \partial K)$ defined as follows. Given a class in $\overline{H}^k_2(\mathcal{U})$ choose a smooth representative, α, and choose any smooth extension of α to an element $\widetilde{\alpha} \in \mathcal{C}^\infty\Omega^k M$. Since $d\alpha = 0$ in $\mathcal{U}$, we know that $d\widetilde{\alpha}$ is a closed smooth form with support in K and defines a relative cohomology class,

$$b([\alpha]) = [d\widetilde{\alpha}] \in H^{k+1}(K, \partial K),$$

that is independent of the choices involved. Clearly $j^*_{\mathcal{U}} \circ b = 0$, so we only need to show $\ker b \subseteq \operatorname{Im}(j^*_{\mathcal{U}})$.

Thus we need to show that, if a form α has an extension $\widetilde{\alpha}$ to M such that $d\widetilde{\alpha}$ is exact as an element of $H^{k+1}(K, \partial K)$, then there is actually an extension of α to M that is exact as an element of $H^k(M)$. Let $i : \partial K \to K$ denote the inclusion of the boundary of K, and choose $\beta \in \mathcal{C}^\infty\Omega^k K$ with $i^*\beta = 0$ satisfying $d\widetilde{\alpha} = d\beta$ in K, then the form

$$\widehat{\alpha} = \begin{cases} \widetilde{\alpha} - \beta & \text{on } K \\ \alpha & \text{on } \mathcal{U} \end{cases}$$

is the required extension of α. Indeed, it is an extension of α to a form in $L^2\Omega^k M$, and, for any $\phi \in \mathcal{C}^\infty_c\Omega^{k+1}M$, we have

$$\begin{aligned} \int_M (\widehat{\alpha}, \delta\phi) &= \int_K (\widetilde{\alpha} - \beta, \delta\phi) + \int_{\mathcal{U}} (\alpha, \delta\phi) \\ &= -\int_{\partial K} (i^*(\widetilde{\alpha} - \beta), \mathrm{i}_\nu \phi) + \int_{\partial K} (i^*\alpha, \mathrm{i}_\nu\phi) = 0, \end{aligned}$$

hence $\widetilde{\alpha}$ is (weakly) closed.

Next we recall the maps involved in (3.6). The first map $e : H^k(\mathcal{U}, \partial\mathcal{U}) \to \overline{H}^k_2(M)$ consists of extending a form by zero, i.e.,

$$e(\alpha) = \begin{cases} \alpha & \text{on } \mathcal{U} \\ 0 & \text{on } K \end{cases}$$

the second map is induced by the inclusion $j_K : K \to M$. Clearly $j^*_K \circ e = 0$, so we only need to show that $\ker j^*_K \subseteq \operatorname{Im}(e)$.

Thus we need to show that any form on M that is exact when pulled back to K is cohomologous to a closed form supported in $\mathcal{U}$. So assume that α is a smooth, harmonic form in $L^2\Omega^k M$ such that $j^*_K\alpha = d\beta$ for some $\beta \in \mathcal{C}^\infty\Omega^{k-1}K$. Let $\overline{\beta} \in \mathcal{C}^\infty_c\Omega^{k-1}M$ be an extension of β. Clearly, α is cohomologous to $\alpha - d\overline{\beta}$ in $\overline{H}^k_2(M)$, and

$$\alpha - d\overline{\beta} = e(j^*_{\mathcal{U}}(\alpha - d\overline{\beta}))$$

so we only need to check that $j^*_{\mathcal{U}}(\alpha - d\overline{\beta})$ is an element of $Z^k_2(\mathcal{U}, \partial\mathcal{U})$. Let $\phi \in \mathcal{C}^\infty_c\Omega^k\mathcal{U}$, and let $\overline{\phi} \in \mathcal{C}^\infty_c\Omega^k M$ be an extension of ϕ, then we have

$$\int_{\mathcal{U}} (j^*_{\mathcal{U}}(\alpha - d\overline{\beta}, \delta\phi) = \int_M (j^*_{\mathcal{U}}(\alpha - d\overline{\beta}, \delta\overline{\phi}) = 0.$$

□

This lemma allows us to prove Lott's result. We say that two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are **isometric at infinity** if there are compact sets $K_i \subseteq M_i$ such that

$$(M_1 \setminus K_1, g_1) \text{ is isometric to } (M_2 \setminus K_2, g_2).$$

Theorem 3.13 (Lott). *Let (M_1, g_1) and (M_2, g_2) be complete oriented manifolds of dimension m which are isometric at infinity, then for any $k \geq 0$,*

$$\dim \overline{H}^k_2(M_1) < \infty \iff \dim \overline{H}^k_2(M_2) < \infty.$$

Proof. It suffices to show that $\dim \overline{H}^k_2(M) < \infty \iff \dim \overline{H}^k_2(\mathcal{U}) < \infty$ with M and $\mathcal{U}$ as above. Notice that, since $H^{k+1}(K, \partial K)$ is always finite dimensional, (3.5) shows that

$$\dim \overline{H}^k_2(M) < \infty \implies \dim \overline{H}^k_2(\mathcal{U}) < \infty.$$

Similarly, since $H^k(K)$ is always finite dimensional, (3.6) shows that

$$\dim \overline{H}^k_2(\mathcal{U}, \partial\mathcal{U}) < \infty \implies \dim \overline{H}^k_2(M) < \infty.$$

Since the manifold is oriented, we can define the Hodge star map and use it to see that

$$\dim \overline{H}^k_2(\mathcal{U}, \partial\mathcal{U}) < \infty \iff \dim \overline{H}^{m-k}_2(\mathcal{U}) < \infty,$$

which finishes the proof. □

3.4. Fredholm operators on open manifolds.

In the previous section we have seen that some of the difficulties in working with L^2 cohomology stem from the fact that the image of d (acting on its maximal domain) is not closed. This is closely related to whether or not the de Rham operator $d + \delta$ is Fredholm. In this section we look at a characterization of Fredholm operators on complete manifolds due to Nicolae Anghel.

In the following couple of results we work with arbitrary separable Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$, to highlight that the results are not geometric. In applications these will always be Hilbert spaces of sections of a vector bundle over a manifold.

Lemma 3.14. *A closed operator $T : \mathcal{D}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ has closed range if and only if there is a positive number $C > 0$ such that*

$$\|Tu\| \geq C d(u, \text{null}(T)) \textit{ for all } u \in \mathcal{D}(T). \tag{3.7}$$

An equivalent way of stating (3.7) is

$$\|Tu\| \geq C\,\|u\| \text{ for all } u \in \mathcal{D}(T) \cap (\operatorname{null} T)^{\perp},$$

which, in turn, is equivalent to saying that the generalized inverse of T is a bounded operator.

Proof. If (3.7) holds, and $(u_n) \subseteq \mathcal{D}(T)$ are such that $Tu_n \to v$, then $d(u_n, \operatorname{null}(T))$ is a Cauchy sequence and we can pick $(z_n) \subseteq \operatorname{null}(T)$ so that $(u_n - z_n)$ is a Cauchy sequence, say with limit u. Since T is closed, $u \in \mathcal{D}(T)$ and $Tu = v$, thus $\operatorname{Im}(T)$ is closed.

To prove the converse, consider the operator $\widetilde{T} : \mathcal{H}_1/\operatorname{null}(T) \to \mathcal{H}_2$ induced by T. This operator is clearly injective and linear. It is also closed. Indeed, if

$$[u_n] \to [u] \text{ and } \widetilde{T}[u_n] \to v,$$

then we can find $z_n \in \operatorname{null}(T)$ such that $u_n - z_n \to u$ and $T(u_n - z_n) = \widetilde{T}[u_n] \to v$. Then, since T is closed, we have $u \in \mathcal{D}(T)$ and $Tu = v$, hence $[u] \in \mathcal{D}(\widetilde{T})$ and $\widetilde{T}[u] = v$, so $\widetilde{T}$ is closed.

The estimate (3.7) is equivalent to saying that the inverse of $\widetilde{T}$ is a bounded operator $\operatorname{Im}(T) \to \mathcal{D}(T)$ - indeed the largest C is $\left\|\widetilde{T}^{-1}\right\|^{-1}$ - and if the range of T is closed, then the closed graph theorem implies that $\widetilde{T}^{-1}$ is bounded. □

We say that a closed operator with closed range T is **semi-Fredholm** if either its null space of its cokernel are finite dimensional. In the former case we say that T is **essentially injective** and in the latter **essentially surjective**. For these operators one can talk about $\operatorname{ind}(T)$ as an extended integer. If $\mathcal{D}(T)$ is dense - so that we can define T^* - then $\operatorname{Im}(T)$ is closed if and only if $\operatorname{Im}(T^*)$ is closed and T^* is semi-Fredholm if and only if T is semi-Fredholm and moreover $\operatorname{ind}(T) = -\operatorname{ind}(T^*)$. Clearly a self-adjoint operator is semi-Fredholm precisely when it is Fredholm.

Lemma 3.15. *A closed operator $T : \mathcal{D}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ is essentially injective if and only if*

$$\begin{cases} (s_k) \subseteq \mathcal{D}(T) \text{ bounded} \\ (Ts_k) \text{ convergent} \end{cases} \implies (s_k) \text{ has a convergent subsequence.} \tag{3.8}$$

Proof. If T is essentially injective, then its generalized inverse, G, is a bounded operator and a left parametrix. Hence $s_k = GTs_k + \Pi s_k$ has a convergent subsequence because Πs_k has a convergent subsequence.

The proof of the converse is much like the proof of Lemma 2.2. First, the null space of T must be finite dimensional since (3.8) implies that its unit ball is sequentially compact. Next, notice that any sequence of vectors (u_k) orthogonal to the null space of T, such that (Tu_k) converges, must have a bounded subsequence. Otherwise the sequence $\widetilde{u}_k = \frac{1}{\|u_k\|}u_k$ is bounded and $T\widetilde{u}_k \to 0$ and so by (3.8) has a convergent subsequence. But, if $\widetilde{u}$ is the limit of such a subsequence, then $\|\widetilde{u}\| = 1$ and $\widetilde{u}$ is simultaneously in $\mathrm{null}(T)$ and $\mathrm{null}(T)^{\perp}$, which clearly can not happen. Thus (u_k) has a bounded subsequence and so from (3.8) a convergent subsequence. Denote a limit point by u and note that, since T is closed, $u \in \mathcal{D}(T)$ and $Tu = \lim Tu_k$, hence $\mathrm{Im}(T)$ is closed. □

Now we return to the setting of complete manifolds and we can establish Anghel's criterion for Fredholm operators.

Theorem 3.16. *Let M be a complete Riemannian manifold, and let $P : L^2(M;E) \to L^2(M;F)$ be an essentially self-adjoint elliptic differential operator. Then P is Fredholm if and only if there is a compact set $K \subseteq M$ and a constant $C > 0$ such that*

$$\|Pu\|_{L^2(M,F)} \geq C \|u\|_{L^2(M;E)} \text{ for all } u \in \mathcal{C}^\infty_c(M \setminus K; E). \tag{3.9}$$

Proof.
The first implication is true more generally:
Let $\mathcal{H}(E)$ and $\mathcal{H}(F)$ be Hilbert spaces of sections of E and F respectively, with $\mathcal{C}^\infty_c(M \setminus K; E) \subseteq \mathcal{H}(E)$ and $\mathcal{H}(E)$ continuously included in $L^2_{\mathrm{loc}}(M,E)$. If $P : \mathcal{D}(P) \subseteq \mathcal{H}(E) \to \mathcal{H}(F)$ is a closed, essentially injective operator, and K_n is a sequence of compact subsets of M such that $M \setminus \cup_n K_n$ has measure zero then, for some $n \in \mathbb{N}$ we have

$$\|Pu\|_{\mathcal{H}(F)} \geq \frac{1}{n} \|u\|_{\mathcal{H}(E)} \text{ for all } u \in \mathcal{C}^\infty_c(M \setminus K_n; E). \tag{3.10}$$

Indeed, otherwise for each n we can find $u_n \in \mathcal{C}^\infty_c(M \setminus K_n; E)$ such that $\|u_n\| = 1$ and $\|Pu_n\| \leq \frac{1}{n}$. Using Lemma 3.15 we can assume, without loss of generality, that u_n converges to some element $u \in \mathcal{H}(E)$. Clearly $\|u\| = 1$ and, since P is closed, $u \in \mathcal{D}(P)$ and $Pu = 0$. On the other hand u_n converges weakly to zero in L^2, so using the inclusion $\mathcal{H}(E) \to L^2_{\mathrm{loc}}(M;E)$, and the fact that the u_n are compactly supported, u_n also converges weakly to zero in $\mathcal{H}(E)$. Thus, if $\Pi_{\ker P}$ is the projection onto the null space of P, we know that $\Pi_{\ker P} u_n \to 0$ since $\Pi_{\ker P}$ is compact. Finally, using G, the generalized inverse of P, we see that $u_n = GPu_n + \Pi_{\ker P} u_n \to 0$, which contradicts $\|u\| = 1$. This contradiction proves (3.10). To apply this to a complete manifold we can use a smooth proper function with bounded gradient $\mu : M \to [0, \infty)$ and let $K_n = \mu^{-1}([0,n])$.

For the converse, let (s_k) be a sequence of elements in $\mathcal{D}(P)$ that are bounded in $L^2(M;E)$ and such that (Ps_k) converges in $L^2(M;F)$, by Lemma 3.15 if we show that (s_k) has a convergent subsequence, we will know that P is semi-Fredholm and self-adjoint, hence Fredholm. Since P is essentially self-adjoint its domain is equal to $\mathcal{D}_{\min}(P)$, the closure of $\mathcal{C}^\infty_c(M;E)$ with respect to the graph norm $\|u\|_P = \|u\|_{L^2} + \|Pu\|_{L^2}$. Thus we can assume that the sequence (s_k) is made up of elements of $\mathcal{C}^\infty_c(E)$ (though note that the union of their supports need not be compact).

Let $\mathcal{U}$ be an open, precompact subset of M such that $K \subseteq \mathcal{U} \subseteq \overline{\mathcal{U}}$, and let $\chi \in \mathcal{C}_c^\infty(M, [0,1])$ be identically equal to one on K and zero on $M \setminus \overline{\mathcal{U}}$. The benefit of restricting to $(s_k) \subseteq \mathcal{C}_c^\infty(E)$ is that χs_k is still in $\mathcal{D}(P)$.

Since $\overline{\mathcal{U}}$ is compact, we can use Rellich's lemma to find a subsequence of s_k such that $(s_k\big|_{\overline{\mathcal{U}}})$ converges. Since $[P, \chi]$ is a differential operator of order lower than P, supported in $\overline{\mathcal{U}}$, another appeal to Rellich's lemma and passage to a subsequence allows us to assume that $[P, \chi]s_k$ converges. Finally, from

$$\begin{aligned} C \left\|(1-\chi)(s_k - s_\ell)\right\| &\leq \left\|P\left((1-\chi)(s_k - s_\ell)\right)\right\| \\ &= \left\|(1-\chi)P(s_k - s_\ell) + [P,\chi](s_k - s_\ell)\right\| \\ &\leq \left\|P(s_k - s_\ell)\right\| + \left\|[P,\chi](s_k - s_\ell)\right\|, \end{aligned}$$

we see that $(s_k\big|_{M\setminus\overline{\mathcal{U}}})$ converges, and hence (s_k) converges. □

It is useful to have Anghel's criterion for Hilbert spaces other than L^2, e.g., one may want to use weighted Sobolev spaces. For this reason we extract below as a corollary the more general statement covered by the proof above. We recall that a section $s \in \mathcal{C}^{-\infty}(M; E)$ is in $H^p_{\mathrm{loc}}(M; E)$ if its restriction to any open sets on which M and E are trivialized is an element of $H^p(\mathbb{R}^m; \mathbb{C}^k)$. The H^p-norms of the restrictions to compact subsets of these trivializing neighborhoods yield a family of semi-norms which we use to topologize $H^p_{\mathrm{loc}}(M; E)$. Elliptic regularity shows that if a section s of E is in $\mathcal{D}(P)$ then $s \in H^p_{\mathrm{loc}}(M; E)$.

Corollary 3.17. *Let M be a complete manifold, P an elliptic differential operator of order $p > 0$, and let $\mathcal{H}(E)$ and $\mathcal{H}(F)$ be Hilbert spaces of sections of E and F respectively, such that*

i) *$\mathcal{C}_c^\infty(M; E)$ is dense in $\mathcal{H}(E)$ and $\mathcal{C}_c^\infty(M; F) \subseteq \mathcal{H}(F)$*
ii) *$\mathcal{H}(E)$ includes into $H^p_{\mathrm{loc}}(M; E)$ and $\mathcal{H}(F)$ into $L^2_{\mathrm{loc}}(M; F)$ continuously*
iii) *P extends by continuity from an operator $\mathcal{C}_c^\infty(M; E) \to \mathcal{C}_c^\infty(M; F)$ to a continuous operator $\mathcal{H}(E) \to \mathcal{H}(F)$.*

Then P is essentially injective as an operator $\mathcal{H}(E) \to \mathcal{H}(F)$ if and only if there is a compact set $K \subseteq M$ and a constant $C > 0$ such that

$$\left\|Pu\right\|_{\mathcal{H}(F)} \geq C \left\|u\right\|_{\mathcal{H}(E)} \text{ for all } u \in \mathcal{C}_c^\infty(M \setminus K; E). \tag{3.11}$$

Proof. The forward implication was proved explicitly in this more general context above. For the converse implication, we can extend the inequality (3.11) to elements of $\mathcal{H}(E)$ using (i), and our use of Rellich's lemma on compact subsets of M is justified by (ii), so the proof above applies verbatim. □

This leads us to Gilles Carron's concept of an operator that is **non-parabolic at infinity** by requiring that Anghel's criterion 'hold locally'. Precisely, an elliptic operator $P : L^2(M, E) \to L^2(M, F)$ is non-parabolic at infinity if there is a compact subset K of M, so that, for any bounded open subset $\mathcal{U} \subseteq M \setminus K$ and any $s \in \mathcal{C}^\infty_c(M \setminus K, E)$, we have

$$\|Ps\|_{L^2(M\setminus K;F)} \geq C(\mathcal{U}) \ \|s\|_{L^2(\mathcal{U};E)}$$

for some constant $C(\mathcal{U}) > 0$ depending on $\mathcal{U}$ but not on s. Anghel's criterion shows that every Fredholm operator is non-parabolic at infinity. We will see below that these are precisely the operators that are Fredholm in an extended sense.

For any such P, and a compact subset of M, $\widehat{K}$, that contains K in its interior, we define a new norm on $\mathcal{C}^\infty_c(M, E)$ by

$$\|s\|^2_W = \|s\|^2_{L^2(\widehat{K},E)} + \|Ps\|^2_{L^2(M,F)}.$$

Note that P non-parabolic at infinity and $s \in \mathcal{C}^\infty_c(M, E)$ together imply that $\|s\|^2_L \leq C_s \|s\|^2_W$ for some constant depending on the support of s; in particular, $\|s\|_W = 0$ implies $\|s\|_{L^2} = 0$. We denote by $W_P(E)$ the completion of $\mathcal{C}^\infty_c(M, E)$ with respect to this norm; clearly $\mathcal{D}(P) \subseteq W_P(E)$.

Lemma 3.18. *Let $\widehat{K}_1$ and $\widehat{K}_2$ be compact subsets of M that contain K in their interior. The norms $\|\cdot\|_W$ defined using $\widehat{K}_1$ or $\widehat{K}_2$ are equivalent and hence the space $W_P(E)$ is independent of the choice of $\widehat{K}$.*

Proof. Denote by $\|\cdot\|_i$ the norm defined using K_i. If $\widehat{K}_1 \subseteq \widehat{K}_2$ then clearly $\|\cdot\|_1 \leq \|\cdot\|_2$. Also, because P is non-parabolic at infinity,

$$\|s\|^2_{L^2(\widehat{K}_2;E)} = \|s\|^2_{L^2(\widehat{K}_1;E)} + \|s\|_{L^2(\widehat{K}_2\setminus\widehat{K}_1;E)} \leq \|s\|^2_{L^2(\widehat{K}_1;E)} + C\|Ps\|_{L^2(M;F)},$$

for some $C > 0$, and hence $\|\cdot\|_2 \leq C\|\cdot\|_1$. For general $\widehat{K}_i$ this argument shows that both $\|\cdot\|_i$ are equivalent to the norm induced by $\widehat{K}_1 \cap \widehat{K}_2$. □

Theorem 3.19 (Carron)**.** *Let M be a complete manifold and P be an essentially self-adjoint elliptic operator of order $p > 0$. Then P is non-parabolic at infinity if and only if there is a Hilbert space $W(E)$ satisfying (i), (ii), (iii) such that $P : W(E) \to L^2(M; F)$ is Fredholm.*

Proof. If P is non-parabolic at infinity then we claim that $W_P(E)$ satisfies (i), (ii), (iii) and that $P : W_P(E) \to L^2(M; F)$ is Fredholm. The first and third properties are clear from the definition of $W_P(E)$. The second property is equivalent to asking that, for any compact subset $\widetilde{K}$ of M, there is a constant $C = C(\widetilde{K})$ such that

$$\|s\|_{H^p(\widetilde{K};E)} \leq C\|s\|_W, \quad \text{for all } s \in \mathcal{C}^\infty_c(\widetilde{K}; E). \tag{3.12}$$

However, since P is elliptic and $\widetilde{K}$ is compact, we have

$$\|s\|_{H^p(\widetilde{K};E)} \leq C(\|Ps\|_{L^2(\widetilde{K};E)} + \|s\|_{L^2(\widetilde{K};E)}),$$

and, using that P is non-parabolic at infinity, this is bounded by a multiple of $\|s\|_W$. Finally to see that P is Fredholm, note that if $u \in \mathcal{C}^\infty_c(M \setminus K; E)$ then $\|u\|_W = \|Pu\|_{L^2(M;F)}$ so the criterion of Corollary 3.17 is satisfied.

Conversely, if we assume that there is a Hilbert space as above, it is easy to check that P is non-parabolic at infinity. Indeed, let K be as in Corollary 3.17, let $\mathcal{U}$ be a bounded open subset of $M \setminus K$ and let $u \in \mathcal{C}_c^\infty(\mathcal{U}; E)$. Using (i), $u \in W(E)$, and hence using (ii) in the form (3.12), there is a constant $C > 0$ such that

$$\|u\|_{L^2(\mathcal{U};E)} \leq \|u\|_{H^p(\mathcal{U};E)} \leq C \|u\|_W ,$$

then, from (3.11), this is bounded above by a multiple of $\|Pu\|_{L^2(M;F)}$ - which implies that P is non-parabolic at infinity.

□

If P is non-parabolic at infinity, then its L^2 kernel and co-kernel are finite. Indeed, $\mathcal{D}(P) \subseteq W_P(E)$, so the L^2 kernel of P is a subspace of the kernel of P as an operator $W_P(E) \to L^2(F)$.

3.4.1. *Relation with L^2 cohomology.* We have seen how to characterize operators with closed image, operators that are Fredholm, and operators that are Fredholm in an extended sense. Before passing to more specific situations, we want to return to the long exact sequence in L^2 cohomology.

First note that if the range if $D = d + \delta$ is closed, then the range of d is closed. Indeed, from Lemma 3.14, this is the same as saying

$$\begin{aligned} \|Du\| \geq C \|u\| , &\quad \text{for every } u \in (\ker D)^\perp \\ \Longrightarrow \|du\| \geq C \|u\| , &\quad \text{for every } u \in (\ker d)^\perp, \end{aligned}$$

and this is clear as on $(\ker d)^\perp$, $d = D$. Thus if D has closed range, then reduced L^2 cohomology is the same as unreduced L^2 cohomology and so the sequences (3.3)-(3.4) hold in reduced cohomology.

Instead assume that D is non-parabolic at infinity. Since D is Fredholm as a map $W_P\Omega^* \to L^2\Omega^*$, we can restate the Kodaira decomposition as

$$L^2\Omega^k = L^2\mathcal{H}^k \oplus dW_P\Omega^{k-1} \oplus \delta W_P\Omega^{k+1}.$$

Indeed, we know that the range of D is closed and that $\mathcal{C}_c^\infty\Omega^*$ is dense in $W_P\Omega^*$, hence

$$\overline{d\mathcal{C}_c^\infty\Omega^{k-1}} = dW_P\Omega^{k-1}, \text{ and } \overline{\delta\mathcal{C}_c^\infty\Omega^{k+1}} = \delta W_P\Omega^{k+1}.$$

This yields exactness of

$$H^k(K, \partial K) \to \overline{H}_2^k(M) \to \overline{H}_2^k(\mathcal{U}).$$

Given an operator non-parabolic at infinity, we refer to elements of $\ker P \setminus L^2$ as **extended solutions of** P. If D is non-parabolic at infinity, and there are no extended solutions of D, then we have exactness of

$$\overline{H}_2^k(\mathcal{U}) \to H^{k+1}(K, \partial K) \to \overline{H}_2^k(M).$$

Hence in this case the long exact sequences in unreduced L^2 cohomology are also exact in reduced L^2 cohomology.

In the next few sections we will compute the reduced L^2 cohomology of certain spaces making use either of Lemma 3.14 to guarantee closed range of d or of Theorem 3.19 to guarantee non-parabolicity at infinity.

3.5. L^2 cohomology of conformally compact manifolds.

Our first example is the class of conformally compact manifolds. Although these are not non-parabolic at infinity, we will see that the exterior derivative has closed range.

Let M be the interior of a manifold with boundary $\overline{M}$. A boundary defining function (or **bdf**) for M is a non-negative function $x : \mathcal{C}^\infty(\overline{M}, \mathbb{R})$ such that $\partial M = \{x = 0\}$ and dx has no zeroes on ∂M. A metric g on M is called conformally compact if $\overline{g} = x^2 g$ extends smoothly to a metric on $\overline{M}$. Such a metric is automatically complete.

An important feature of conformally compact metrics is their relation to the conformal geometry of the boundary. Indeed, this stems from the simple observation that if x and $\widehat{x}$ are two bdfs for M, and we denote $x^2 g\big|_{\partial M}$ and $\widehat{x}^2 g\big|_{\partial M}$ by h and $\widehat{h}$ respectively, then

$$\widehat{x} = e^{2\omega} x \implies \widehat{h} = e^{2\omega}\big|_{\partial M} h.$$

Since any two bdfs are related in this way, we see that g naturally determines a conformal class of metrics on the boundary.

The motivating example of a conformally compact metric is the unit ball model of hyperbolic space. It is well-known that this metric is conformally equivalent to the flat metric on the ball and that there is a close relation between the conformal geometry of the sphere and the geometry of hyperbolic space.

Conformally compact metrics have been actively studied as a tool for conformal geometry and as a backdrop for the conjectural AdS/CFT (Anti-de-Sitter/Conformal field theory) correspondence in physics. There is a very rich analytic theory on these manifolds, dating from the MIT thesis of Rafe Mazzeo. We will study the 'zero calculus' of pseudo-differential operators adapted to these metrics and the corresponding scattering theory in later lectures. The L^2 cohomology of a conformally compact metric was computed by Mazzeo, in this section we present a method of computing it due to Nader Yeganefar.

First we present some general facts about conformally compact metrics.

Lemma 3.20. *Let M be the interior of a compact manifold with boundary, let g be a conformally compact metric on M, and let $[h]$ be the conformal class of metrics induced by g on ∂M.*

a) *Any other conformally compact metric g' on M is quasi-isometric to g - hence they have the same reduced L^2 cohomology.*
b) *If x is any bdf, and $\overline{g} = x^2 g$, then $|dx|_{\overline{g}}\big|_{\partial M}$ is independent of the choice of x.*

c) *Let α be any smooth extension of $|dx|_{\overline{g}}\big|_{\partial M}$ off of ∂M, and let $\widehat{h}$ be any metric in $[h]$, then there is a unique boundary defining function $\widehat{x}$ such that*

$$\widehat{x}^2 g\big|_{\partial M} = \widehat{h}, \quad \text{and} \quad |d\widehat{x}| = \alpha \text{ near } \partial M.$$

d) *The curvature of g is asymptotically isotropic - in fact, the limiting sectional curvatures at $q \in \partial M$ are all equal to $-|dx|^2_{\overline{g}}(q)$.*

A conformally compact metric is called **asymptotically hyperbolic** if $|dx|\big|_{\partial M} = 1$, or, equivalently by part (d) of the lemma, if the sectional curvature is asymptotic to -1. For an asymptotically hyperbolic metric, a boundary defining function is called **special** or geodesic if $|dx|_{\overline{g}} = 1$ in a neighborhood of the boundary. It follows from part (b) of the lemma that each metric in the conformal class is associated to a unique special bdf.

Proof.
a) For any fixed bdf, x, both $x^2 g$ and $x^2 g'$ are metrics on the compact manifold with boundary $\overline{M}$ and hence quasi-isometric, i.e., there is a $C > 0$ such that

$$Cx^2 g \leq x^2 g' \leq \tfrac{1}{C} x^2 g.$$

Dividing through by x^2, we conclude that g and g' are quasi-isometric.
b) If $\widehat{x} = e^{\omega} x$, and $\widehat{g} = \widehat{x}^2 g$, then

$$|d\widehat{x}|^2_{\widehat{g}} = |dx + x d\omega|^2_{\overline{g}} = |dx|^2_{\overline{g}} + 2x(\nabla_{\overline{g}} x)(\omega) + x^2 |d\omega|^2, \tag{3.13}$$

hence $|d\widehat{x}|_{\widehat{g}}\big|_{\partial M} = |dx|_{\overline{g}}\big|_{\partial M}$.
c) From (3.13) the first condition on $\widehat{x}$ is equivalent to

$$2(\nabla_{\overline{g}} x)(\omega) + x|d\omega|^2 = \frac{\alpha^2 - |dx|^2_{\overline{g}}}{x}$$

which is a non-characteristic first-order PDE for ω, so there is a solution near ∂M for every choice of ω_0, hence $\widehat{h}$.
d) Let x be any boundary defining function on M, and let $\overline{g} = x^2 g$. Using the well-known formulas for conformal change of metric (see [Besse]) the Levi-Civita connections and the curvature tensors of $\overline{g}$ and g are related by

$$\begin{aligned} &g(\nabla_U V, W) \\ &= g(\overline{\nabla}_U V, W) - \tfrac{dx(U)}{x} g(V,W) - \tfrac{dx(V)}{x} g(U,W) + \tfrac{dx(W)}{x} g(U,V) \end{aligned} \tag{3.14}$$

$$x^2 R = \overline{R} + \overline{g} \owedge (\mathrm{Hess}_{\overline{g}}(\log x) + \tfrac{dx}{x} \circ \tfrac{dx}{x} - \tfrac{1}{2} |\tfrac{dx}{x}|^2_{\overline{g}} \overline{g}). \tag{3.15}$$

So let $\mathcal{U}$ be a coordinate chart (for $\overline{g}$) centered at a point $q \in \partial M$, and let V and W be vector fields in $\mathcal{U}$ that are linearly independent at q. The sectional curvature of the plane spanned by V and W at any point not on the boundary is given by

$$\frac{g(R(U,V)V,U)}{g(U,U)g(V,V) - g(U,V)^2} = \frac{x^4 g(R(U,V)V,U)}{\frac{1}{2}\overline{g} \owedge \overline{g}(U,V)(U,V)} = -|dx|^2_{\overline{g}} + \mathcal{O}(x).$$

Thus we see that the sectional curvature is asymptotically equal to $-|dx|^2_{\overline{g}}$. $\square$

Let x be a bdf and $\alpha = |dx|$, the flow of the vector field $\nabla_{\overline{g}} x$ defines a diffeomorphism between a neighborhood of the boundary ∂M and a product $[0, \varepsilon)_x \times \partial M$, on which the metric takes the form[4]

$$g = \frac{dx^2}{\alpha^2 x^2} + \frac{h(x, y, dy)}{x^2},$$

where y represents directions tangent to the boundary. Since the L^2 cohomology does not change under quasi-isometry, we can assume without loss of generality that the metric has the form

$$g = \frac{dx^2 + h(y, dy)}{x^2} = dr^2 + e^{2r} h(y, dy) \tag{3.16}$$

near the boundary.

The key step in Yeganefar's proof of Mazzeo's result is an integration by parts identity of Donnelly-Xavier, which will be used to establish that the range of d is closed (in most cases).

Lemma 3.21. *If $\mathcal{U}$ is a neighborhood of the boundary where the metric takes the form* (3.16) *and $\omega \in \mathcal{C}^\infty_c \Omega^k \mathcal{U}$, then, with $m = \dim M$,*

$$\sqrt{2} \left\| d\omega + \delta\omega \right\|_{L^2} \left\| \omega \right\|_{L^2} \geq \left(\frac{m-1}{2} - k \right) \left\| \omega \right\|^2_{L^2} + \int_{\partial \mathcal{U}} \left[\tfrac{1}{2} |\omega|^2 - |x \partial_x \lrcorner\, \omega|^2 \right]. \tag{3.17}$$

Proof. First let N be any Riemannian manifold with smooth boundary ∂N and inward pointing normal vector field ν. A simple computation using Cartan's identity shows that, for any vector field X and k-form ω,

$$\mathcal{L}_X \omega = \nabla X . \omega + \nabla_X \omega, \qquad \text{where } \nabla X . \omega = \sum \theta^i \wedge (\nabla_{E_i} X \lrcorner\, \omega).$$

Then a judicious use of Green's theorem proves that, for $\omega \in \mathcal{C}^\infty_c \Omega^*$,

$$\int_N (\nabla X . \omega, \omega) + \tfrac{1}{2} (\mathrm{div} X) |\omega|^2 = \int_N \left[(X \lrcorner\, \omega, \delta\omega) + (X \lrcorner\, d\omega, \omega) \right] + \int_{\partial N} \left[\tfrac{1}{2} (X, \nu) |\omega|^2 - (\nu \lrcorner\, \omega, X \lrcorner\, \omega) \right]. \tag{3.18}$$

Indeed,

$$\int_N (\mathrm{div} X) |\omega|^2 + \int_{\partial N} (X, \nu) |\omega|^2 = \int_N X(\omega, \omega) = 2 \int_N (\mathcal{L}_X \omega - \nabla X . \omega, \omega)$$

[4] See [Joshi-Sa Barreto, Prop. 2.1]

and

$$\int_N (\mathcal{L}_X\omega,\omega) = \int_N [(X \lrcorner d\omega,\omega) + (d(X \lrcorner \omega),\omega)]$$
$$= \int_N [(X \lrcorner d\omega,\omega) + (X \lrcorner \omega,\delta\omega)] - \int_{\partial N} (\nu \wedge X \lrcorner \omega,\omega).$$

For the case at hand, set $X = x\partial_x = \nu$. From (3.14) and (3.16) it is immediate that $\nabla_X X = 0$ and $\nabla_Y X = -Y$ whenever $Y \perp X$. Thus, for any $\omega \in \mathcal{C}_c^\infty\Omega^k\mathcal{U}$,

$$(\nabla X.\omega,\omega) = -k|\omega|^2 + |x\partial_x \lrcorner \omega|^2 \quad \text{and} \quad \mathrm{div}X = m-1,$$

and (3.18) becomes

$$\int_{\mathcal{U}} \left[\left(\frac{m-1}{2} - k\right)|\omega|^2 + |x\partial_x \lrcorner \omega|^2\right] = \int_{\mathcal{U}} [(x\partial_x \lrcorner \omega,\delta\omega) + (x\partial_x \lrcorner d\omega,\omega)] - \int_{\partial\mathcal{U}} \left[\tfrac{1}{2}|\omega|^2 - |x\partial_x \lrcorner \omega|^2\right]. \tag{3.19}$$

Next we use Cauchy-Schwarz to see

$$\int_{\mathcal{U}} [(x\partial_x \lrcorner \omega,\delta\omega) + (x\partial_x \lrcorner d\omega,\omega)] \le \|\omega\|\left(\|d\omega\| + \|\delta\omega\|\right)$$
$$\le \sqrt{2}\,\|\omega\|\left(\|d\omega\|^2 + \|\delta\omega\|^2\right)^{1/2} = \sqrt{2}\,\|\omega\|\,\|d\omega+\delta\omega\|,$$

where we have used that the image of d is orthogonal to the image of δ. Finally (3.17) follows by omitting $\int_N |x\partial_x \lrcorner \omega|^2$ in (3.19). □

Proposition 3.22. *Let $\mathcal{U}$ be an open neighborhood of the boundary as above. For $k \le \frac{m-1}{2}$ we have $\overline{H}_2^k(\mathcal{U}) = 0$, and*

$$\|d\omega+\delta\omega\|_{L^2} \ge \frac{1}{\sqrt{2}}\left(\frac{m-1}{2} - k\right)\|\omega\|_{L^2}, \tag{3.20}$$

for every $\omega \in \mathcal{C}_c^\infty\Omega^k\mathcal{U}$.

Proof. For the second assertion, we have $\omega \in \mathcal{C}_c^\infty\Omega^k\mathcal{U}$. In particular, since $\mathcal{U}$ is open, ω vanishes in a neighborhood of $\partial\mathcal{U}$, so the lemma implies (3.20) directly.

To prove the first assertion we need to show that any $\omega \in L^2\Omega^k\mathcal{U}$ satisfying

$$d\omega = \delta\omega = 0, \quad x\partial_x \lrcorner \omega\big|_{\partial\mathcal{U}} = 0$$

is identically zero. Choose $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^+,[0,1])$ such that ρ is equal to one near zero, set $\rho_N(t) = \rho(t/N)$ and $\omega_N = \rho_N(\log x)\omega$. Then, we have

$$\|d\omega_N+\delta\omega_N\|_{L^2}\|\omega_N\|_{L^2} = \left\|(\partial_t\rho_N)(\tfrac{dx}{x} \wedge -x\partial_x \lrcorner)\omega\right\|_{L^2}\|\omega_N\|_{L^2}$$
$$\le \frac{\|\rho'\|_\infty}{N}\left\|(\tfrac{dx}{x} \wedge -x\partial_x \lrcorner)\omega\right\|_{L^2}\|\omega\|_{L^2} \to 0,$$

hence, from the lemma,

$$\left(\frac{m-1}{2}-k\right)\int_{\mathcal{U}}|\omega|^2+\frac{1}{2}\int_{\partial\mathcal{U}}|\omega|^2=0.$$

Thus if $2k<m-1$ we have $\omega=0$. If $2k=m-1$ we have $\omega\big|_{\partial\mathcal{U}}=0$ and so a unique continuation result shows that $\omega=0$ (e.g., extend ω by zero into M, or instead analyze the de Rham operator as in the next section).

□

Thanks to this lemma and Anghel's criterion for Fredholm operators, we know that $d+\delta$ is Fredholm as an operator

$$L^2\Omega^kM\to L^2\Omega^{k+1}M\oplus L^2\Omega^{k-1}M,\ \text{for all } k<\frac{m-1}{2}$$

In particular, the image of d is closed and so $\overline{H}_2^k(M)=H_2^k(M)$ for all $k\leq\frac{m-1}{2}$. Thus the long exact sequence in unreduced L^2 cohomology is a long exact sequence in reduced L^2 cohomology for these values of k.

Theorem 3.23 (Mazzeo)**.** *Let (M^m,g) be a conformally compact manifold, then*

$$\overline{H}_2^k(M)\cong L^2\mathcal{H}^k(M)\cong\begin{cases}H^k(M,\partial M) & \text{if } k<\frac{m}{2}\\ H^k(M) & \text{if } k>\frac{m}{2}\end{cases}$$

and, if m is even, $\dim\overline{H}_2^{m/2}(M)=\infty$.

Proof. The space of harmonic middle degree forms is conformally invariant by Proposition 3.9 and hence infinite dimensional by Proposition 3.5.

Since reduced L^2 cohomology satisfies Poincaré Duality, we only need to prove the result for $k<\frac{m}{2}$. However, we have the truncated long exact sequence

$$\begin{aligned}0\to H^0(K,\partial K)\to\overline{H}_2^0(M)\to\overline{H}_2^0(\mathcal{U})\to\dots\\ \to H^{\lfloor\frac{m-1}{2}\rfloor}(K,\partial K)\to\overline{H}_2^{\lfloor\frac{m-1}{2}\rfloor}(M)\to\overline{H}_2^{\lfloor\frac{m-1}{2}\rfloor}(\mathcal{U}),\end{aligned}$$

and, from Proposition 3.22, we know that $\overline{H}_2^k(\mathcal{U})=0$ for $k\leq\frac{m-1}{2}$ so the result follows. □

3.6. **Asymptotically cylindrical and Euclidean manifolds.**

In this section we consider complete Riemannian manifolds, M, such that outside a compact set, K, M is isometric to a warped product $\mathbb{R}^+\times\partial K$ with metric $dr^2+f(r)^2d\theta^2$.

We start out by establishing an elementary inequality (similar to Hardy's inequality on $\mathbb{R}^m$).

Lemma 3.24. *For any* $u \in \mathcal{C}_c^\infty((0,\infty))$ *we have*

$$\frac{1}{4}\int \frac{u(r)^2}{r^2}\,dr \leq \int (u'(r))^2\,dr \tag{3.21}$$

$$\frac{1}{4}\int \frac{u(r)^2}{r(\log r)^2}\,dr \leq \int (u'(r))^2 r\,dr \tag{3.22}$$

$$\frac{(1-k)^2}{4}\int u(r)^2 r^{k-2}\,dr \leq \int (u'(r))^2 r^k\,dr, \quad k \neq 1. \tag{3.23}$$

Proof. The first inequality is a consequence of Cauchy-Schwartz and integration by parts:

$$\int \frac{u^2}{r^2} = -\int u^2\partial_r(\tfrac{1}{r}) = \int \frac{2uu'}{r} \leq \left(\int \frac{4u^2}{r^2}\right)^{1/2}\left(\int (u')^2\right)^{1/2}$$
$$\implies \int \frac{u^2}{4r^2}\,dr \leq \int (u')^2\,dr.$$

The second and third follow from the first by a change of variable. Note that if $f, g \in \mathcal{C}^\infty(\mathbb{R}^+,\mathbb{R}^+)$ satisfy $f'(s) = g(f(s))$ then

$$\int (u'(r))^2 g(r)\,dr = \int [(u\circ f)'(s)]^2\,ds$$
$$\geq \int \frac{(u\circ f)^2(s)}{4s^2}\,ds = \int \frac{u^2(r)}{4(f^{-1}(r))^2 g(r)}\,dr.$$

So we get (3.22) by substituting $g(r) = r$, $f(s) = e^s$, and we get (3.23) by substituting $g(r) = r^k$ and $f(s) = [(1-k)s]^{1/(1-k)}$. □

To study the Gauss-Bonnet operator, we point out that for any warped product metric $dr^2 + f(r)^2 d\theta$ we can identify $\Omega^{\text{even}}\mathcal{U}$ (or $\Omega^{\text{odd}}\mathcal{U}$) with one parameter families of forms in $\Omega^*\partial\mathcal{U}$, via

$$\mathcal{C}_c^\infty\Omega^{\text{odd}}\mathcal{U} \xrightarrow{\quad\mathcal{L}\quad} \mathcal{C}_c^\infty(\mathbb{R}^+,\Omega^*\partial\mathcal{U})$$
$$\alpha = \alpha_1 + dr\wedge\alpha_2 \longmapsto f^{\mathbf{N}-(m-1)/2}\alpha_2 - f^{\mathbf{N}-(m-1)/2}\alpha_1$$

where $\mathbf{N}$ is the number operator on forms on $\partial\mathcal{U}$, i.e., if $\beta \in \Omega^k\partial\mathcal{U}$, then $\mathbf{N}\beta = k\beta$. The extension of $\mathcal{L}$ from smooth forms with compact support, where it is a bijection, to L^2-forms is an isometry. The action of $d+\delta$ is conjugated by $\mathcal{L}$ to an operator D on $\Omega^*\partial\mathcal{U}$ given by

$$D\alpha = \mathcal{L}(d+\delta)\mathcal{L}^{-1}\alpha = (-1)^{\mathbf{N}}\left(\frac{\partial}{\partial r} + \frac{(d+\delta)_{\partial\mathcal{U}} + f'\mathcal{J}}{f}\right)\alpha$$

where $\mathcal{J} = (-1)^{\mathbf{N}}(\mathbf{N} - \frac{m-1}{2})$.

Proposition 3.25 (Carron)**.** *If M is a complete manifold and, outside a compact set K, it is isometric to a warped product $([1,\infty)\times\partial K, dr^2 + f(r)^2 h)$ with $f(r) = 1$ or $f(r) = r$, then the Gauss-Bonnet operator is non-parabolic at infinity.*

Proof. With the notation as above, let $A = (d+\delta)_{\partial\mathcal{U}} + f'\mathcal{J}$. In both cases considered here, A is an elliptic self-adjoint operator independent of r. Denote the eigenvalues of A by λ and by ϕ_λ an orthonormal eigenbasis. We can write any section $s \in \mathcal{C}^\infty_c(\mathbb{R}^+ \times \partial K; E)$ as

$$s(r,\theta) = \sum_{\lambda \in \mathrm{Spec}(A)} s_\lambda(r)\phi_\lambda(\theta). \tag{3.24}$$

Then

$$\|s\|^2_{L^2} = \sum_\lambda \int |s_\lambda|^2 \, dr$$

and

$$\begin{aligned} \|Ds\|^2_{L^2} &= \sum_\lambda \int \left| s'_\lambda + \frac{\lambda}{f} s_\lambda \right|^2 \, dr \\ &= \sum_\lambda \int \left[(s'_\lambda)^2 + \frac{\lambda^2 + \lambda f'}{f^2} |s_\lambda|^2 \right] \, dr. \end{aligned} \tag{3.25}$$

In the first case above, we use (3.21) to see that

$$\|Ds\|^2_{L^2} = \sum_\lambda \int \left[(s'_\lambda)^2 + \frac{\lambda^2}{f^2}|s_\lambda|^2 \right] \, dr \geq \sum_\lambda \int (s'_\lambda)^2 \, dr \geq \sum_\lambda \int \frac{(s_\lambda)^2}{4r^2} \, dr$$

so, if $\operatorname{supp} s \subseteq [0,T] \times \partial K$, we have $\|Ds\|^2 \geq (2T)^{-2} \|s\|$. Hence in this case D is non-parabolic at infinity.

In the second case above, (3.25) combined with (3.22) yields

$$\begin{aligned} \|Ds\|^2_{L^2} &= \sum_\lambda \int \left[|s'_\lambda|^2 + \frac{\lambda^2 + \lambda a}{a^2 r^2} |s_\lambda|^2 \right] \, dr \\ &= \sum_\lambda \int \left[|s'_\lambda|^2 - \frac{1}{4r^2}|s_\lambda|^2 + \frac{(\lambda + \frac{a}{2})^2}{a^2 r^2} |s_\lambda|^2 \right] \, dr \\ &\geq \sum_\lambda \int \left[|s'_\lambda|^2 - \frac{1}{4r^2}|s_\lambda|^2 \right] \, dr \xrightarrow{\widetilde{s}_\lambda = \sqrt{r} s_\lambda} \sum_\lambda \int |\widetilde{s}'_\lambda|^2 r \, dr \\ &\geq \sum_\lambda \int \frac{|\widetilde{s}_\lambda|^2}{4r(\log r)^2} \, dr = \sum_\lambda \int \frac{|s_\lambda|^2}{(2r\log r)^2} \, dr, \end{aligned}$$

hence, if $\operatorname{supp} s \subseteq [1,T] \times \partial K$, we have $\|Ds\|^2 \geq (2T\log T)^{-2} \|s\|$. So, also in this case, D is non-parabolic at infinity. $\square$

Similar proofs work for other warping functions (e.g., the first proof works whenever $f(r)/r$ is a decreasing function), and for other operators such as Dirac and Dirac-type (see [Carron, indice relatif]). Next we apply this proposition to computing the reduced L^2 cohomology.

Theorem 3.26. *If M is a complete manifold quasi-isometric outside a compact set K to a half-cylinder $\mathbb{R}^+ \times \partial K$, then*

$$\overline{H}_2^k(M) \cong \operatorname{Im}(H_c^k(M) \to H^k(M)).$$

Proof. Since quasi-isometry does not change the reduced L^2 cohomology, we can assume that $\mathcal{U} = M \setminus K$ is isometric to the half-cylinder, so by Proposition 3.25, the Gauss-Bonnet operator $D = d + \delta$ is non-parabolic at infinity.

Consider the full cylinder $X = \mathcal{U} \coprod \mathcal{U}$. The reflection across $\partial\mathcal{U}$, τ, is an isometric involution and induces an isometric involution τ^* of $L^2\Omega^k X$. Denote by $L^2_\pm\Omega^k X$ the ± 1 eigenspaces of τ^*. Since τ^* commutes with the exterior derivative, it induces an isometric involution of the L^2 harmonic forms, and we denote the ± 1 eigenspaces by $L^2_\pm\mathcal{H}^k$. It now follows, just as for forms on compact manifolds with boundary, that $L^2_+\Omega^k X$ and $L^2_-\Omega^k X$ are naturally isometric to $L^2\Omega^k\mathcal{U}$ with, respectively, absolute and relative boundary conditions. This identification descends to L^2 harmonic forms so that

$$L^2\mathcal{H}^k\mathcal{U} \cong L^2_+\mathcal{H}^k X, \quad L^2\mathcal{H}^k(\mathcal{U}, \partial\mathcal{U}) \cong L^2_-\mathcal{H}^k X.$$

Every harmonic form on X has an expansion in eigensections of $(d+\delta)_{\partial K}$

$$s(r, \theta) = \sum_\lambda s_\lambda(0) e^{-\lambda r} \phi_\lambda(\theta).$$

From which we immediately see that $L^2\mathcal{H}^k X = \{0\}$ for every k, and hence $L^2\mathcal{H}^k\mathcal{U} = L^2\mathcal{H}^k(\mathcal{U}, \partial\mathcal{U}) = \{0\}$.

Because $d + \delta$ is non-parabolic at infinity we know that the sequences

$$H^k(K, \partial K) \to \overline{H}_2^k(M) \to \overline{H}_2^k(\mathcal{U}) = 0,$$
$$0 = \overline{H}_2^k(\mathcal{U}, \partial\mathcal{U}) \to \overline{H}_2^k(M) \to H^k(K),$$

are both exact. Hence, since $H^k(K) = H^k(M)$ and $H^k(K, \partial K) = H_c^k(M)$, in the commutative diagram

$$\begin{array}{ccc} H_c^k(M) & \longrightarrow & H^k(M) \\ & \searrow \qquad \nearrow & \\ & \overline{H}_2^k(M) & \end{array}$$

we know that the downward arrow is surjective and the upward arrow is injective, so the result follows. □

In the next theorem we compute the reduced L^2 cohomology of strongly asymptotically Euclidean manifolds. Later, with the b-calculus, we will compute these groups for the larger class of asymptotically locally Euclidean (i.e., scattering) metrics.

Theorem 3.27. *If M is a complete manifold that is isometric at infinity to $\mathbb{R}^m$ then*

$$\overline{H}_2^k(M) \cong \begin{cases} H_c^k(M) & \text{if } k < m-1 \\ H^k(M) & \text{if } k > 1 \end{cases} \tag{3.26}$$

Proof. We prove a slightly more general result. We will show that (3.26) holds if we assume that there is a compact set $K \subseteq M$ with connect boundary ∂K so that $M \setminus K$ is (quasi-)isometric to $([1,\infty) \times \partial K, dr^2 + r^2 h)$, that the spectrum of the Laplacian on p-forms on $(\partial K, h)$ does not intersect $[-1/2, 1/2]$ for $0 < p < m-1$, and that $\overline{H}_2^k(\mathcal{U}) = 0$ for all $k < m-1$.

First we point out that the absence of harmonic p-forms on ∂K for $0 < p < m-1$ implies, by the long exact sequence

$$\begin{aligned} &H^0(K, \partial K) \to H^0(K) \to H^0(\partial K) \to \ldots \to H^{m-1}(K, \partial K) \\ &\quad \to H^{m-1}(K) \to H^{m-1}(\partial K) \to H^m(K, \partial K) \to H^m(K) \to H^m(\partial K), \end{aligned}$$

that $H^p(K) = H^p(K, \partial K)$ for $0 < p < m$, and hence $H^p(M) = H_c^p(M)$ for these p.

We know, from Proposition 3.25, that $D = d + \delta$ is non-parabolic at infinity, and our assumptions will allow us to rule out extended solutions. As in (3.24), we can write any form on $\mathcal{U}$ as

$$s(r,\theta) = \sum_\lambda \widetilde{s}_\lambda(r)\phi_\lambda(\theta) = \sum_\lambda r^{-\lambda} s_\lambda(r)\phi_\lambda(\theta).$$

The benefit of the second representation is that

$$Ds(r,\theta) = \sum_\lambda r^{-\lambda} s_\lambda'(r)\phi_\lambda(\theta).$$

Thus we see that a harmonic form is in the L^2 kernel if its expansion only involves $\lambda > 1/2$, and, using (3.23), a harmonic form is in the W kernel if its expansion only involves $\lambda > -1/2$. It follows that, if the spectral condition is satisfied, there are no extended solutions to $d+\delta$ acting on forms of degree $0 < k < m$.

Thus we can use the long exact sequences to compute $\overline{H}_2^k(M)$. From the sequence induced by the inclusion of $\mathcal{U}$ into M,

$$\begin{aligned} &H^0(K, \partial K) \to \overline{H}_2^0(M) \to \overline{H}_2^0(\mathcal{U}) \to \ldots \to H^{m-1}(K, \partial K) \\ &\quad \to \overline{H}_2^{m-1}(M) \to \overline{H}_2^{m-1}(\mathcal{U}) \to H^m(K, \partial K) \to \overline{H}_2^m(M) \to \overline{H}_2^m(\mathcal{U}) \end{aligned}$$

we see that $\overline{H}_2^k(M) = H^k(K, \partial K) = H_c^k(M)$ for all $k < m-1$, while from the sequence induced by the inclusion of K into M we see that $\overline{H}_2^k(M) = H^k(K) = H^k(M)$ for all $k > 1$.

Finally we point out that if M is isometric outside a compact set to $\mathbb{R}^m \setminus B_r(0)$ then the hypotheses above are satisfied. Indeed, polar coordinates on $\mathbb{R}^m$ are of the required form, the spectrum of the Hodge Laplacian on the sphere does not intersect $[-1/2, 1/2]$ except on functions and on $(m-1)$-forms[5], and the long exact sequence induced by the inclusion of $\mathbb{R}^m \setminus B_r(0)$ into M, together with the fact that $\overline{H}_2^k(\mathbb{R}^m) = 0$, shows that $\overline{H}_2^k(\mathcal{U}) = 0$ for $k < m-1$. □

Once we have developed the b-calculus we will show, following Hausel-Hunsciker-Mazzeo, that for a more general class of asymptotically locally Euclidean metrics,

$$L^2\mathcal{H}^k \cong \begin{cases} H^k(M, \partial M) & \text{if } k < \frac{m}{2} \\ \mathrm{Im}(H^k(M, \partial M) \to H^k(M)) & \text{if } k = \frac{m}{2} \\ H^k(M) & \text{if } k > \frac{m}{2} \end{cases}$$

3.7. **Other manifolds.**

Before moving on from L^2 cohomology we briefly summarize some of the results for other manifolds. This is not meant as a survey, but rather a suggestion of other things to look at. Among the topics we will not have time to even mention are L^2 cohomology of Galois covers of compact manifolds - where things are even homotopy invariant - and L^2 cohomology of complex manifolds.

The approach we have espoused above is based on work of Gilles Carron. Whereas we have computed the reduced L^2 cohomology of a manifold isometric to $\mathbb{R}^m$ at infinity, Carron computes the L^2 cohomology of arbitrary complete manifolds whose curvature vanishes outside a compact set.

Let M be the interior of a compact manifold with boundary, x a boundary defining function, and g a metric on M of the form

$$\frac{dx^2}{x^{2a+2}} + \frac{h(y, dy)}{x^{2b}},$$

near ∂M. These metrics include asymptotically cylindrical or b-metrics ($a = b = 0$) and asymptotically locally Euclidean or scattering metrics ($a = b = 1$). Yakov Shapiro has shown (in his 2007 MIT thesis) that, if $a \geq b \geq 0$,

$$L^2\mathcal{H}^k \cong \begin{cases} H^k(M, \partial M) & \text{if } k < \frac{m+1-a/b}{2} \\ \mathrm{Im}(H^k(M, \partial M) \to H^k(M)) & \text{if } \frac{m+1-a/b}{2} \leq k \leq \frac{m-1+a/b}{2} \\ H^k(M) & \text{if } k > \frac{m-1+a/b}{2} \end{cases}$$

Notice that $\frac{1}{2}(m+1-a/b) \leq \frac{m}{2} \leq \frac{1}{2}(m-1+a/b)$ with equality if $a = b$. The general case $b > a$ is still open, though conformally compact manifolds are one such example.

Notice that in these examples $\overline{H}_2^0(M) = H^0(M, \partial M) = 0$ which is consistent with the fact that

$$\dim \overline{H}_2^0(M) = \#\text{ of finite-volume connected components of } M.$$

[5] See [Iwasaki-Katase] for a discussion of the Hodge spectrum of $\mathbb{S}^n$.

However this suggests that if we use complete metrics of finite volume the rôles of relative and absolute cohomology should be reversed. Indeed, a theorem of Mazzeo-Phillips shows that the L^2 cohomology of non-compact hyperbolic manifolds of finite volume is given by

$$L^2\mathcal{H}^k \cong \begin{cases} H^k(M) & \text{if } k < \frac{m-1}{2} \\ \operatorname{Im}(H^k(M,\partial M) \to H^k(M)) & \text{if } \frac{m-1}{2} \leq k \leq \frac{m+1}{2} \\ H^k(M,\partial M) & \text{if } k > \frac{m+1}{2} \end{cases}$$

This has been generalized by Nader Yeganefar to apply[6] to complete manifolds with finite volume whose sectional curvatures satisfy

$$-b^2 \leq K \leq -a^2 < 0, \qquad \text{where } ma - (m-2)b > 0.$$

A class of manifolds extending manifolds with boundary is that of stratified manifolds. A stratification of $\overline{X}$ is a filtration by closed subsets,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m = \overline{X}.$$

so that the strata, $S_i = X_i \setminus X_{i-1}$ and $X^{\text{reg}} = \overline{X} \setminus X_{m-1}$, are smooth manifolds of (pure) dimension i and satisfy

$$X_{i-1} \subseteq \overline{S_i^\circ}.$$

Each stratum S_i has a 'tubular neighborhood' $\mathcal{T}_i$ in $\overline{X}$ which is the total space of a fibration

$$C(\widehat{L}_i) - \mathcal{T}_i \xrightarrow{\phi_i} S_i,$$

with fiber the cone over a stratified space $\widehat{L}_i$, and comes equipped with a continuous function $\rho_i : \mathcal{T}_i \to [0,2)$ which can be identified with the radial variable along the cone.

Intersection homology was introduced by Goresky-MacPherson as a way to define characteristic numbers for stratified manifolds. A generic cycle will intersect the strata transversally, but an arbitrary cycle need not. For intersection homology, one restricts to cycles such that, for some *preversity function* p,

> An i-cycle is allowed, for each k, to intersect X_{m-k} in a set of dimension at most $i - k + p(k)$.

Goresky-MacPherson restricted to stratifications with $X_{m-1} = X_{m-2}$ and pervesity functions satisfying

$$p(2) = 0 \quad \text{and} \quad p(k+1) \in \{p(k), p(k)+1\}.$$

The resulting groups are denoted $IH_k^{\mathbf{p}}(X)$. They are finitely generated and coincide with the ordinary cohomology groups when $p(k) \equiv 0$ and the ordinary homology groups when $p(k) = k - 2$. The associated cohomology theory is known as **intersection cohomology** and the groups are denoted $IH^k_{\mathbf{p}}(X)$.

[6]For $2k = m \pm 1$, Yeganefar's proof only applies when $a = b$.

If X^{reg} is oriented, intersection cohomology satisfies Poincaré Duality in that there is, for every k, a non-degenerate pairing

$$IH^k_{\mathbf{p}}(X) \times IH^{m-k}_{\mathbf{q}}(X) \to \mathbb{R},$$

where $\mathbf{q}$ is the *complementary perversity* to $\mathbf{p}$ namely $q(k) + p(k) = k - 2$. The upper and lower middle pervesity functions are, respectively,

$$\underline{m}(k) = \lfloor \tfrac{k-2}{2} \rfloor, \quad \overline{m}(k) = \lceil \tfrac{k-2}{2} \rceil$$

and, if all of the strata have even codimension (or the space is 'Witt') the middle pervesity intersection cohomology groups satisfy Poincaré Duality.

For the simplest non-trivial case of a stratification, consider a stratified manifold with only one singular stratum. We can think of these as manifolds with boundary whose boundary is the total space of a fibration:

$$Z - \partial M \xrightarrow{\phi} Y,$$

and endow them with metrics that near the boundary have, for instance, one of the forms

$$\begin{aligned} g_e &= \frac{dx^2}{x^2} + \frac{\phi^* g_Y}{x^2} + g_Z, \\ \text{or } g_\phi &= \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z, \\ \text{or } g_d &= \frac{dx^2}{x^2} + \phi^* g_Y + x^2 g_Z. \end{aligned}$$

These are associated to the stratified space X obtained by collapsing the fibers Z of ∂M. Hausel-Hunsicker-Mazzeo computed the reduced L^2 cohomology of metrics asymptotically like g_ϕ and g_d and found (with $b = \dim Y$, $f = \dim Z$)

$$L^2\mathcal{H}^k(g_\phi) = \begin{cases} \mathrm{Im}\left(IH^k_{f+\frac{b}{2}-k+\frac{1}{2}}(X) \to IH^k_{f+\frac{b}{2}-k-\frac{1}{2}}(X) \right) & \text{if } b \text{ odd} \\ IH^k_{f+\frac{b}{2}-k}(X) & \text{if } b \text{ even} \end{cases}$$

$$L^2\mathcal{H}^k(g_d) = \mathrm{Im}\left(IH^k_{\underline{m}}(X) \to IH^k_{\overline{m}}(X) \right).$$

Similarly, Hunsicker-Mazzeo computed the reduced L^2 cohomology of metrics asymptotically like g_e and found

$$L^2\mathcal{H}^k(g_e) = IH^k_{f+\frac{b}{2}-k}(X).$$

provided that k is not of the form $j + \frac{b+1}{2}$ with $H^j(Z) \neq 0$.

Among incomplete metrics we mention the seminal work of Cheeger, who showed that on a stratified manifold the L^2 cohomology of piecewise-flat metrics[7] coincides with the middle perversity intersection cohomology of the

[7] Also more general iterated 'admissable' metrics, see [Cheeger, On the Hodge Theory of Riemannian pseudomanifolds]

underlying space. Also, for incomplete metrics with one singular stratum, we mention the work of Hunsicker-Mazzeo for metrics of the form

$$dx^2 + \phi^* g_Y + x^2 g_Z$$

and the work of Hunsicker on metrics of the form $dx^2 + \phi^* g_Y + x^{2c} g_Z$. In all of these works the lack of completeness often requires the introduction of boundary conditions.

Finally, as an instance where more complicated spaces show up, we recall the Zuker conjecture (which was established by Looijenga and by Saper-Stern). Here the object of study is a Hermitian locally symmetric space X and the result is that the reduced L^2 cohomology of X is the middle perversity intersection cohomology of the Bailey-Borel-Satake compactification of X, denoted X^*. This space is obtained from X by first compactifying to a manifold with corners (Borel-Serre compactification), then collapsing boundary strata so that equivariant metrics extend non-degeneratively (reductive Borel-Serre compactification), and finally collapsing more strata until the complex structure extends non-degeneratively. These steps can be thought off as respectively topological, geometric, and algebraic. One reason for the interest in this result is that it is involved in extending the Langland's program to non-compact manifolds[8].

[8]See for instance [Kleiman, The development of intersection homology theory].

4. The b and zero calculi

In this section we will return to pseudo-differential operators. We start by discussing the extension of the symbolic calculus to manifolds of bounded geometry due to Meladze-Shubin and Kordyukov. This class of manifolds is large enough to include the asymptotically cylindrical and asymptotically hyperbolic manifolds. As we saw before, the symbolic calculus is often sufficient for the study of elliptic operators on closed manifolds. This is not true on non-compact manifolds partly because smoothing operators are not compact. This will motivate us to restrict the metrics further and introduce refined calculi of pseudo-differential operators.

4.1. **Bounded geometry and uniform differential operators.**

A Riemannian manifold has bounded geometry if its injectivity radius is positive and all covariant derivatives of its curvature tensor are uniformly bounded.

In more detail, recall that, at any point $\zeta \in M$, the exponential map is a diffeomorphism between a ball $B_r(0) \subseteq T_\zeta M$ and a neighborhood $\mathcal{U}_r(\zeta)$ of ζ in M. The injectivity radius at ζ is the supremum of the radii for which this is true; the injectivy radius of M, r_{inj} is the infimum over all $\zeta \in M$ of the injectivity radius at ζ. Thus a positive injectivity radius means that we can find a single $r > 0$ such that

$$\exp_\zeta : B_r(0) \to \mathcal{U}_r(\zeta)$$

will be a diffeomorphism for every $\zeta \in M$. This is very useful for patching together local constructions on a non-compact manifold. We point out that manifolds with positive injectivity radius are automatically complete. The second condition is simply that, for any $k \in \mathbb{N}_0$, there is a constant $C_k > 0$ such that

$$|\nabla^k R|_g(\zeta) \leq C_k, \text{ for all } \zeta \in M. \tag{4.1}$$

Examples of manifolds with bounded geometry include compact manifolds, Galois covers of compact manifolds (with pulled-back metrics), and homogeneous spaces (with metrics induced by an invariant metric on the underlying group). Two examples that we will deal with at length are b and zero metrics. Let M be the interior of a manifold with boundary, x a boundary defining function, y local coordinates along the boundary, and $h(x, y, dx, dy)$ a family of symmetric two tensors on M such that $h(0, y, dx, dy) = h_0(y, dy)$ is a metric on ∂M. Then an (exact) b-metric on M is one that in a collar neighborhood of the boundary takes the form

$$g_b = \frac{dx^2}{x^2} + h(x, y, dx, dy),$$

and an (exact) 0-metric is one that in a collar takes the form

$$g_0 = \frac{dx^2}{\alpha^2 x^2} + \frac{h(x, y, dx, dy)}{x^2}.$$

A 0-metric is another name for the conformally compact metrics we have already encountered in §3.5, while b metrics are a generalization of the cylindrical ends we studied in §3.6. Both of these metrics have bounded geometry[9].

We say that $f \in \mathcal{C}^\infty(M)$ is in $\mathcal{BC}^\infty(M)$ if the derivatives of f in normal coordinates in a neighborhood $\mathcal{U}_r(\zeta)$ as above are bounded

$$|D^\alpha f| \leq C_{|\alpha|},$$

by constants $C_{|\alpha|} > 0$ independent of the point ζ. A theorem of Eichhorn[10] shows that for manifolds with positive injectivity radius, (4.1) is equivalent to demanding that, in any neighborhood $\mathcal{U}_r(\zeta)$ the metric tensor in normal coordinates satisfies

$$|D^\alpha g_{ij}| \leq C_K, \quad |D^\alpha g^{ij}| \leq C_K. \tag{4.2}$$

That is, in normal coordinates g and g^{-1} are matrices valued in $\mathcal{BC}^\infty(M)$.

Most geometric constructions have 'uniform' or 'bounded' analogues. For instance, if E be a vector bundle on M then it is called a **bundle of bounded geometry** if it has trivializations over $\mathcal{U}_r(\zeta)$ (for small enough $r > 0$) whose transition functions are matrices valued in $\mathcal{BC}^\infty(M)$. The tangent bundle of a manifold with bounded geometry, as well as its tensor powers, are bundles of bounded geometry. A metric on a bundle of bounded geometry is a **bounded metric** if, in these trivializations, it satifies (4.2). A differential operator $\mathrm{Diff}(M; E, F)$ is **uniform** if in these trivializations it has the form

$$\sum a_\alpha(\zeta) D^\alpha$$

with a_α a matrix valued in $\mathcal{BC}^\infty(M)$. We denote the space of such operators by $\mathrm{Diff}_{\mathcal{B}}(M; E, F)$. Geometric operators associated to a metric of bounded geometry, such as the exterior derivative, its adjoint, or the Hodge Laplacian, are automatically uniform. The symbol of a uniform differential operator is an element of $\mathcal{BC}^\infty(S^*M, \pi^* \hom(E, F))$ – we say that $A \in \mathrm{Diff}_{\mathcal{B}}(M; E, F)$ is **uniformly elliptic** if its symbol is invertible with inverse in $\mathcal{BC}^\infty(S^*M, \pi^* \hom(F, E))$. In particular for these operators there are positive constants C_1 and C_2 (independent of $\zeta \in M$) such that

$$C_1 \leq |\sigma(A)(\zeta, \chi)| \leq C_2, \text{ for all } \chi \in S^*_\zeta M.$$

In the case of b and 0 metrics, matters are simplified by considering the compactification of M to a manifold with boundary, $\overline{M}$. [11] For instance,

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[9]See [Ammann-Lauter-Nistor, On the geometry of Riemannian manifolds with a Lie structure at infinity] for a discussion of bounded geometry in the context of 'boundary fibration structures'.

[10]See for instance [Schick, Manifolds with boundary and of bounded geometry] for a proof.

[11]For instance, for a b-metric, $\mathcal{BC}^\infty(M)$ is equal to the space $\mathcal{A}^0(\overline{M})$ described in §4.4.2.

vector fields of bounded pointwise length, such as those involved in normal coordinate charts are constrained to be in

$$\mathcal{V}_b = \mathrm{Span}_{\mathcal{C}^\infty(\overline{M})}\{x\partial_x, \partial_y\}, \quad \mathcal{V}_0 = \mathrm{Span}_{\mathcal{C}^\infty(\overline{M})}\{x\partial_x, x\partial_y\},$$

in the b and 0 cases respectively. We point out that $\mathcal{V}_*$ is a Lie subalgebra of the vector fields on M and (by the Serre-Swan theorem or Melrose's rescaling construction) is the space of sections of a vector bundle on M, denoted in each case by

$$\mathcal{V}_b = \Gamma({}^bTM), \quad \mathcal{V}_0 = \Gamma({}^0TM).$$

These vector bundles, known as the b-tangent bundle or the 0-tangent bundle, extend non-degenerately to $\overline{M}$ with simple descriptions:

$$\begin{gathered} {}^bTM = \left\{ V \in TM : V\big|_{\partial M} \text{ is tangent to } \partial M \right\}, \\ {}^0TM = \left\{ V \in TM : V\big|_{\partial M} = 0 \right\}. \end{gathered}$$

They are each isomorphic to TM, canonically over M, but not canonically over $\overline{M}$. A b-metric (and similarly a 0-metric) extends from TM to a non-degenerate metric on ${}^bTM$ over $\overline{M}$, hence (4.2) follows from compactness of $\overline{M}$. Bundles of bounded geometry and bounded metrics are just bundles and metrics that extend to $\overline{M}$.

A differential operator of order k is uniform with respect to a b-metric if in local coordinates it has the form

$$A = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j(\partial_y)^\alpha,$$

with $a_{j,\alpha} \in \mathcal{C}^\infty(\overline{M})$, such operators are known as b-differential operators and the space of such is denoted $\mathrm{Diff}^k_b(M;E,F)$. The symbol of a b-differential operator is

$$\sigma_b(A) = \sum_{j+|\alpha|=k} a_{j,\alpha}(x,y)\xi^j\eta^\alpha.$$

It is invariantly defined as an element of $\mathcal{C}^\infty({}^bS^*M, \pi^*\hom(E,F))$ when ξ is dual to $x\partial_x$ and η is dual to ∂_y. An operator $A \in \mathrm{Diff}^k_b(M;E,F)$ is uniformly elliptic as an element of $\mathrm{Diff}^k_{\mathcal{B}}(M;E,F)$ precisely when its b-symbol is invertible - in which case we say that A is b-elliptic. The same is true, *mutatis mutandis*, for 0-metrics – i.e., for these metrics the space of uniform differential operators is the same as the space of 0-differential operators and the uniformly elliptic operators are the 0-elliptic operators.

Finally, let us consider the case of forms, say for a 0-metric. Notice that

$$\begin{aligned} d(\alpha dy^I + \beta dx \wedge dy^J) &= (\partial_x\alpha)dx \wedge dy^I + (\partial_{y_i}\alpha)dy^i \wedge dy^I + (\partial_{y_i}\beta)dy^i \wedge dy^J \\ &= (x\partial_x\alpha)\frac{dx}{x} \wedge dy^I + (x\partial_{y_i}\alpha)\frac{dy^i}{x} \wedge dy^I + (x\partial_{y_i}\beta)\frac{dy^i}{x} \wedge dy^J, \end{aligned}$$

so to think of the exterior derivative as a 0-differential operator, we consider a non-standard extension of $\Omega^* M$ to $\overline{M}$. Namely, the bundle of zero forms

$$ {}^0\Omega^* M = \mathrm{Span}_{\mathcal{C}^\infty(\overline{M})} \left\{ \frac{dx}{x}, \frac{dy}{x} \right\} $$

is a bundle over $\overline{M}$ which is canonically isomorphic to $\Omega^* M$ in the interior of M, and the exterior derivative is a 0-differential operator of order one,

$$ d \in \mathrm{Diff}_0^1(M; {}^0\Omega^k M, {}^0\Omega^{k+1} M). $$

Sometimes, for emphasis, we might denote d by ${}^0 d$ when thinking of it as an operator between these bundles. It is easy to see that in the same way, δ, the formal adjoint of d, and Δ, the Hodge Laplacian, define operators

$$ \delta \in \mathrm{Diff}_0^1(M; {}^0\Omega^{k+1} M, {}^0\Omega^k M), \quad \Delta \in \mathrm{Diff}_0^2(M; {}^0\Omega^k M, {}^0\Omega^k M), $$

and that the latter is 0-elliptic. Naturally the same is true, *mutatis mutandis*, for b-metrics.

Remark 4.1. In the following when we wish to make a statement that is true for both the b and 0 calculus[12], we will use the moniker e. For instance, we may make a statement about ${}^eT^*M$ when we wish to make a statement that is true for both ${}^bT^*M$ and ${}^0T^*M$.

4.2. **Bounded geometry and uniform pseudo-differential operators.**

In general the difficulty with pseudo-differential operators on a non-compact manifold is that without restrictions on the growth of the Schwartz kernels they will not compose. On a manifold with bounded geometry we can get around this by restricting the support of the operators following Shubin[13].

We start with a localization result due to Gromov.

Lemma 4.2. *Let (M, g) be a manifold with bounded geometry. For every $0 < \varepsilon < r_{\mathrm{inj}}/3$ there is a countable covering of M by balls of radius ε,*

$$ M = \bigcup \mathcal{U}_\varepsilon(\zeta_i), $$

such that the cover by balls of double the radius with the same centers has bounded, finite multiplicity.

Proof. Let $\mathcal{T} = \{\mathcal{U}_{\varepsilon/2}(\zeta_i)\}$ be a maximal set of non-intersecting balls of radius $\varepsilon/2$. Notice that $\mathcal{T}$ is countable (because M is separable). Let $\zeta \in M$, since $\mathcal{T}$ is maximal, $\mathcal{U}_{\varepsilon/2}(\zeta)$ must intersect $\mathcal{U}_{\varepsilon/2}(\zeta_{i_0})$ for some $i_0 \in \mathbb{N}$ and hence $\zeta \in \mathcal{U}_\varepsilon(\zeta_{i_0})$. Since ζ was arbitrary we have $M = \bigcup \mathcal{U}_\varepsilon(\zeta_i)$.

To see that $\mathcal{U}_{2\varepsilon}(\zeta_i)$ is locally finite, choose $\widehat{\zeta} \in M$ and assume that $\widehat{\zeta} \in \mathcal{U}_{2\varepsilon}(\zeta_{i_k})$ for some subset $\{i_k\} \subseteq \mathbb{N}$. Then the balls $\mathcal{U}_{\varepsilon/2}(\zeta_{i_k})$ are disjoint

[12] There is an 'edge' calculus that includes both the b and 0 calculi as particular cases, see [Mazzeo: Elliptic edge operators].

[13] See [Shubin, Weak Bloch property and weight estimates for elliptic operators].

and contained in $\mathcal{U}_{3\varepsilon}(\widehat{\zeta})$. A simple consequence of bounded geometry is the existence of constants $c(r)$, $C(r)$ for every $r < r_{\text{inj}}$ such that

$$c(r) \leq \operatorname{Vol}(\mathcal{U}_r(\zeta)) \leq C(r),$$

independently of the point ζ. In particular, we see that there can not be more than $C(3\varepsilon)/c(\varepsilon/2)$ elements in $\{i_k\}$. □

This allows us to define a pair of partitions of unity $\{\phi_i\}$ and $\{\widetilde{\phi}_i\}$ with

$$\operatorname{supp}\phi_i \subseteq \mathcal{U}_\varepsilon(\zeta_i), \quad |D^\alpha \phi_i| < C_{|\alpha|}$$
$$\operatorname{supp}\widetilde{\phi}_i \subseteq \mathcal{U}_{2\varepsilon}(\zeta_i), \quad |D^\alpha \widetilde{\phi}_i| < \widetilde{C}_{|\alpha|}$$

for constants $C_{|\alpha|} > 0$, $\widetilde{C}_{|\alpha|} > 0$ independent of i and such that

$$\widetilde{\phi}_i\big|_{\operatorname{supp}(\phi_i)} \equiv 1.$$

We can use these to 'transfer' pseudo-differential operators from $\mathbb{R}^m$ to M, as follows.

An operator $A : \mathcal{C}^\infty_c(M; E) \to \mathcal{C}^\infty_c(M; F)$ is called a **uniform pseudo-differential operator** of order $s \in \mathbb{R}$, $A \in \Psi^s_{\mathcal{B}}(M; E, F)$, if its Schwartz kernel $\mathcal{K}_A \in \mathcal{C}^{-\infty}(M^2; \operatorname{Hom}(E, F))$ satisfies:

i) There is a $C_A > 0$ such that

$$\mathcal{K}_A(\zeta, \zeta') = 0 \text{ if } d(\zeta, \zeta') > C_A.$$

ii) Outside of the diagonal $\mathcal{K}_A$ is smooth uniformly in that for every $\delta > 0$, and any multi-indices α, β there is a constant $C_{\alpha\beta\delta} > 0$ such that

$$|D^\alpha_\zeta D^\beta_{\zeta'} \mathcal{K}_A(\zeta, \zeta')| \leq C_{\alpha\beta\delta}, \text{ whenever } d(\zeta, \zeta') > \delta.$$

iii) For any $i \in \mathbb{N}$, $\widetilde{\phi}_i A \phi_i$ is a pseudo-differential operator of order s in $B_{2\varepsilon}(0)$, whose full symbol satisfies the usual symbol estimates (1.2) with bounds independent of i.

We always assume that the symbols are (one-step) polyhomogeneous.

As usual, we denote $\cap_s \Psi^s_{\mathcal{B}}(M; E, F)$ by $\Psi^{-\infty}_{\mathcal{B}}(M; E, F)$. An operator $A : \mathcal{C}^\infty_c(M; E) \to \mathcal{C}^\infty_c(M; F)$ is in $\Psi^{-\infty}_{\mathcal{B}}(M; E, F)$ precisely when its Schwartz kernel is in $\mathcal{BC}^\infty(M^2; \operatorname{Hom}(E, F))$ and there is a constant $C_A > 0$ such that $\mathcal{K}_A(\zeta, \zeta') = 0$ if $d(x, y) > C_A$. These operators are smoothing (i.e., they map $\mathcal{C}^{-\infty}(M; E)$ to $\mathcal{C}^\infty(M; F)$) but not every smoothing operator is in $\Psi^{-\infty}_{\mathcal{B}}(M; E, F)$.

Theorem 4.3. *Uniform pseudo-differential operators compose - i.e., if E, F, and G are bounded vector bundles, then, for any $s, t \in \mathbb{R} \cup \{-\infty\}$,*

$$\Psi^s_{\mathcal{B}}(M; E, F) \circ \Psi^t_{\mathcal{B}}(M; F, G) \subseteq \Psi^{s+t}_{\mathcal{B}}(M; E, G).$$

Furthermore the principal symbol is a homomorphism and fits into a short exact sequence

$$0 \to \Psi^{s-1}(M; E, F) \to \Psi^s(M; E, F) \xrightarrow{\sigma} \mathcal{BC}^\infty(S^*M, \pi^* \hom(E, F)) \to 0.$$

Proof. Let $A \in \Psi^s_{\mathcal{B}}(M; E, F)$ and $B \in \Psi^t_{\mathcal{B}}(M; F, G)$, then we have

$$A \circ B = \sum_{i,j,k,\ell} (\widetilde{\phi}_i A \phi_j) \circ (\widetilde{\phi}_k B \phi_\ell)$$

and both results follow from the corresponding results on $\mathbb{R}^m$ (or $\mathbb{T}^m$). □

Corollary 4.4.

1) *If $P \in \Psi^s_{\mathcal{B}}(M; E, F)$ is uniformly elliptic then there is an operator Q in $\Psi^{-s}_{\mathcal{B}}(M; F, E)$ such that*

$$QP - \mathrm{Id}_E \in \Psi^{-\infty}_{\mathcal{B}}(M; E) \text{ and } PQ - \mathrm{Id}_F \in \Psi^{-\infty}_{\mathcal{B}}(M; F). \tag{4.3}$$

2) *If $P \in \Psi^0_{\mathcal{B}}(M; E, F)$ then P defines a bounded operator*

$$P : L^2(M; E) \to L^2(M; F).$$

3) *If $P \in \Psi^s_{\mathcal{B}}(M; E, F)$, $s > 0$, and P is uniformly elliptic, then, as an unbounded operator on L^2 with core domain $\mathcal{C}^\infty_c(M; E)$, P has a unique closed extension, i.e., we have*

$$\mathcal{D}_{\min}(P) = \mathcal{D}_{\max}(P).$$

In particular if a uniformly elliptic operator of order $s > 0$ is formally self-adjoint, then it is essentially self-adjoint.

Remark 4.5. If P and Q satisfy (4.3), we say that Q is an **interior parametrix** for P. It turns out that operators in $\Psi^{-\infty}_{\mathcal{B}}(M)$ are not necessarily compact so Q is not a parametrix for P in the functional analytic sense.

Proof. Having established the symbolic calculus, these properties follow as on closed manifolds. As explained in Proposition 2.1, (1) is a direct consequence of the symbolic calculus and asymptotic completeness (which also holds for the uniform calculus). As explained in Proposition 2.4, (2) is reduced by the symbolic calculus to the case $P \in \Psi^{-\infty}_{\mathcal{B}}(M; E, F)$, which follows from the following lemma and the description of $\Psi^{-\infty}_{\mathcal{B}}(M; E, F)$ above.

Lemma 4.6 (Schur's test)**.** *If P is an integral operator whose integral kernel $\mathcal{K}_P(\zeta, \zeta')$ is a measurable function satisfying*

$$\int |\mathcal{K}(\zeta, \zeta')| \, d\zeta \leq C \text{ for almost every } \zeta',$$
$$\int |\mathcal{K}(\zeta, \zeta')| \, d\zeta' \leq C \text{ for almost every } \zeta$$

then P is a bounded operator on L^2, with operator norm bounded by C.

Proof. For any $h \in L^2(M)$,

$$\begin{aligned}
\|Ph\|^2 &\leq \int \left(\int |\mathcal{K}(\zeta, \zeta')h(\zeta')| \, d\zeta' \right)^2 \, d\zeta \\
&\leq \int \left(\int |\mathcal{K}(\zeta, \zeta')| \, d\zeta' \right) \left(\int |\mathcal{K}(\zeta, \zeta')||h(\zeta')|^2 \, d\zeta' \right) \, d\zeta \\
&\leq C \int \int |\mathcal{K}(\zeta, \zeta')||h(\zeta')|^2 \, d\zeta' \, d\zeta \leq C^2 \int |h(\zeta')|^2 \, d\zeta' = C^2 \|h\|^2
\end{aligned}$$

□

Finally, (3) follows from (1) as explained in Corollary 2.9. □

Before continuing, we pause to define Sobolev spaces as we did on closed manifolds. For any $s \in \mathbb{R}^+$, we define

$$H^s_{\mathcal{B}}(M, E) = \left\{ u \in L^2(M, E) : Pu \in L^2(M, E) \text{ for every } P \in \Psi^s_{\mathcal{B}}(M, E) \right\}$$

and then, as in Corollary 2.8, we have

$$\text{(4.4)} \quad H^s_{\mathcal{B}}(M, E) = \{ u \in L^2(M, E) : \\ Pu \in L^2(M, E) \text{ for some uniformly elliptic } P \in \Psi^s_{\mathcal{B}}(M, E) \}.$$

For $s < 0$ we define $H^s_{\mathcal{B}}(M, E)$ by duality from $H^{-s}_{\mathcal{B}}(M; E)$. (Note that in (4.4) we can not just start with $u \in \mathcal{C}^{-\infty}(M, E)$ as we could on a closed manifold since, for instance, this would not exclude constants.) Also as before we can put a norm on these spaces by choosing a fixed family of invertible uniform pseudo-differential operators of order s, say D_s, and defining

$$\|u\|_{H^s_{\mathcal{B}}} = \|D_s u\|_{L^2} .$$

An alternate approach is to define Sobolev norms locally and patch them together. Thus we can use a partition of unity as above to define

$$(\|u\|'_{H^s_{\mathcal{B}}})^2 = \sum \|\phi_i u\|^2_{H^s(B_\varepsilon(\zeta_i))} ,$$

and then define $H^s_{\mathcal{B}}(M, E)$ as the completion of $\mathcal{C}^\infty_c(M, E)$ with respect to this norm. Another approach would define the norm, for $s \in \mathbb{N}$, as

$$(\|u\|''_{H^s_{\mathcal{B}}})^2 = \sum_{k=0}^{s} \int_M |\nabla^k u(\zeta)|^2 \, d\zeta,$$

and again define $H^s_{\mathcal{B}}(M, E)$ as the completion of $\mathcal{C}^\infty_c(M, E)$ with respect to this norm. Because the operators D_s, ∇^k are uniform it is easy to see that these three approaches define the same space with equivalent norms. With these definitions we can continue listing some of the consequences of the symbolic calculus for metrics of bounded geometry.

Corollary 4.7.

4) *If $P \in \Psi^s_{\mathcal{B}}(M;E,F)$ then, for any $t \in \mathbb{R}$, P defines a bounded operator*
$$P : H^t_{\mathcal{B}}(M;E) \to H^{t-s}_{\mathcal{B}}(M;F).$$
5) *If $P \in \Psi^s_{\mathcal{B}}(M;E,F)$ is uniformly elliptic, then*
$$Pu \in H^t(M,F) \iff u \in H^{t+s}(M,E).$$
In particular, if $s > 0$ then, as an unbounded operator on $L^2(M,E)$, the unique closed extension of P has domain $H^s(M,E)$.
6) *If $P \in \Psi^s_{\mathcal{B}}(M;E,F)$ is uniformly elliptic, then there is a constant $C > 0$ such that*

$$\|u\|_{H^{t+s}(M,E)} \leq C(\|Pu\|_{H^s(M,F)} + \|u\|_{H^s(M,E)}). \tag{4.5}$$

Remark 4.8. The basic elliptic estimate (4.5) exhibits the need for uniform estimates. Recall that we obtain it from a symbolic parametrix via

$$\|u\|_{H^{k+s}(M,E)} = \|QPu + Ru\|_{H^{k+s}(M,E)}$$
$$\leq \|QPu\|_{H^{k+s}(M,E)} + \|Ru\|_{H^{k+s}(M,E)} \leq C_1 \|Pu\|_{H^k(M,F)} + C_2 \|u\|_{H^k(M,E)}.$$

Since Q is well-defined up to an element of $\Psi^{-\infty}_{\mathcal{B}}(M;E)$ the analysis of Corollary 2.6 and Remark 2.7 shows that the largest value of C_1 is

$$\sup_{S^*M} |\sigma(Q)(\zeta,\chi)| = \inf_{S^*M} |\sigma(P)(\zeta,\chi)|^{-1},$$

hence we need to demand that P be *uniformly* elliptic to guarantee that this constant is greater than zero.

It is often useful to work on weighted versions of the Sobolev spaces defined above. For any positive uniform function $\omega \in \mathcal{BC}^\infty(M)$[14], and any $t \in \mathbb{R}$ we define

$$\omega H^t_{\mathcal{B}}(M,E) = \{\omega u : u \in H^t_{\mathcal{B}}(M,E)\}, \quad \|v\|_{\omega H^t_{\mathcal{B}}} = \left\|\omega^{-1} v\right\|_{H^t_{\mathcal{B}}}.$$

Since multiplication by ω defines a uniformly elliptic element of $\Psi^0_{\mathcal{B}}(M;E)$, the analogues of Corollaries 4.4 and 4.7 hold for weighted Sobolev spaces. (For instance, $P \in \Psi^s_{\mathcal{B}}(M;E)$ implies $\omega^{-1}P\omega \in \Psi^s_{\mathcal{B}}(M;E)$, and one of these is uniformly elliptic iff the other one is.)

In the next section we explain the conspicuous lack of properties involving Fredholmness from Corollaries 4.4 and 4.7.

4.3. **Compact operators on non-compact manifolds.**

The important difference between the parametrix construction on a manifold with bounded geometry and that on a closed manifold is that, in the former, the error, though smoothing, is not compact! Indeed, we will see that the naïve extension of Rellich's theorem to non-compact manifolds fails and find a replacement in Corollary 4.10 below.

[14] More generally, we could allow $\omega \in \mathcal{BC}^\infty(M;\hom(E))$ as long as ω is invertible with inverse in $\mathcal{BC}^\infty(M;\hom(E))$.

In the next theorem we will characterize pre-compact subsets of $L^2(M)$ for M complete. We formulate the theorem so as to include weighted Sobolev spaces as well as spaces such as $W_P(E)$.

Theorem 4.9. *Let M be a manifold, E a bundle over M, and $\mathcal{H}(E)$ a Hilbert space of sections of E such that sections with disjoint support are orthogonal. Then a bounded subset, $\mathcal{S}$, of $\mathcal{H}(E)$ is precompact if*

i) *The restriction of $\mathcal{S}$ to any compact subset of M is pre-compact,*

ii) *For every $\varepsilon > 0$, there is a compact subset $K \subseteq M$ such that*

$$\left\| s\big|_{M\setminus K} \right\|_{\mathcal{H}(E)} \leq \varepsilon \quad \textit{for every } s \in \mathcal{S}.$$

And, if $\mathcal{C}_c^\infty(M, E)$ is dense in $\mathcal{H}(E)$, then every precompact subset of $\mathcal{H}(E)$ is bounded and satisfies (i) and (ii).

If $\mathcal{H}(E)$ is a weighted L^2 space, then (ii) is equivalent to:

ii') *$\mathcal{S}$ is a bounded subset of $\omega\mathcal{H}(E)$ for some continuous positive function ω such that*

$$\lim_{\zeta\to\infty} \omega(\zeta) = 0.$$

Proof. First we show that if $\mathcal{S}$ satisfies (i) and (ii) then it is pre-compact by showing that any sequence $(s_k) \subset \mathcal{S}$ has a Cauchy subsequence. Let K_ℓ be an increasing sequence of compact subsets of M that exhaust M, without loss of generality we can assume that

$$\left\| s\big|_{M\setminus K_\ell} \right\|_{\mathcal{H}(E)} \leq \frac{1}{\ell} \quad \text{for every } s \in \mathcal{S}.$$

The restriction of (s_k) to any K_ℓ is pre-compact by (i), so by iteratively restricting to a subsequence and finally choosing a diagonal subsequence we can assume that (s_k) is Cauchy on each K_ℓ. Let $\varepsilon > 0$, $\ell > \frac{1}{\varepsilon}$ and $N \in \mathbb{N}$ large enough so that

$$\left\| (s_i - s_j)\big|_{K_\ell} \right\|_{\mathcal{H}(E)} \leq \varepsilon \text{ for all } i, j > N,$$

then, for any such i, j, we have

$$\|s_i - s_j\|^2_{\mathcal{H}(E)} \leq \left\|(s_i - s_j)\big|_K\right\|^2_{\mathcal{H}(E)} + \left\|(s_i - s_j)\big|_{M\setminus K}\right\|^2_{\mathcal{H}(E)} \leq \varepsilon^2 + 4\varepsilon^2,$$

which shows that (s_n) is Cauchy.

Next we assume that $\mathcal{C}_c^\infty(M; E)$ is dense in $\mathcal{H}(E)$ and we prove the converse. Clearly, if $\mathcal{S}$ is pre-compact, then (i) is true because restriction to a compact set is a uniformly continuous map in $\mathcal{H}(E)$ (e.g., because sections with disjoint support are orthogonal so $\|s\|_{\mathcal{H}(E)} \geq \left\|s\big|_K\right\|$). Furthermore, since $\mathcal{H}(E)$ is a complete metric space, we know that $\mathcal{S}$ will be pre-compact if and only if it is totally bounded. That is, for every $\varepsilon > 0$ there is a finite set $\{f_1, \ldots, f_N\} \subseteq \mathcal{H}(E)$ so that

$$s \in \mathcal{S} \implies \min\left\{\|s - f_i\|_{\mathcal{H}(E)}\right\} < \varepsilon.$$

Since $\mathcal{C}_c^\infty(M)$ is dense in $\mathcal{H}(E)$, the f_i can always be chosen in $\mathcal{C}_c^\infty(M)$. Thus, for any $\varepsilon > 0$, we choose $f_i \subseteq \mathcal{C}_c^\infty(M)$ as above and then $K = \cup\,\mathrm{supp}(f_i)$ satisfies (ii).

Finally we show that (ii) and (ii') are equivalent for weighted L^2 spaces. The key property we need is that, for these spaces,

$$\text{if } h \text{ satisfies } |h| \leq C \text{ then } \|hs\|_{\mathcal{H}(E)} \leq C\,\|s\|_{\mathcal{H}(E)}\,. \tag{4.6}$$

Assume (ii) and let μ be a smooth proper function $M \to \mathbb{R}^+$, we can find an increasing sequence of integers n_ℓ so that $\widehat{K}_\ell = \mu^{-1}([0, n_\ell])$ satisfies

$$\left\| s\big|_{M\setminus\widehat{K}_\ell} \right\|_{\mathcal{H}(E)} \leq \frac{1}{2^\ell} \quad \text{for every } s \in \mathcal{S}$$

(i.e., n_ℓ large enough so that $K_{2^\ell} \subseteq \mu^{-1}([0,n_\ell])$). Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be any continuous function such that

$$f([0,n_1]) = 1, \quad f([n_\ell, n_{\ell+1}]) \subseteq [2^{\ell-1}, 2^\ell],$$

then using (4.6) we have, for any $s \in \mathcal{S}$,

$$\begin{aligned} \|(f\circ\mu)s\|^2_{\mathcal{H}(E)} &= \left\|(f\circ\mu)s\big|_{K_1}\right\|^2_{\mathcal{H}(E)} + \sum_{\ell\geq 1}\left\|(f\circ\mu)s\big|_{K_{\ell+1}\setminus K_\ell}\right\|^2_{\mathcal{H}(E)} \\ &\leq \|s\|^2_{\mathcal{H}(E)} + \sum_{\ell\geq1} 2^\ell \left\| s\big|_{K_{\ell+1}\setminus K_\ell}\right\|^2_{\mathcal{H}(E)} \\ &\leq \|s\|^2_{\mathcal{H}(E)} + \sum_{\ell\geq 1}\frac{2^\ell}{4^\ell} = \|s\|^2_{\mathcal{H}(E)} + 1. \end{aligned}$$

Hence $\mathcal{S}$ is a bounded subset of $\omega\mathcal{H}(E)$ with

$$\omega = (f\circ\mu)^{-1/2},$$

which clearly satisfies $\omega \to 0$ as $\mu \to \infty$.

On the other hand, if we assume (4.6) and (ii'), so that $\|\omega^{-1}s\|_{\mathcal{H}(E)} \leq \mathcal{L}$ for some $\mathcal{L} > 0$ and all $s \in S$, then for any $\varepsilon > 0$, we can find a compact set K such that

$$\zeta \in M\setminus K \implies \omega(\zeta) < \frac{\varepsilon}{\mathcal{L}}.$$

Thus, for any $s \in \mathcal{S}$, we have

$$\left\| s\big|_{M\setminus K}\right\|_{\mathcal{H}(E)} = \left\|\omega\frac{s}{\omega}\big|_{M\setminus K}\right\|_{\mathcal{H}(E)} < \frac{\varepsilon}{\mathcal{L}}\left\|\frac{s}{\omega}\big|_{M\setminus K}\right\|_{\mathcal{H}(E)} \leq \varepsilon,$$

and hence (ii) is satisfied. □

With this theorem we have the replacement of Rellich's theorem for manifolds with bounded geometry.

Corollary 4.10. *The inclusions $H^s_{\mathcal{B}}(M) \hookrightarrow H^t_{\mathcal{B}}(M)$ for $s > t$ are not compact, but, if $\omega \in \mathcal{BC}^\infty(M)$ is positive and satisfies $\lim_{\zeta\to\infty}\omega(\zeta) = 0$, then the inclusion*

$$\omega H^s_{\mathcal{B}}(M) \hookrightarrow H^t_{\mathcal{B}}(M)$$

is compact for any $s > t$.

Proof. First assume that $t = 0$.

For any $s > 0$ we can find a sequence $(u_k) \subseteq \mathcal{C}_c^\infty(M)$ with:

$$\|u_k\|_{H^s_{\mathcal{B}}(M)} \leq 1, \quad \|u_k\|_{L^2(M)} \geq C,$$
$$\text{and } \operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset \text{ whenever } i \neq j,$$

e.g., by using the open balls $\mathcal{U}_{\varepsilon/2}(\zeta_i)$ from the proof of Lemma 4.2. Since u_k conveges weakly to zero in L^2, it can not have a strongly convergent subsequence, and hence the unit ball of $H^s_{\mathcal{B}}(M)$ is not compact in $L^2(M)$.

On the other hand, $\omega H^s_{\mathcal{B}}(M)$ satisfies property (i) of Theorem 4.9 by Rellich's theorem for compact manifolds and clearly satisfies property (ii'), hence $\omega H^s_{\mathcal{B}}(M)$ includes compactly into $L^2(M)$ if $s > 0$.

For general $t \in \mathbb{R}$, we can factor the inclusion using any invertible $D_t \in \Psi^t_{\mathcal{B}}(M)$,

$$\begin{array}{ccc} \omega H^s_{\mathcal{B}}(M) & \hookrightarrow & H^t_{\mathcal{B}}(M) \\ {\scriptstyle D_t}\downarrow\cong & & {\scriptstyle D_t}\uparrow\cong \\ \omega H^{s-t}_{\mathcal{B}}(M) & \hookrightarrow & L^2(M) \end{array}$$

and thus reduce to the case $t = 0$. □

4.4. **Stretched products and normal operators.**

We have seen that on any Riemannian manifold with bounded geometry we have a calculus of uniform pseudo-differential operators. The uniformly elliptic operators have symbolic parametrices, but these are not parametrices in the functional analytic sense because the residues - though smoothing - are not compact. In order for the residue terms to be compact we need for them to vanish at infinity. It is very helpful when the manifold has a natural compactification, since then we have the more concrete problem of studying the behavior of the operators at the boundary. Such is the case for b and 0 metrics.

Let M be the interior of a compact manifold with boundary. The Schwartz kernel $\mathcal{K}_A$ of a pseudo-differential operator, say a uniform pseudo-differential operator for either a b or a 0-metric, is a distribution on M^2 that is singular along diag_M and supported in a metric neighborhood of diag_M. We can compactify M^2 naturally to $\overline{M}^2$ and consider $\mathcal{K}_A$ as a distribution on $\overline{M}^2$, but the behavior at the corner ∂M^2 is complicated as the support of $\mathcal{K}_A$ is 'collapsing' here (see Figure 1). Not just that, but as a distribution $\mathcal{K}_A$ is conormal with respect to the diagonal and it is not clear what this means at a corner.

In the same spirit as the introduction of ${}^bTM$, ${}^0TM$ and associated bundles, the solution is to consider an alternate compactification of M^2. We

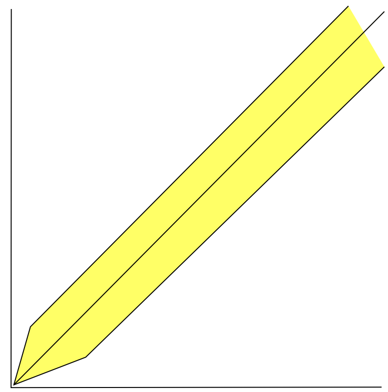


FIGURE 1. The support of $\mathcal{K}_A$ in $\overline{M}^2$

will describe a manifold with corners in which the (extension of the) diagonal is suitably transverse to the boundary. The picture we are aming for is represented in Figure 2.

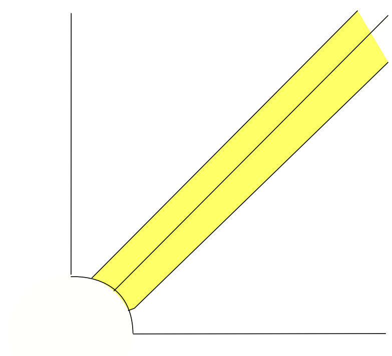


FIGURE 2. The support of $\mathcal{K}_A$ in an alternate compactification of M^2

In either case we obtain the alternate compactification of M^2 by 'blowing-up' submanifolds of the corner of $\overline{M}^2$. The simplest example of a blow-up is the introduction of polar coordinates around the origin in the plane. This effectively replaces $\mathbb{R}^2$ with $\mathbb{R}^+ \times \mathbb{S}^1$ with the main change being that the origin in $\mathbb{R}^2$ is replaced with a whole circle of points in $\mathbb{R}^+ \times \mathbb{S}^1$. The return to Cartesian coordinates is an example of a 'blow-down map'.

More generally we will need to blow-up submanifolds of manifolds with corners. This is defined for a submanifold Y of X if it is a **p-submanifold**. This means that, any point $p \in Y$ has a neighborhood $\mathcal{U} \subseteq X$, such that

$$\begin{aligned} X \cap \mathcal{U} &= X' \times X'', \quad \text{where } \partial X'' = \emptyset, \\ Y \cap \mathcal{U} &= X' \times \{p''\}, \quad \text{for some } p'' \in X''. \end{aligned} \tag{4.7}$$

Thus the diagonal in Figure 1 is *not* a p-submanifold, but the diagonal in Figure 2 is. Also the origin in $\mathbb{R}^2$ or in $(\mathbb{R}^+)^2$ is a p-submanifold. If Y is a p-submanifold of X, the blow-up of Y in X, denoted $[X;Y]$ is defined to be

$$[X;Y] = (X \setminus Y) \bigsqcup (S^+Y),$$

where S^+Y is the interior spherical normal bundle of Y in X. This is endowed with the unique minimal differential structure with respect to which smooth functions on X and polar coordinates around Y are smooth. Blowing-up Y introduces a new boundary face instead of Y[15]. There is a natural blow-down map

$$\beta : [X;Y] \to X$$

given by collapsing the fibers of S^+Y. For the example of polar coordinates around the origin we have

$$\begin{aligned} [\mathbb{R}^2; \{(0,0)\}] = \mathbb{R}^+ \times \mathbb{S}^1 &\xrightarrow{\quad\beta\quad} \mathbb{R}^2 \\ (r,\theta) &\longmapsto (r\cos\theta, r\sin\theta) \end{aligned}$$

What should we blow-up to understand the integrals kernels of pseudo-differential operators for b and 0 metrics?

Consider first the case of a 0 metric. Points in the support of $\mathcal{K}_A$ are close to each other in the interior. As we let these points evolve via the geodesic flow in the direction $-x\partial_x$ they will either get exponentially closer to each other or exponentially further away from each other (since the curvature at infinity is negative). In the latter case they will leave the support of $\mathcal{K}_A$, so it is only the former situation that is interesting. This suggests that at the boundary the integral kernel will be supported on $\mathrm{diag}_{\partial M}$, so this is the set we should blow-up to understand these kernels. It is easily checked that $\mathrm{diag}_{\partial M}$ is a p-submanifold of $\overline{M}^2$, so we define

$$M_0^2 = [\overline{M}^2; \mathrm{diag}_{\partial M}],$$

and denote the corresponding blow-down map by $\beta_0 : M_0^2 \to \overline{M}^2$. This space is often reffered to as the **0-stretched product** or the **0-double space**. An important (p-)submanifold of M_0^2 is the 'zero diagonal', diag_0, defined by

$$\mathrm{diag}_0 = \mathrm{closure}\left(\beta_0^{-1}(\mathrm{diag}_{M^\circ})\right),$$

where we use M° to emphasize that we mean the interior of $\overline{M}$.

The geometry of a b-metric is asymptotically cylindrical. Thus points that are close together will stay close together. This suggests that to resolve the kernels of uniform pseudo-differential operators associated to b-metrics

[15]This process is sometimes reffered to as 'radial blow-up' to dinstinguish from the blow-ups used in algebraic geometry. In the latter, one would glue in a projective bundle and not introduce any new boundary faces.

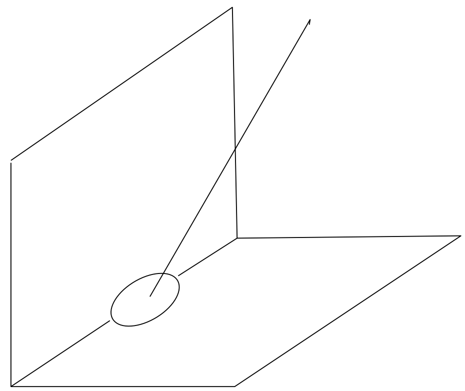


FIGURE 3. The 0-stretched product and the 0-diagonal

we should blow-up the whole corner $(\partial M)^2$. It is easily checked that $(\partial M)^2$ is a p-submanifold of $\overline{M}^2$ and we define

$$M_b^2 = [\overline{M}^2; (\partial M)^2],$$

and denote the corresponding blow-down map $\beta_b : M_b^2 \to \overline{M}^2$. As expected, this space is reffered to as the b**-stretched product** or the b**-double space**. The b-diagonal, diag_b, is defined analogously to diag_0.

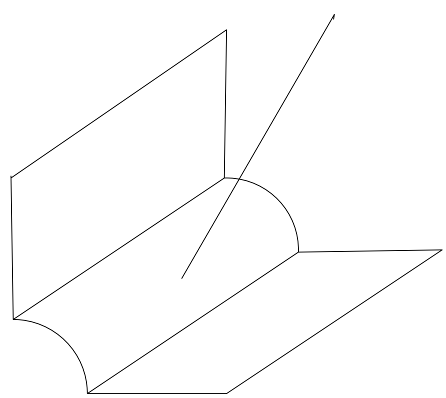


FIGURE 4. The b-stretched product and the b-diagonal

Each of these spaces has three boundary hypersurfaces, we denote the left and right boundary faces by $\mathcal{B}_{10}$ and $\mathcal{B}_{01}$ respectively and the front face (the only one that intersects the lifted diagonal) by $\mathcal{B}_{11}$. Similarly we denote boundary defining functions of these faces by ρ_{10}, ρ_{01} and ρ_{11} respectively.

In order to define the b and 0 pseudo-differential operators, we need to discuss the allows behavior of a distribution on a stretched double-space. We will require that distributions be conormal to the lifted diagonals and have polyhomogeneous expansions at the boundary faces.

4.4.1. *Behavior at the diagonal.* First, let us recall the definition of pseudo-differential operators on a closed manifold Z (of dimension n) in terms of their Schwartz kernels. If $\Omega^{1/2}$ denotes the trivial bundle of half-densities on Z, then

$$A \in \Psi^s(Z) \iff \mathcal{K}_A \in I^s(Z^2, \mathrm{diag}_Z; \Omega^{1/2}).$$

Here $I^s(Z^2, \mathrm{diag}_Z; \Omega^{1/2})$ denotes the space of distributions on Z^2 with values in $\Omega^{1/2}$ that are conormal with respect to diag_Z. This means that in any coordinate chart on Z^2 that does not intersect the diagonal $\mathcal{K}_A$ is a smooth section of $\Omega^{1/2}$, while, in any coordinate chart $\mathcal{U}$ that intersects the diagonal (and a corresponding trivialization of $\Omega^{1/2}$), in local coordinates $(\overline{\zeta}', \overline{\zeta}'')$ such that $\mathrm{diag}_Z \cap \mathcal{U} = \{\overline{\zeta}' = 0\}$ we have

$$\mathcal{K}_A\big|_{\mathcal{U}} = \frac{1}{(2\pi)^n} \int e^{i\overline{\zeta}'\cdot\xi} a(\overline{\zeta}'', \xi)\, d\xi,$$

with a a symbol of order $s + \frac{n}{4}$. Although this description involves coordinate charts, the space of conormal distributions is invariantly defined.

In order for this description to apply to a submanifold of a manifold with corners, we need to be able to choose coordinates $(\overline{\zeta}', \overline{\zeta}'')$ so that the vanishing of the first few coordinates $\overline{\zeta}'$ defines the submanifold – this is precisely the condition that defines a p-submanifold. So given a manifold with corners X, an emdedded p-submanifold Y, and a vector bundle E over X, we define the space of distributional sections **conormal to** Y, $I^s(X, Y; E) \subseteq \mathcal{C}^{-\infty}(X; E)$, as above. i.e., by requiring that elements in $I^s(X, Y; E)$ restricted to $X \setminus Y$ be smooth sections of E and restricted to neighborhoods of points $p \in Y$ as in (4.7) be smooth functions valued in $I^{s+\frac{n}{4}}(X'', \{p''\}; E)$.

4.4.2. *Behavior at the boundary.* We shall also impose conditions on the Schwartz kernels at the boundary, namely that they be **polyhomogeneous conormal** functions. These are distributions that at each boundary face have asymptotic expansions in a corresponding bdf, possibly involving logarithms and non-integer powers.

Let X be a manifold with corners and let $\mathcal{B}_1, \dots, \mathcal{B}_N$ be the boundary hypersurfaces of X. We assume that each $\mathcal{B}_k$ is embedded so that there are boundary defining functions ρ_k and collar neighborhoods of $\mathcal{B}_k$ in X,

$$\mathcal{U} \cong [0, 1)_{\rho_k} \times \mathcal{B}_k. \tag{4.8}$$

A function u on X is conormal to the boundary, $u \in \mathcal{A}^0(X)$, if, whenever $V_1, \dots, V_\ell$ is a collection of vector fields tangent to all of the boundary faces of X we have

$$V_1 V_2 \cdots V_\ell u \in L^\infty(X).$$

Given multi-indices $\mathbf{s} \in \mathbb{C}^N$, $\mathbf{p} \in \mathbb{N}_0^N$, we define $\mathcal{A}^{\mathbf{s},\mathbf{p}}(X) = \rho^{\mathbf{s}}(\log \rho)^{\mathbf{p}} \mathcal{A}^0(X)$. The union over all $\mathbf{s}$ and $\mathbf{p}$ is the space of functions conormal to the boundary, and we denote it by $\mathcal{A}^*(X)$.

If $u \in \mathcal{A}^0(X)$ extends smoothly to a manifold containing X in its interior, then u has 'Taylor expansions' at each boundary face and we write $u \in \mathcal{A}^0_{\mathrm{phg}}(X)$. If $u \in \mathcal{A}^0_{\mathrm{phg}}(X)$ and the Taylor expansion of u at each boundary face are identically zero, i.e., if u vanishes to infinite order at each boundary face, we write $u \in \dot{\mathcal{C}}^\infty(X)$. If $u \in \mathcal{A}^0_{\mathrm{phg}}$ vanishes to infinite order at every boundary hypersurface expect say $\mathcal{B}$, then we write $u \in \dot{\mathcal{C}}^\infty_{\mathcal{B}}(X)$.

The larger space of **polyhomogeneous conormal functions**, $\mathcal{A}^*_{\mathrm{phg}}(X)$, is defined inductively using the maximum codimension of the corners of X. If X does not have boundary, then $\mathcal{A}^*_{\mathrm{phg}}(X) = \mathcal{A}^*(X) = \mathcal{C}^\infty(X)$. If X is a manifold with corners with maximum codimension ℓ, then its boundary hypersurfaces are manifolds with corners with maximum codimension $\ell - 1$, thus we can assume inductively that $\mathcal{A}^*_{\mathrm{phg}}(\mathcal{B}_k)$ has been defined for any $\mathcal{B}_k$. We define $\mathcal{A}^*_{\mathrm{phg}}(X)$ to be those functions that, in each collar neighborhood (4.8), have an expansion of the form

$$u \sim \sum_{\mathrm{Re}\, s_j \to \infty} \sum_{p=0}^{p_j} \rho_k^{s_j} (\log \rho_k)^p u_{s_j,p}, \text{ with } u_{s_j,p} \in \mathcal{A}^*_{\mathrm{phg}}(\mathcal{B}_k). \tag{4.9}$$

The meaning of $\sim$ in this expression is that, for any $N \in \mathbb{N}$,

$$u - \sum_{\mathrm{Re}\, s_j \leq N} \sum_{p=0}^{p_j} \rho_k^{s_j} (\log \rho_k)^p u_{s_j,p} \in \dot{\mathcal{C}}^N([0,1), \mathcal{A}^*_{\mathrm{phg}}(\mathcal{B}_k),$$

where $\dot{\mathcal{C}}^N$ denotes functions in $\mathcal{C}^N$ that vanish to order N at the boundary.

In practice it is useful to keep track of the exponents ocurring these expansions. An **index set** is a discrete subset $E \subseteq \mathbb{C} \times \mathbb{N}_0$ with the property that

$$(s_j, p_j) \in E, \quad |(s_j, p_j)| \to \infty \implies \mathrm{Re}\, s_j \to \infty.$$

We will often use the notation

$$\mathrm{Re}\, E = \min\{\mathrm{Re}\, s : \exists (s,p) \in E\}.$$

An index set E is said to be *smooth* if $(s,p) \in E$ implies $(s+k, p-\ell) \in E$ for any $k, \ell \in \mathbb{N}_0$ with $\ell \leq p$ – we will always assume that our index sets are smooth. Given a collection of index sets $\mathcal{E} = \{E_1, \dots, E_N\}$, one per boundary hypersurface, we denote by $\mathcal{E}(k)$ or $\mathcal{E}(\mathcal{B}_k)$ the subcollection of $\mathcal{E}$ of index sets corresponding to boundary faces that intersect $\mathcal{B}_k$ (other than $\mathcal{B}_k$), and by $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X)$ the subspace of $\mathcal{A}^*_{\mathrm{phg}}(X)$ whose expansions (4.9) have the form

$$u \sim \sum_{(s,p) \in E_k} \rho_k^s (\log \rho_k)^p u_{s,p}, \qquad \text{with } u_{s,p} \in \mathcal{A}^{\mathcal{E}(k)}_{\mathrm{phg}}(\mathcal{B}_k). \tag{4.10}$$

Finally we point out that the individual coefficients $u_{s,p}$ are generally not invariantly defined. That is, if we choose a different product decomposition near a boundary face the coefficients will change. It is to avoid having the exponents change that we restrict to *smooth* index sets. A change of coordinates will replace each coefficient with a linear combination of 'lower-order' coefficients. In particular, for an expansion corresponding to an index set E, the leading coefficient is invariantly defined, as are the coefficients of the first occurence of any particular power of log, i.e., the coefficients corresponding to

$$\mathrm{lead}(E) = \{(s,p) \in E : k \in \mathbb{N} \implies (s-k,p) \notin E\}.$$

4.4.3. *The calculi.* Next, we put these two notions together. If X is a manifold with corners, Y is a p-submanifold, $\mathcal{E}$ is a collection of index sets for the boundary hypersurfaces of X, and G is a vector bundle over X, then a distributional section of G is in $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^s(X,Y;G)$ if it is conormal to Y and at each boundary hypersurface of X it has an expansion of the form (4.10) with exponents determined by $\mathcal{E}$ and coefficients conormal to the intersection of Y with that boundary face.

When listing index sets for either M^2_b or M^2_0 we use the ordering $\{\mathfrak{B}_{10}, \mathfrak{B}_{01}, \mathfrak{B}_{11}\}$. Recall that ρ_{ij} denotes a boundary defining function for $\mathfrak{B}_{ij}$. Given any manifold with corners X, let $\overline{\rho}$ be a product over a complete set of bfds for X (known as a *total bdf*), we define the b density bundles of X to be

$$\Omega_b(X) = \overline{\rho}^{-1}\Omega(X). \tag{4.11}$$

Similarly, for X equal to M, M^2, or M^2_0, we define the 0-density bundle of X to be

$$\Omega_0(X) = \overline{\rho}^{-m}\Omega(X),$$

where $m = \dim M$. On M these coincide with the Riemannian density bundles for b and 0 metrics respectively.

We define the **small b-calculus** by

$$\begin{aligned} A \in \Psi^s_b(M) &\iff \mathcal{K}_A \in \mathcal{A}^{\{\infty,\infty,\mathbb{N}_0\}}_{\mathrm{phg}} I^s(M^2_b, \mathrm{diag}_b; \Omega_b^{1/2}) \\ &\iff \mathcal{K}_A \in \dot{\mathcal{C}}^\infty_{\mathfrak{B}_{11}} I^s(M^2_b, \mathrm{diag}_b; \Omega_b^{1/2}), \end{aligned}$$

where $s \in \mathbb{R}$ is known as the *order* of A. In words, A is in $\Psi^s_B(M)$ if its Schwartz kernel is a distribution on M^2_b, conormal with respect to the b-diagonal, with a 'Taylor series' at the front face and vanishing to infinite order at both side faces[16].

We define the **large b-calculus** by

$$A \in \Psi^{s,\mathcal{E}}_b(M) \iff \mathcal{K}_A \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^s(M^2_b, \mathrm{diag}_b; \Omega_b^{1/2}),$$

where $\mathcal{E}$ is any collection of smooth index sets for the boundary faces of M^2_b. All b-differential operators define elements of the small b-calculus, but we

[16]The small b-calculus is *almost* the same as the uniform calculus of pseudo-differential operators corresponding to a b-metric, however the kernel of an operator in $\Psi^s_{\mathcal{B}}(M)$ vanishes in a neighborhood of the side faces and need not have an expansion at the front face (i.e., involves $\mathcal{A}^{\{\infty,\infty,\mathbb{N}_0\}}$ instead of $\mathcal{A}^{\{\infty,\infty,\mathbb{N}_0\}}_{\mathrm{phg}}$).

will see that paramatrices of Fredholm operators are elements of the large b-calculus.

Similarly, the **small 0-calculus** is defined by

$$A \in \Psi_0^s(M) \iff \mathcal{K}_A \in \dot{\mathcal{C}}^\infty_{\mathfrak{B}_{11}} I^s(M_0^2, \mathrm{diag}_0; \Omega_0^{1/2}),$$

and the **large 0-calculus** is defined by

$$A \in \Psi_0^{s,\mathcal{E}}(M) \iff \mathcal{K}_A \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^s(M_0^2, \mathrm{diag}_0; \Omega_0^{1/2}).$$

The existence of the interior symbol map is a feature of conormal distributions. A distribution in $I^s(X,Y)$ has a symbol defined invariantly on the conormal bundle to Y in X. Thus the following lemma is evidence that we are considering the 'correct' spaces.

Lemma 4.11. *The conormal bundle to* diag_b *in* M_b^2 *is* ${}^bT^*M$*, and similarly the conormal bundle to* diag_0 *in* M_0^2 *is* ${}^0T^*M$*.*

Proof. We will prove this for M_0^2, the proof for M_b^2 is similar. We denote the dimension of M by m, the blow-down map $M_0^2 \to M^2$ by β and the composition of β with the projection to the left or right factor of M by β_L and β_R respectively.

It suffices to compute in normal coordinates near the front face. Let x be a boundary defining function and let y_i denote coordinates along the boundary, so that $\mathcal{V}_0$ is locally spanned by $x\partial_x$, $x\partial_{y_i}$. We denote the lift of these coordinates to the left factor of M in M^2 again by x and y_i and the lift to the right factor by x' and y_i'. In these coordinates

$$\mathrm{diag}_{\partial M} = \{x = 0, x' = 0, y = y'\},$$

so the projective coordinates

$$s = \frac{x}{x'},\ u_i = \frac{y_i - y_i'}{x'},\ y_i',\ x'$$

are coordinates for M_0^2 near $\mathfrak{B}_{11}$ and away from $\mathfrak{B}_{01}$ (the lift of $x' = 0$). In these coordinates,

$$\mathrm{diag}_0 = \{s = 1, u = 0\},$$

and $\beta_L^*(x\partial_x) = s\partial_s$, $\beta_L^*(x\partial_{y_i}) = s\partial_{u_i}$ thus the normal bundle to diag_0 is naturally isomorphic to ${}^0TM$ and dually $N^*\,\mathrm{diag}_0 \cong {}^0T^*M$. □

Thus elements of these calculi fit into short exact sequences such as

$$0 \to \Psi_0^{s-1,\mathcal{E}}(M) \to \Psi_0^{s,\mathcal{E}}(M) \xrightarrow{\sigma_0} \mathcal{A}^{E_{11}}_{\mathrm{phg}}({}^0S^*M \otimes N_s) \to 0$$

(with N_s a bundle carrying the homogeneity) and similarly for the b operators[17]. It also follows that the differential operators in the b and 0 calculi - thought of as those pseudo-differential operators whose Schwartz kernel is supported *on* the diagonal - are respectively the b and 0 differential operators.

[17]Implicit in this sequence is the fact that the relevant density bundles 'cancel out' and allow us to consider the symbol as a function (see [Mazzeo: edge, pg 1631]).

Borel's lemma extends to asymptotic completeness of the small b and 0 calculi with respect to the interior symbol. In particular, given any element of either large calculus, $A \in \Psi_e^{s,\mathcal{E}}(M)$, we can find $A' \in \Psi_e^s(M)$ so that the full symbols of A and A' coincide. Thus we have the following decomposition

$$\Psi_e^{s,\mathcal{E}}(M) = \Psi_e^{s,(\infty,\infty,E_{11})}(M) + \Psi_e^{-\infty,\mathcal{E}}(M), \tag{4.12}$$

where we can further assume that the first term vanishes in neighborhood of the side faces and so can be considered an element of the corresponding uniform calculus.

One reason to demand expansions of the form (4.10) is that then we can talk about the leading order term at each boundary face. We refer to the leading term at each boundary face as the **normal operator** at that face. The most important of these is the normal operator at the front face (which we usually refer to as *the* normal operator). Thus for the small b-calculus we have a short exact sequence

$$0 \to x\Psi_b^s(M) \to \Psi_b^s(M) \xrightarrow{N_b} \mathcal{S}I^{s+\frac{1}{4}}(\mathfrak{B}_{11}, \operatorname{diag}_b \cap \mathfrak{B}_{11}; \Omega_b^{1/2}\big|_{\mathfrak{B}_{11}}) \to 0, \tag{4.13}$$

where $\mathcal{S}$ stands for Schwartz functions which appear since elements in the small calculus vanish to infinite order at the side faces (the change in the order of the conormal distribution is mostly a matter of convention). In (4.13) we are implicitly identifying

$$\operatorname{null}(N_b) \leftrightarrow \rho_{11}\mathcal{A}_{\text{phg}}^{\infty,\infty,\mathbb{N}_0} I^s(M_b^2, \operatorname{diag}_b; \Omega_b^{1/2}) \text{ with } x\Psi_b^s(M),$$

which is an immediate consequence of $\beta_L^* x = \rho_{10}\rho_{11}$ and the action of an element of the b calculus as described below. Since the front face is the inward pointing spherical normal bundle of $(\partial M)^2$, we can rewrite the final space in (4.13) as

$$\mathcal{S}I^{s+\frac{1}{4}}((\partial M)^2 \times \mathbb{R}^+, \operatorname{diag}_{\partial M} \times \{1\}; \Omega_b^{1/2}(\mathfrak{B}_{11})).$$

We will see below that the normal operator map N_b is a homomorphism when the image is interpreted as $\mathbb{R}^+$-invariant pseudo-differential operators on the (compactified) normal bundle to ∂M. The $\mathbb{R}^+$-invariance suggests taking the Mellin transform – the resulting family of operators on ∂M is known as the **indicial family** and will be discussed later.

The small zero calculus has the same short exact sequence, but in this case the front face is the inward pointing spherical normal bundle to $\operatorname{diag}_{\partial M}$, which means we can write the image space of the sequence as

$$\mathcal{S}I^{s+\frac{1}{4}}(\partial M \times \mathbb{R}^{m-1} \times \mathbb{R}^+, \partial M \times \{(0,1)\}; \Omega_0^{1/2}(\mathfrak{B}_{11})).$$

The normal operator map N_0 is also a homomorphism but now the image should be interpreted as a family of pseudo-differential operators on $\mathbb{R}^{m-1} \times \mathbb{R}^+$, parametrized by ∂M, that are invariant with respect to the semidirect product on $\mathbb{R}^{m-1} \times \mathbb{R}^+$:

$$(a, \alpha) \cdot (b, \beta) \mapsto (a + \alpha b, \alpha\beta).$$

If this were the direct product we would take Fourier transform in $\mathbb{R}^{m-1}$ and then Mellin transform in $\mathbb{R}^+$ and end up with a family of scalar symbols. However, because it is a semidirect product, the symmetries only allow us to take the Fourier transform and then 'mod out' by dilation invariance in $\mathbb{R}^+$ ending up with a family of pseudo-differential operators on $\mathbb{R}^+$, parametrized by $S^*\partial M$. A natural compactification of $\mathbb{R}^+$ allows us to identify this with a family of 'b, sc operators' on the unit interval.

Finally we point out a simple example of operators with vanishing symbol and normal operator.

Lemma 4.12. *Let $\phi \in \mathcal{C}^\infty(M)$.*
If $P \in \Psi_0^s(M)$ then $[P, \phi] \in x\Psi_0^{s-1}(M)$.
If $P \in \Psi_b^s(M)$ and $\phi\big|_{\partial M} = 0$ then $[P, \phi] \in x\Psi_b^{s-1}(M)$.

Proof. The distributional kernel of $[P, \phi]$ is

$$\mathcal{K}_P \beta_R^* \phi - \beta_L^* \phi \mathcal{K}_P = \mathcal{K}_P (\beta_R^* \phi - \beta_L^* \phi).$$

Since the principal symbol of $[P, \phi]$ *as an element of* $\Psi_e^s(M)$ is $[\sigma(P), \phi] = 0$ we only need to check that the distributional kernel of $[P, \phi]$ vanishes at the front face.

In M_0^2 we use coordinates

$$r = x + x', \quad \tau = \frac{x - x'}{x + x'}, \quad U = \frac{y - y'}{x + x'}, \quad y'$$

to find

$$\beta_R^* \phi - \beta_L^* \phi = \phi\left(\frac{r}{2}(1 - \tau), y'\right) - \phi\left(\frac{r}{2}(1 + \tau), rU + y'\right) = \mathcal{O}(r).$$

While in M_b^2 we use coordinates

$$r = x + x', \quad \tau = \frac{x - x'}{x + x'} \quad y, \quad , y'$$

to find

$$\begin{aligned} \beta_R^* \phi - \beta_L^* \phi &= \phi\left(\frac{r}{2}(1 - \tau), y'\right) - \phi\left(\frac{r}{2}(1 + \tau), y\right) \\ &= \phi(0, y') - \phi(0, y) + \mathcal{O}(r), \end{aligned}$$

which is $\mathcal{O}(r)$ if $\phi\big|_{\partial M} = 0$.

□

4.5. **Operations on polyhomogeneous conormal distributions.**

In this section we follow Melrose's geometric approach to the mapping properties and the composition of b and 0 pseudo-differential operators. We will follow [Mazzeo, Elliptic theory of differential edge operators I] and [Epstein-Melrose-Mendoza, Resolvent of the Laplacian on strictly pseudoconvex domains, Appendix B] and refer to [Grieser, Basics of the b-calculus] and [Melrose, Differential analysis on manifolds with corners] for, respectively, a more leisurely and a more complete account. In particular we refer to the latter for proofs of the statements of this section.

For general distributions and a smooth map between closed manifolds $f : X_1 \to X_2$, the push-forward of a distribution $u \in \mathcal{C}^{-\infty}(X_1)$, when it exists, is denoted $f_* u \in \mathcal{C}^{-\infty}(X_2)$ and characterized by

$$(f_* u, \phi)_{X_2} = (u, f^* \phi)_{X_1} = (u, \phi \circ f)_{X_1}, \text{ for every } \phi \in \mathcal{C}_c^\infty(X_2).$$

It is easy to see that one need only require that f be proper for push-forward to be well-defined.

Pull-back of a distribution $v \in \mathcal{C}^{-\infty}(X_2)$ should similarly be characterized by

$$(f^* v, \phi)_{X_1} = (v, f_* \phi)_{X_2}, \text{ for every } \phi \in \mathcal{C}_c^\infty(X_1).$$

However, it is not always possible to carry this out (e.g., the push-forward of a smooth density need not be smooth). Generally one needs to make sure that the map f 'avoids' the singularities of v[18]. For instance, if Y_2 is a submanifold of X_2 that is transverse to f, then

$$f^* : I^s(X_2, Y_2) \to I^{s+\frac{1}{4}(\dim X_1 - \dim X_2)}(X_1, f^{-1}(Y_2)).$$

Alternately, if f is a fibration or just a submersion then $f^* v$ is defined for every $v \in \mathcal{C}^{-\infty}(X_2)$. In this case, $f_* \phi$ is the 'integral along the fibers' of ϕ and is always smooth (it is useful to think of ϕ as a density instead of a function).

Not surprisingly, if the X_i are manifolds with corners, then we need to impose conditions on f at the boundary. First and foremost, we say that a smooth map $f : X_1 \to X_2$ is a ***b*-map** if for any choice of bdf's on X_1, $\{r_j\}$, and bdfs $\{\rho_i\}$ on X_2 we have

$$f^* \rho_i = h \prod r_j^{e(i,j)} \text{ with } h \in \mathcal{C}^\infty(X_1) \text{ non-vanishing.} \tag{4.14}$$

The matrix $(e(i,j))$ is called *the lifting matrix* of f and consists of non-negative integers. This condition guarantees a sort of uniformity on the various boundary hypersurfaces, e.g., the pre-image of a boundary hypersurface is a union of boundary hypersurfaces, and that the degree of vanishing of df at a boundary hypersurface is locally constant. Examples of b maps include blow-down maps and the inclusions of p-submanifolds. It is easy to see that the composition of two b-maps is again a b-map, and that b-maps induce linear maps on the b-tangent and b-cotangent bundles

$${}^b f_* : {}^b T X_1 \to {}^b T X_2, \quad {}^b f^* : {}^b T^* X_2 \to {}^b T^* X_1.$$

(Thus the set of b-maps is a natural choice of *Hom* for the category of manifolds with corners.)

Assume we know that a function on X_2 has an expansion of the form (4.10) and we pull-back by a map satisfying (4.14) then substituting $\prod r_j^{e(i,j)}$ for ρ_i tells us the form of the expansion for the pull-back. Thus, for any b-map

[18] In the sense that the 'normal bundle to f' and the wave-front set of v do not intersect [Hörmander, vol. 1, Thm. 8.2.4].

f, and a collection of index sets $\mathcal{F}$ for the boundary hypersurfaces of X_2, we define a collection of index sets on X_1 via

$$\mathcal{E} = f^\sharp(\mathcal{F}), \quad E_j = \left\{ \left(\sum e(i,j)s_i, \sum_{e(i,j)\neq 0} p_i \right) : (s_i, p_i) \in F_i \right\}$$

(if E_j is not smooth, we replace it with the smallest smooth index superset without changing the notation).

If $f : X_1 \to X_2$ is a b-map and Y_2 is a p-submanifold of X_2, we say that f is **b-transverse** to Y_2 if

$${}^b f_*({}^b T_\zeta X_1) + {}^b T_{f(\zeta)} Y_2 = {}^b T_{f(\zeta)} X_2, \text{ for every } \zeta \in f^{-1}(Y_2),$$

in which case $f^{-1}(Y_2)$ is a p-submanifold of X_1. If Y_1 is a p-submanifold of X_1, we say that f is b-transverse to Y_1 if

$$\mathrm{null}({}^b f_*)_\zeta + {}^b T_\zeta Y_1 = {}^b T_\zeta X_1, \text{ for every } \zeta \in Y_1.$$

Also, if Y_1 and Y_1' are both p-submanifolds of X_1 we say that they are b-transverse if

$${}^b T_\zeta Y_1 + {}^b T_\zeta Y_1' = {}^b T_\zeta X_1, \text{ for every } \zeta \in Y_1 \cap Y_1'.$$

Theorem 4.13 (Pull-back theorem)**.** *Let f be a b-map between manifolds with corners (with embedded boundary hypersurfaces), $f : X_1 \to X_2$, then, for any collection of index sets $\mathcal{F}$ for the boundary hypersurfaces on X_2, the pull-back of smooth functions in the interior extends to a continuous linear map*

$$f^* : \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_2) \to \mathcal{A}^{f^\sharp\mathcal{F}}_{\mathrm{phg}}(X_1).$$

If furthermore f is b-transverse to a p-submanifold $Y_2 \subseteq X_2$, then pull-back extends to

$$f^* : \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^s(X_2, Y_2) \to \mathcal{A}^{f^\sharp\mathcal{F}}_{\mathrm{phg}} I^{s+\frac{1}{4}(\dim X_1 - \dim X_2)}(X_1, f^{-1}(Y_2)).$$

As above for push-forward it is simplest to restrict to fibrations. A b-map is a **b-fibration** if it does not map any boundary hypersurface to a corner and it is a fibration in the interior of each boundary hypersurface. Thus any non-trivial blow-down map is *not* a b-fibration, but the maps $\beta_L : M_0^2 \to M$ and $\beta_R : M_0^2 \to M$ are. The push-forward by a b-fibration of a function that is polyhomogeneous conormal to the boundary in X_1 will be a polyhomogeneous conormal function on X_2, but it is perhaps less clear how the index sets should change. The results are simplest when expressed in term of b-densities (4.11).

Remark 4.14. Consider the simplest example. Let $X = [(\mathbb{R}^+)^2, \{0,0\}]$, $\beta_L : X \to (\mathbb{R}^+)^2 \to \mathbb{R}^+$ be the blow-down map followed by projection onto the left factor, and consider $v \in \mathcal{A}^{\mathcal{E}}(X; \Omega_b)$ and $(\beta_L)_* v$. Computing $(\beta_L)_* v$ in local coordinates near $\mathfrak{B}_{01}$ but away from $\mathfrak{B}_{10}$,

$$\left[\int_{\mathbb{R}^+} v\left(x, \frac{x'}{x}\right) \frac{dx'}{x'} \right] \frac{dx}{x},$$

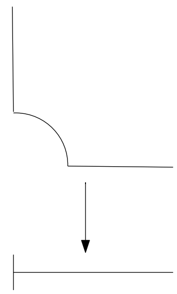


FIGURE 5. The b-fibration, β_L

we see that the push-forward will exist if $\operatorname{Re} E_{01} > 0$. To see which terms occur in the expansion, assume $\operatorname{supp} v \subseteq \beta^*([0,C]^2)$ and consider local coordinates near $\mathfrak{B}_{10}$ but away from $\mathfrak{B}_{01}$,

$$\left[\int_{x/C}^1 v\left(\frac{x}{x'}, x'\right)\frac{dx'}{x'}\right]\frac{dx}{x}$$
$$\sim \sum C_{s,p,t,q}\left[\int_{x/C}^1 \left(\frac{x}{x'}\right)^s\left(\log\frac{x}{x'}\right)^p (x')^t(\log x')^q\frac{dx'}{x'}\right]\frac{dx}{x}$$

and, since

$$\begin{aligned}
x^s\int_{x/C}^1 (x')^{t-s}\frac{dx'}{x'} &= \frac{1}{t-s}(x^s-(x/C)^t) \text{ if } t \neq s\\
x^s\int_{x/C}^1 \frac{dx'}{x'} &= -x^s\log(x/C)\\
x^s\int_{x/C}^1 (x')^{t-s}(\log x')^r\frac{dx'}{x'} &= -\frac{x^t}{C^{t-s}(t-s)}(\log(x/C))^r\\
&\quad -\frac{rx^s}{t-s}\int_{x/C}^1 (x')^{t-s}(\log x')^{r-1}\frac{dx'}{x'} \text{ if } t \neq s\\
x^s\int_{x/C}^1 (\log x')^r\frac{dx'}{x'} &= -\frac{1}{r+1}x^s(\log(x/C))^{r+1},
\end{aligned}$$

we see that the index set of $(\beta_L)_*v$ is the union of those of v at $\mathfrak{B}_{10}$ and $\mathfrak{B}_{11}$ except that whenever these coincide at a power of the bdf, we get an extra power of a logarithm.

Given two index sets E and F, their **extended union** is the index set defined by

$$E\overline{\cup}F = E \cup F \cup \{(s, p+q+1) : (s,p) \in E, (s,q) \in F\}.$$

Thus in the example above we found that the index set of $(\beta_L)_*v$ at the boundary of $\mathbb{R}^+$ was $E_{11}\overline{\cup}E_{10}$. For any b-fibration $f : X_1 \to X_2$ and any collection $\mathcal{E}$ of index sets for X_1 we define a collection of index sets on X_2,

$$\mathcal{F} = f_\sharp \mathcal{E}, \quad F_i = \overline{\bigcup_{\mathcal{B}_j(X_1)\subseteq f^{-1}(\mathcal{B}_i(X_2))}} \left\{ \left(\frac{s}{e(i,j)}, p \right) : (s,p) \in E_j \right\}.$$

Theorem 4.15 (Push-forward theorem)**.** *Let f be a b-fibration between manifolds with corners (with embedded boundary hypersurfaces), $f : X_1 \to X_2$. Then for every collection of index sets $\mathcal{E}$ on X_1 such that*

$$\operatorname{Re} E_j > 0 \text{ whenever } f(\mathcal{B}_i) \not\subseteq \partial X_2,$$

i.e., whenever $e(i,j) = 0$ for every i, push-forward gives a continuous map

$$f_* : \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X_1; \Omega_b) \to \mathcal{A}^{f_\sharp \mathcal{E}}_{\text{phg}}(X_2; \Omega_b).$$

If furthermore f is b-transverse to a p-submanifold $Y_1 \subseteq X_1$, then push-forward extends to

$$f_* : \mathcal{A}^{\mathcal{E}}_{\text{phg}} I^s(X_1, Y_1; \Omega_b) \to \mathcal{A}^{f_\sharp \mathcal{E}}_{\text{phg}}(X_2; \Omega_b). \tag{4.15}$$

Finally, we need a result for the product of two conormal distributions. It does not always make sense to multiply two distributions, however it is sometimes possible to define the product by first taking the tensor product (which always makes sense) and then pulling back via the diagonal map (which does not always make sense)[19]. For conormal distributions we have the following theorem.

Theorem 4.16. *Let f be a b-fibration between manifolds with corners (with embedded boundary hypersurfaces), $f : X_1 \to X_2$. Let Y_1 and Y_1' be b-transverse p-submanifolds of X_1 that are both b-transverse to f. Assume that f embedds $Y_1 \cap Y_1'$ onto a p-submanifold $Y_2 \subseteq X_2$. Then, for any two collections of index sets $\mathcal{E}$, $\mathcal{E}'$ on X_1 such that*

$$\operatorname{Re}(E_j + E_j') > 0 \text{ whenever } f(\mathcal{B}_i) \not\subseteq \partial X_2,$$

and any two real numbers $s, s' \in \mathbb{R}$, multiplication on X_1 followed by push-forward to X_2 gives a separately continuous bilinear form

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}} I^s(X_1, Y_1; \Omega_b^{1/2}) \times \mathcal{A}^{\mathcal{E}'}_{\text{phg}} I^{s'}(X_1, Y_1'; \Omega_b^{1/2}) \to \mathcal{A}^{f_\sharp(\mathcal{E}+\mathcal{E}')}_{\text{phg}} I^{s+s'}(X_2, Y_2; \Omega_b).$$

4.6. **Mapping properties and composition.**
Armed with the pull-back and push-forward theorems of the previous section we can deduce the action of b and 0 pseudo-differential operators on polyhomogeneous conormal distributions and also establish composition properties of these operators.

The motivation is to consider the action of an integral operator on M,

$$Pf(\zeta) = \int_M \mathcal{K}_P(\zeta, \zeta') f(\zeta') d\zeta'. \tag{4.16}$$

[19]This will work for distributions u and v as long as whenever (ζ, ξ) is in the wave front set of u, $(\zeta, -\xi)$ is *not* in the wave-front set of v [Hörmander, vol. 1, Thm. 8.2.10].

This suggests analyzing Pf for, say, b-operators by writing

$$Pf = (\beta_L)_*(\mathcal{K}_P \cdot \beta_R^* f).$$

Immediately from the push-forward and pull-back theorems we have the following theorem.

Proposition 4.17. *Let M be a manifold with boundary, F and index set on M and $P \in \Psi_b^{s,\mathcal{E}}(M)$ then*

$$\operatorname{Re}(E_{01} + F) > 0 \implies P : \mathcal{A}^F_{\mathrm{phg}}(M; \Omega_b^{1/2}) \to \mathcal{A}^{E_{10}\overline{\cup}(E_{11}+F)}_{\mathrm{phg}}(M; \Omega_b^{1/2}).$$

Similarly, if $P \in \Psi_0^{s,\mathcal{E}}(M)$ then

$$\operatorname{Re}(E_{01} + F) > m - 1 \implies P : \mathcal{A}^F_{\mathrm{phg}}(M; \Omega_0^{1/2}) \to \mathcal{A}^{E_{10}\overline{\cup}(E_{11}+F)}_{\mathrm{phg}}(M; \Omega_0^{1/2}).$$

In particular elements of the small calculus preserve spaces of polyhomogeneous conormal half-densities[20]

$$P \in \Psi_e^s(M) \implies P : \mathcal{A}^F_{\mathrm{phg}}(M; \Omega_e^{1/2}) \to \mathcal{A}^F_{\mathrm{phg}}(M; \Omega_e^{1/2}).$$

Proof. This will follow directly from the push-forward and pull-back theorems after we have taken the densities into account. In general we have:

Lemma 4.18. *Let X be a manifold with corners, Y a p-submanifold, and let $\beta : [X, Y] \to X$ be the blow-down map, then*

$$\beta^*\Omega(X) = \rho^{\dim X - \dim Y - 1}\Omega([X; Y]), \tag{4.17}$$

where ρ is a boundary defining function for the boundary hypersurface replacing Y.

We can motivate the factor (and indeed prove the lemma) by considering the model case $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$, since then $[\mathbb{R}^n; \mathbb{R}^k]$ is the same as $\mathbb{R}^k \times [\mathbb{R}^{n-k}; \{0\}]$ and (4.17) is just the usual change of variables for polar coordinates.

In particular we have

$$\beta_b^*\Omega_b(M^2) = \Omega_b(M_b^2), \text{ and } \beta_0^*\Omega_0(M^2) = \Omega_0(M_0^2).$$

It is convenient to trivialize the density bundles by picking a non-vanishing section γ_b of $\Omega_b(M)$. This induces a non-vanishing section of $\Omega(M_b^2)$,

$$\mu_b = \beta_L^*\gamma_b\beta_R^*\gamma_b.$$

Writing the kernel of P as $\kappa_P\sqrt{\mu_b}$ we have

$$P(f\sqrt{\gamma_b}) = (\beta_L)_*(\kappa_P\sqrt{\mu_b}\beta_R^*(f\sqrt{\gamma_b})).$$

To facilitate applying the push-forward theorem we multiply both sides by γ_b to get

$$P(f\sqrt{\gamma_b})\sqrt{\gamma_b} = (\beta_L)_*\left[\kappa_P\sqrt{\mu_b}\beta_R^*(f\sqrt{\gamma_b})\beta_L^*\sqrt{\gamma_b}\right] = (\beta_L)_*\left[\kappa_P\beta_R^* f\mu_b\right].$$

[20] Recall that we use the moniker e to make statements that apply to both the b and 0 calculi.

Since this is the push-forward of a b-density we can apply the push-forward theorem as indicated in Figure 6. Finally dividing both sides by $\sqrt{\gamma_b}$ the

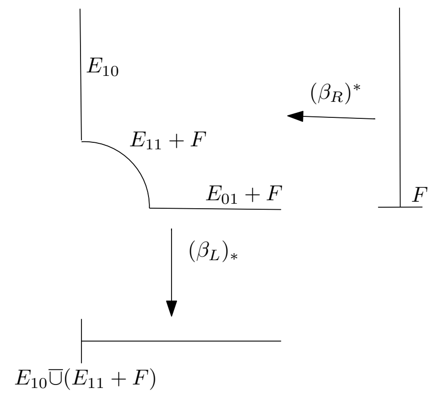


FIGURE 6. The index sets of f, $\mathcal{K}_P \cdot \beta_R^* f$, and its push-forward

result follows.

For the zero-calculus we have a similar argument. Let $\gamma_0 = \rho^{-(m-1)}\gamma_b$, and $\mu_0 = \beta_L^*\gamma_0\beta_R^*\gamma_0$, then we have

$$P(f\sqrt{\gamma_0})\sqrt{\gamma_0} = (\beta_L)_*\left[\kappa_P\beta_R^* f\mu_0\right] = (\beta_L)_*\left[\kappa_P\beta_R^* f(\rho_{11}\rho_{10}\rho_{01})^{-(m-1)}\mu_b\right].$$

By the push-forward theorem, if $\mathrm{Re}(E_{01}+F) > m-1$ this is an element of

$$\mathcal{A}^{[E_{10}-(m-1)]\overline{\cup}[E_{11}+F-(m-1)]}(M,\Omega_b) = \mathcal{A}^{E_{10}\overline{\cup}(E_{11}+F)}(M,\Omega_0),$$

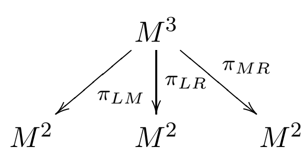
so dividing both sides by $\sqrt{\gamma_0}$ the result follows. $\square$

From here we see that it often makes sense to compose elements of the large calculus as, say, operators on polyhomogeneous conormal distributions. Below we will describe the action of elements of the large calculus on Sobolev spaces, and again see that it often makes sense to compose them. We now show that, when it makes sense, the composition of two elements of either large calculus is again an element of that calculus.

Directly from (4.16) if P_1 and P_2 are integral operators then the integral kernel of their composition is

$$\mathcal{K}_{P_1\circ P_2}(\zeta,\zeta'') = \int \mathcal{K}_{P_1}(\zeta,\zeta')\mathcal{K}_{P_2}(\zeta',\zeta'')\,d\zeta'.$$

For the corresponding statement for b operators, suppose we are able to extend the natural projections



(where L, M, and R stand for 'left', 'medium', and 'right', respectively) from the interior of M^3 to a compactification, M_e^3, with three b-fibrations

(4.18)
$$\begin{array}{ccccc} & & M_e^3 & & \\ & \swarrow^{\beta_{LM}} & \downarrow^{\beta_{LR}} & \searrow^{\beta_{MR}} & \\ M_e^2 & & M_e^2 & & M_e^2 \end{array}$$

Then for operators P_1, P_2 in the large calculus whose composition makes sense, the distributional kernel of the composition is given by

$$\mathcal{K}_{P_1 \circ P_2} = (\beta_{LR})_* \left[\beta_{LM}^* \mathcal{K}_{P_1} \beta_{MR}^* \mathcal{K}_{P_2} \right],$$

and hence the push-forward and pull-back theorems will determine both if the composition makes sense and the asymptotics of the composition.

We construct M_b^3 from $\overline{M}^3$ by means of appropriate blow-ups. In each of the three pairs of copies of M in $\overline{M}^3$ there is a copy of $(\partial M)^2$ which we denote S_{ij}, $i, j \in \{L, M, R\}$, $i \neq j$. In order for the fibrations $\pi_{..}$ to extend from the interior to a b-fibration $\beta_{..}$ we will need to blow-up all three $S_{..}$. However as these have non-trivial intersection, the resulting space will depend on the order in which we blow them up.

One way around this which preserves the symmetries is to first blow-up their intersection $T = S_{LM} \cap S_{LR}$. This intersection is a p-submanifold of $\overline{M}^3$, so $[\overline{M}^3; T]$ makes sense, and each $S_{..}$ lifts to a p-submanifold of this blown-up space.

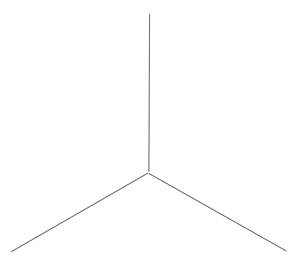


Figure 7. $\overline{M}^3$

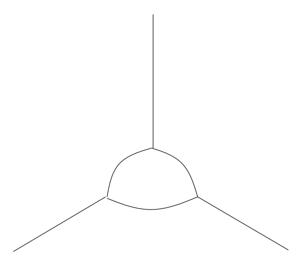


Figure 8. $[\overline{M}^3; T]$

Thus we can define the triple b-space to be

$$M_b^3 = [\overline{M}^3; T; S_{LM} \cup S_{LR} \cup S_{MR}].$$

Composing the (total) blow-down map $\beta : M_b^3 \to \overline{M}^3$ with the projections onto the left, middle, or right factors, we get the maps β_L, β_M, and β_R respectively. Let x be a bdf for M, we arrange the boundary hypersurfaces of M_b^3 into seven types denoted $\mathfrak{B}_{ijk}$ with $i, j, k \in \{0, 1\}$ not all equal to zero, with $i = 1$ iff $\beta_L^* x$ vanishes there, $j = 1$ iff $\beta_M^* x$ vanishes there, and $k = 1$ iff $\beta_R^* x$ vanishes there. Thus for instance $\mathfrak{B}_{111}$ denotes the bhs produced by blowing-up T, $\mathfrak{B}_{110}$ denotes the bhs produced by blowing-up S_{LM}, and $\mathfrak{B}_{010}$ corresponds to the lift (from the interior) of $M \times \partial M \times M$.

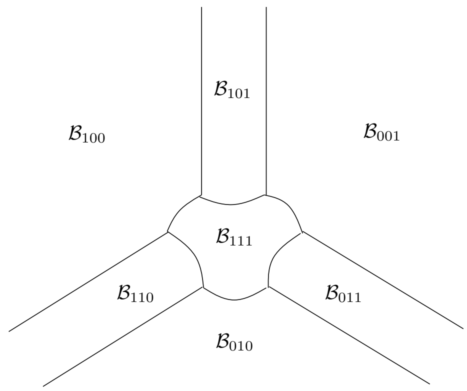


FIGURE 9. The space M_b^3

The construction of the 0-triple space is very similar. The difference is that S_{ij} represents the copy of $\mathrm{diag}_{\partial M}$ inside each of the three pairs of copies of M in $\overline{M}^3$. Otherwise, as above we define

$$M_0^3 = [\overline{M}^3; T; S_{LM} \cup S_{LR} \cup S_{MR}]$$

and divide the boundary hypersurfaces into seven types $\mathfrak{B}_{...}$ in the same way.

In order to define the maps β_{LM}, β_{LR} and β_{MR} we will make use of a *commutation result* for blow-ups. We have already mentioned that blowing-up disjoint p-submanifolds is insensitive to the order in which they are blown-up. More generally we have the following result[21]

Lemma 4.19. *If X is a manifold with corners and Y, Z are p-submanifolds such that either Y and Z are transverse or $Y \subseteq Z$ then*

$$[X; Y; Z] = [X; Z; Y].$$

The proof consists in noting that the identity map in the interior extends to a smooth map in both directions.

Thus we define β_{LM} in either case as follows

$$\beta_{LM} : M_e^3 \to [\overline{M}^3; T; S_{LM}] = [\overline{M}^3; S_{LM}; T] \to [\overline{M}^3; S_{LM}] = M_e^2 \times M \to M_e^2$$

and similarly β_{LR} and β_{MR}. It is easy to see that these are b-fibrations, and so we can proceed as outlined above.

Theorem 4.20. *The composition of two operators $A \in \Psi_b^{s,\mathcal{E}}(M)$ and $B \in \Psi_b^{t,\mathcal{F}}(M)$ is defined whenever $\mathrm{Re}(E_{01} + F_{10}) > 0$ in which case $A \circ B \in$*

[21]See for instance [Hassel-Mazzeo-Melrose, Analytic surgery and the accumulation of eigenvalues, Lemma 2.1]

$\Psi_b^{s+t,\mathcal{G}}(M)$ *with*

$$G_{10} = (E_{11} + F_{10})\overline{\cup}E_{10}, \quad G_{01} = (E_{01} + F_{11})\overline{\cup}F_{01}$$
$$G_{11} = (E_{11} + F_{11})\overline{\cup}(E_{10} + F_{01}).$$

Similarly, if $A \in \Psi_0^{s,\mathcal{E}}(M)$, $B \in \Psi_0^{t,\mathcal{F}}(M)$, *and* $\mathrm{Re}(E_{01} + F_{10}) > m - 1$ *then* $A \circ B \in \Psi_0^{s+t,\mathcal{G}}(M)$ *with* $\mathcal{G}$ *as above.*

Proof. The interior singularity can be handled using Theorem 4.16 or (4.12), composition in the uniform calculus, and (4.15), so we focus on the expansions at the boundary.

As in the proof of Proposition 4.17, it is useful to start by pointing out that (4.17) implies

$$\beta_b^*(\Omega_b(M^3)) = \Omega_b(M_b^3), \quad \beta_0^*(\Omega_0(M^3)) = \Omega_0(M_0^3),$$

where $\Omega_0(M_0^3) = \overline{\rho}^{-m}\Omega(M_0^3)$.

Let μ_b be a nowhere vanishing section of $\Omega_b(M_b^2)$ and let

$$\nu_b = \sqrt{\beta_{LM}^*\mu_b\beta_{LR}^*\mu_b\beta_{MR}^*\mu_b}$$

be the induced nowhere-vanishing section of $\Omega_b(M_b^3)$. If we denote the kernel of A as $\kappa_A\sqrt{\mu_b}$ and similarly for B and C, we have

$$\kappa_{A\circ B}\sqrt{\mu_b} = (\beta_{LR})_* \left[\beta_{LM}^*(\kappa_A\sqrt{\mu_b})\beta_{MR}^*(\kappa_B\sqrt{\mu_b})\right],$$

so if we multiply both sides by $\sqrt{\mu_b}$ we have

$$\kappa_{A\circ B}\mu_b = (\beta_{LR})_* \left[\beta_{LM}^*(\kappa_A)\beta_{MR}^*(\kappa_B)\nu_b\right],$$

and we can apply the push-forward theorem as illustrated in Figure 10.

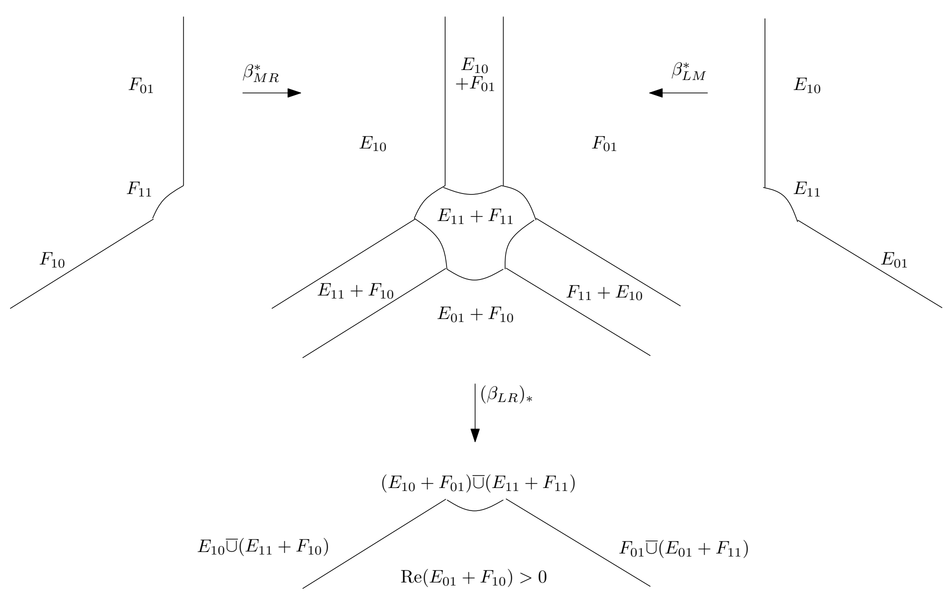


FIGURE 10. Composition via the triple space

For the zero calculus, we define $\mu_0 = (\rho_{11}\rho_{10}\rho_{01})^{-(m-1)}\mu_b$ and

$$\nu_0 = \sqrt{\beta_{LM}^*\mu_0\beta_{LR}^*\mu_0\beta_{MR}^*\mu_0},$$

and find that, with $\overline{\rho}$ a total boundary defining function for M_0^3,

$$\begin{aligned}\kappa_{A\circ B}\mu_0 &= (\beta_{LR})_*\left[\beta_{LM}^*(\kappa_A)\beta_{MR}^*(\kappa_B)\nu_0\right]\\ &= (\beta_{LR})_*\left[\beta_{LM}^*(\kappa_A)\beta_{MR}^*(\kappa_B)\overline{\rho}^{-(m-1)}\nu_0\right].\end{aligned}$$

Applying the push-forward theorem we see that, if

$$\mathrm{Re}(E_{10}+F_{10}-(m-1))>0$$

then this is an element of $\mathcal{A}_{\mathrm{phg}}^{\mathcal{G}'}I^{s+s'}(M_0^2,\mathrm{diag}_0;\Omega_b^{1/2})$ with $\mathcal{G}'=\mathcal{G}-(m-1)$. Changing from b-densities to 0-densities then changes $\mathcal{G}'$ to $\mathcal{G}$, and the theorem follows for the 0-calculus. □

One consequence of this theorem is that the small calculi are graded algebras in the sense that the composition is always defined and satisfies

$$\Psi_e^s(M)\circ\Psi_e^t(M)\subseteq\Psi_e^{s+t}(M).$$

Also notice that elements of the small calculus always compose with elements of the large calculus; the result is particularly nice if $\mathcal{E}$ is a collection of index sets with $E_{11}=\mathbb{N}_0$ as then

$$\Psi_e^s(M)\circ\Psi_e^{t,\mathcal{E}}(M)\subseteq\Psi_e^{s+t,\mathcal{E}}(M).$$

We recall that we have already defined Sobolev spaces and Sobolev norms in the context of b and 0 metrics by means of the uniform calculus. We obtain the same spaces using the small calculus, e.g., for $s>0$,

$$H_e^s(M;\Omega_e^{1/2})=\left\{u\in L^2(M;\Omega_e^{1/2}) : P\in\Psi_e^s(M)\implies Pu\in L^2(M;\Omega_e^{1/2})\right\}.$$

Due to our conventions for b and 0 pseudo-differential operators it is natural to base Sobolev spaces on half-densities instead of functions, though the triviality of the density bundle means that these are essentially the same. We will often use weighted Sobolev spaces,

$$x^aH_e^s(M;\Omega_e^{1/2})=\left\{u=x^av : v\in H_e^s(M;\Omega_e^{1/2})\right\},\quad a\in\mathbb{C}$$

with norms inherited from those on Sobolev spaces, thus

$$\|u\|_{x^aH_e^s}=\left\|x^{-a}u\right\|_{H_e^s}.$$

Theorem 4.21. *An operator in the large b-calculus $A\in\Psi_b^{s,\mathcal{E}}(M)$ extends, for any $a,a'\in\mathbb{C}$, from an operator $\mathcal{C}_c^\infty(M;\Omega_b^{1/2})\to\mathcal{C}^{-\infty}(M;\Omega_b^{1/2})$ to a bounded operator*

$$x^aH_b^t(M;\Omega_b^{1/2})\to x^{a'}H_b^{t'}(M;\Omega_b^{1/2}) \tag{4.19}$$

if $t'\le t-s$, $\mathrm{Re}\,E_{01}>-a$, $\mathrm{Re}\,E_{10}>a'$ and $\mathrm{Re}\,E_{11}\ge a'-a$.

An operator in the large 0*-calculus* $A \in \Psi_0^{s,\mathcal{E}}(M)$ *extends, for any* $a \in \mathbb{C}$, *from an operator* $\mathcal{C}_c^\infty(M;\Omega_0^{1/2}) \to \mathcal{C}^{-\infty}(M;\Omega_0^{1/2})$ *to a bounded operator*

$$x^a H_0^t(M;\Omega_0^{1/2}) \to x^{a'} H_0^{t'}(M;\Omega_0^{1/2})$$

if $t' \le t - s$, $\operatorname{Re} E_{01} > m - 1 - a$, $\operatorname{Re} E_{10} > m - 1 + a'$ *and* $\operatorname{Re} E_{11} \ge a' - a$.

In either case, if $t' < t - s$ *and* $\operatorname{Re} E_{11} > a' - a$ *then the induced operator is compact.*

Proof. Consider first the question of when $P \in \Psi_b^{-\infty,\mathcal{E}}(M)$ defines a bounded operator on $L_b^2(M)$. Applying Schur's test (Lemma 4.6) it is enough to show that there is a constant $C < \infty$ such that

$$(\beta_L)_*|\mathcal{K}_A| \le C, \quad (\beta_R)_*|\mathcal{K}_A| \le C.$$

Applying the push-forward theorem to $|\mathcal{K}_A|$ we see that the first push-forward is defined if $\operatorname{Re} E_{01} > 0$ and is bounded if $\operatorname{Re}(E_{10}\overline{\cup}E_{11}) \ge 0$, while the second push-forward is defined if $\operatorname{Re} E_{10} > 0$ and is bounded if $\operatorname{Re}(E_{01}\overline{\cup}E_{11}) \ge 0$. This is enough to prove the theorem for b-operators when $s = t = t' = 0$ by using (4.12) and the already established fact that elements of the uniform calculus of order $s \le 0$ are bounded on $L^2(M, g_b) = L_b^2(M)$.

We can reduce the general case to this one as follows. First since $t' \le t-s$, $H_b^{t'}$ includes H_b^{t-s} continuously so for the first statement it suffices to assume $t' = t - s$. By composing with invertible pseudo-differential operators in the small calculus we reduce to $s = t = 0$. Finally, note that $A \in \Psi_b^{s,\mathcal{E}}(M)$ induces a bounded map in (4.19) if and only if $x^{-a'}Ax^a$ is a bounded map between H_b^t and $H_b^{t'}$ and that $x^{-a'}Ax^a$ is an element of the large b calculus with index sets obtained from those of A by shifting E_{10} by $-a'$, E_{01} by a, and E_{11} by $a - a'$.

The corresponding statement for the 0-calculus is proved in the same fashion with the extra $m - 1$ summands appearing from switching between 0-densities and b-densities as in the proof of Proposition 4.17.

In the b-calculus, if $t' < t - s$ and $\operatorname{Re} E_{11} > a' - a$ we can find $\widehat{a} > a'$ such that

$$\begin{cases} \operatorname{Re} E_{10} > \widehat{a} \\ \operatorname{Re} E_{11} \ge \widehat{a} - a \end{cases}$$

and the map (4.19) factors through the inclusion

$$\begin{array}{ccc} x^a H_b^t(M;\Omega_b^{1/2}) & \longrightarrow & x^{a'} H_b^{t-s}(M;\Omega_b^{1/2}) \\ & \searrow \qquad \nearrow & \\ & x^{\widehat{a}} H_b^{t'}(M;\Omega_b^{1/2}) & \end{array}$$

which is compact by Corollary 4.10. The 0-case follows similarly. □

An important consequence of this theorem (which already follows from Corollary 4.10) is that an element of $x^a\Psi_e^s(M)$ is compact (as an operator on $L_e^2(M)$) whenever $a > 0$ and $s < 0$. Thus elements of $\Psi_e^0(M)$ that are compact on $L_e^2(M)$ are precisely those whose joint symbol (σ_e, N_e) vanishes[22]. For instance, from Lemma 4.12, this applies to the commutator of an element of the small zero calculus and a smooth function, as well as to the commutator of an element of the small b-calculus and a smooth function that vanishes at the boundary. If the joint symbol of an operator is invertible (as a map on appropriate Sobolev spaces as described in the next section) we say that the operator is **fully elliptic**. We will see soon that an operator induces a Fredholm operator on Sobolev spaces precisely when it is fully elliptic.

Finally we wish to discuss a class of smoothing operators common to the b and 0 calculi. An operator is said to be **very residual**[23] if it is in the space

$$\Psi_e^{-\infty,(E_{10},E_{01},\infty)}(M),$$

that is, if it is a smoothing operator that vanishes to infinite order at the front face. The kernel of such an operator is well-behaved on the blown-down space and is an element of

$$\mathcal{A}_{\mathrm{phg}}^{E_{10},E_{01}}(M^2,\Omega_e^{1/2})$$

that vanishes to infinite order at the submanifold of the corner that is blown-up to form the stretched double-space. The advantage of very residual operators is that they bridge between L^2 statements and polyhomogeneous statements.

Proposition 4.22. *Let A_1 and A_2 be very residual operators.*

i) *If f and $A_1 f$ are elements of weighted e-Sobolev spaces, then $A_1 f \in \mathcal{A}_{\mathrm{phg}}^{E_{10}}(M;\Omega_e^{1/2})$.*

ii) *(* Bi-ideal property *)*
If G is a bounded operator between two weighted e-Sobolev spaces and the composition A_2GA_1 is defined, say as a map

$$A_2GA_1 : x^aH_e^s(M;\Omega_e^{1/2}) \to x^bH_e^t(M;\Omega_e^{1/2}) \tag{4.20}$$

then there is a very residual operator that coincides with A_2GA_1 as a map (4.20).

Proof. For (i) it suffice to note that if we have

$$\mathcal{K}_A(x,y,x',y') - \sum x^s(\log x)^p\mathcal{K}_{s,p}(y,x',y') \in \dot{\mathcal{C}}^N(M;\mathcal{C}^\infty(M))$$

[22] An element of the large calculus $\Psi_e^{s,\mathcal{E}}(M)$ induces a compact operator on $L_e^2(M)$ if $a = \mathrm{Re}(\mathcal{E}) > 0$ and $s < 0$. By Liidsky's theorem this is trace-class if $s < -\dim M$ and a is large enough – so from the composition formula we see that a compact operator in the large calculus is automatically in a Schatten ideal.

[23] In other contexts, the class of very residual operators is taken to be $\dot{\Psi}^{-\infty}(M) = \Psi^{-\infty,(\infty,\infty)}(M)$.

then we also have

$$\int \mathcal{K}_A(x,y,x',y')f(x',y') - \sum x^s(\log x)^p \int \mathcal{K}_{s,p}(y,x',y')f(x',y') \in \dot{\mathcal{C}}^N(M).$$

The same observation shows that the integral kernel of A_2GA_1 (which is smooth since this operator and its adjoint are smoothing) has an expansion at the left face with exponents equal to those of A_2 at the left face, and by looking at its adjoint, that it has an expansion at the right face with exponents equal to those of A_1 at the right face. □

4.7. **Normal operators and Fredholmness.**
As anticipated, the normal operator – which is defined as the leading term of the Schwartz kernel at the front face – can itself be interpreted as an operator.

As a first step we point out that the interiors of the fibers of the front face (recall that the front face is naturally fibered over the submanifold it replaces) have a natural Lie group structure that is consistent with the triple space. Indeed, let $\mathcal{F}$ be the interior of a fiber of the front face in M_e^2 and let β_{LR}, β_{LM}, β_{MR} be the blow-down maps from the blown-up triple space. Then it is easy to see that

$$\beta_{LR}\left(\beta_{LM}^{-1}(\mathcal{F}) \cap \beta_{MR}^{-1}(\mathcal{F})\right) = \mathcal{F},$$

as this follows from the corresponding statement before the blow-ups. However, it is also true that for any $p, q \in \mathcal{F}$ the set

$$\beta_{LR}\left(\beta_{LM}^{-1}(p) \cap \beta_{MR}^{-1}(q)\right)$$

consists of only one point, which we denote $p * q$.

For example, if x is a boundary defining function and y denotes coordinates along the boundary, then near the front face of M_0^2 but away from the left face (corresponding to $x = 0$) we have projective coordinates on M_0^2,

$$(x, y, s', u') = \left(x, y, \frac{x'}{x}, \frac{y'-y}{x}\right). \tag{4.21}$$

Similarly, near the triple face on M_0^3, we have projective coordinates of the form

$$(x, y, s', u') = \left(x, y, \frac{x'}{x}, \frac{y'-y}{x}\right), \quad (x, y, \widehat{s}, \widehat{u}) = \left(x, y, \frac{\widehat{x}}{x}, \frac{\widehat{y}-y}{x}\right),$$
$$\left(x\widehat{s}, y + x\widehat{u}, \frac{s'}{\widehat{s}}, \frac{u'-\widehat{u}}{\widehat{s}}\right)$$

on the double spaces LR, LM, and MR respectively. Then for

$$p = (0, y_0, s, u) \text{ and } q = (0, y_0, \widetilde{s}, \widetilde{u})$$

we have

$$\begin{aligned} p * q &= \beta_{LR}\left(\beta_{LM}^{-1}(0,y_0,s,u) \cap \beta_{MR}^{-1}(0,y_0,\widetilde{s},\widetilde{u})\right) \\ &= \beta_{LR}\left(\left\{x=0, y=y_0, s=\widehat{s}, u=\widehat{u}, \frac{s'}{\widehat{s}}=\widetilde{s}, \frac{u'-\widehat{u}}{\widehat{s}}=\widetilde{u}\right\}\right) \\ &= (0, y_0, s\widetilde{s}, u+s\widetilde{u}) \end{aligned}$$

(just as in [Mazzeo:Hodge]).

Thus we find that if we identify (the interior of) each fiber of the front face in M_0^2, say over $q \in \partial M$, with $T_qM^+ \cong \mathbb{R}^{m-1} \times \mathbb{R}^+$, the product $p * q$ is the affine product

$$\begin{aligned} \left(\mathbb{R}^{m-1} \times \mathbb{R}^+\right)^2 &\longrightarrow \mathbb{R}^{m-1} \times \mathbb{R}^+ \\ (a,b),(c,d) &\longmapsto (a+bc, bd) \end{aligned}$$

and so the group is the semi-direct product $\mathbb{R}^{m-1} \rtimes \mathbb{R}^+$. Moreover, note that right Haar measures for the affine group are (in the coordinates (4.21))

$$C\frac{ds\ du}{s^m}, \quad C > 0 \text{ constant}$$

precisely the zero measures of $\mathbb{R}^{m-1} \times \mathbb{R}^+$ (the left Haar measures are the b measures on $\mathbb{R}^{m-1} \times \mathbb{R}^+$). Similarly, each fiber of the front face of M_b^2 is a (compactified) copy of $\mathbb{R}^+$, the product $p * q$ is the usual multiplication, and the Haar measures (left or right) are precisely the b-measures of $\mathbb{R}^+$.

Remark 4.23. The exponential of a vector field in $\mathcal{V}_0$ is a diffeomorphism of $\overline{M}$ which fixes the boundary point-wise. Any such diffeomorphism induces, for any $q \in \partial M$, a linear map on the inward-pointing 'half' of the tangent space T_q^*M, such a map must be of the form

$$(x,y) \mapsto (xs, y+xu) \text{ for some } s \in \mathbb{R}^+, u \in \mathbb{R}^{m-1},$$

and we see that the zero vector fields are precisely the infinitesimal generators of these maps.

Given any $P \in \Psi_0^{s,\mathcal{E}}(M)$, the leading term at the front face $N_{11}(P)$ defines a family of operators parametrized by points on the boundary. For instance[24], given $q \in \partial M$, and $f \in \mathcal{C}_c^\infty(T_qM^+) \cong \mathcal{C}_c^\infty(\mathbb{R}^{m-1} \times \mathbb{R}^+)$ we set

$$N_q(P)f(x,y) = \int \mathcal{K}_P\big|_{\mathfrak{B}_{11}}(q, (x,y) * (x',y')^{-1}) f(x',y') \frac{dx'\ dy'}{(x')^m}.$$

This defines a map $N_q(P) : \mathcal{C}_c^\infty(T_qM^+) \to \mathcal{C}^{-\infty}(T_qM^+)$ and induces maps, e.g., $\dot{\mathcal{C}}^\infty(T_qM^+) \to \dot{\mathcal{C}}^\infty(T_qM^+)$. Indeed, N_q is a pseudo-differential operator on T_qM^+ of zero type at the boundary and in the next section we will compactify T_qM^+ and consider N_q as an operator in the interior of a compact manifold with boundary. It is convenient to think of a zero metric g on M

[24]We are temporarily neglecting half-density factors for notational convenience.

that has the form $g = \frac{dx^2}{x^2} + \frac{h(x,y)}{x^2}$ near the boundary, as inducing metrics on each T_qM^+ given by

$$\widetilde{g}_q = \frac{dx^2}{x^2} + \frac{h(q)}{x^2}.$$

These are metrics of bounded geometry on each T_qM^+ and the normal operators $N_q(P)$ induce maps between the resulting Sobolev spaces. The upshot of the discussion above is that the normal operator is a homomorphism in that

$$N_q(P_1 \circ P_2) = N_q(P_1) \circ N_q(P_2), \text{ for every } q \in \partial M$$

whenever the composition makes sense.

The maps $N_q(P)$ defined above are so far only formally related to P. The main obstacle with relating them directly is that P acts on functions on M and each $N_q(P)$ acts on functions on T_qM^+. This is easily overcome by identifying a neighborhood $\mathcal{U}$ of $q \in \partial M$ with a neighborhood $\mathcal{V}$ of the origin of T_qM^+. We say that a diffeomorphism ϕ from $\mathcal{U}$ to $\mathcal{U}^+$ is a *normal fibration at* q if

$$\phi(q) = 0 \text{ and } d\phi\big|_q = \mathrm{Id}\,.$$

We say that $\mathcal{U}$ is a **normal neighborhood** of $q \in \partial M$ if there is a normal fibration defined on $\mathcal{U}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two normal neighborhoods of $q \in \partial M$ such that $\overline{\mathcal{U}} \subseteq \mathcal{V}$, and let $\chi, \widetilde{\chi} \in \mathcal{C}_c^\infty(\mathcal{V}, \mathbb{R}^+)$ be equal to one in a neighborhood of q and satisfy

$$\operatorname{supp}\chi \subseteq \mathcal{U} \subseteq \{\widetilde{\chi} = 1\}$$

(we will refer to $\mathcal{U}$, $\mathcal{V}$, $\widetilde{\chi}$, χ collectively as a *normal setup* at q). In local coordinates on $\mathcal{V}$ centered at q, lifted to M^2 , the action of P on $f \in \mathcal{C}_c^\infty(\mathcal{U}, \Omega_0^{1/2})$ is given by

$$Pf(x,y) = \left[\int \mathcal{K}_P\left(\frac{x}{x'}, \frac{y-y'}{x'}, x', y'\right) f(x',y') \frac{dx'\, dy'}{(x')^m}\right] \left|\frac{dx\, dy}{x^m}\right|^{1/2}.$$

If we denote by R_t the dilation by t on $\mathbb{R}^{m-1} \times \mathbb{R}^+$, then notice that (for any f as above)

$$(4.22)\quad R_t^*\widetilde{\chi}P\chi R_{1/t}^* f =$$

$$\left[\int \widetilde{\chi}(tx,ty)\chi(tx',ty')\mathcal{K}_P\left(\frac{x}{x'}, \frac{y-y'}{x'}, tx', ty'\right) f(x',y')\frac{dx'\, dy'}{(x')^m}\right] \left|\frac{ds\, du}{s^m}\right|^{1/2}$$

$$\xrightarrow{t\to 0} \left[\int \mathcal{K}_P\left(\frac{x}{x'}, \frac{y-y'}{x'}, 0, 0\right) f(x',y')\frac{dx'\, dy'}{(x')^m}\right] \left|\frac{ds\, du}{s^m}\right|^{1/2} = N_q(P)f,$$

where we have left off various ϕ's and ϕ^{-1}'s.

For a 0-differential operator, computing the normal operator via the limit (4.22) is particularly simple. Indeed, if in local coordinates near q,

$$P = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha,$$

then the normal operator at $q \in \partial M$ is the differential operator on $T_q M^+ \cong \mathbb{R}^{m-1}_u \times \mathbb{R}^+_s$,

$$N_q(P) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(q)(s\partial_s)^j(s\partial_u)^\alpha.$$

We point out that this expression is invariant under translation in u and dilation in (s,u) as expected.

The situation for the b-calculus is similar but simpler. Whereas the normal operator in the zero calculus yields for each $q \in \partial M$ and operator on the 'upper half' of $\mathbb{R}^m$, the normal operator in the b calculus yields a single operator on the model cylinder $(\partial M)^+ = \mathbb{R}^+_s \times \partial M$. Given a function $f(s,y)$ on the cylinder and a b pseudo-differential operator $A \in \Psi_b^{k,\mathcal{E}}(M)$, we define the action of A on f by

$$N(P)f(s,y) = \int \mathcal{K}_A\big|_{\mathfrak{B}_{11}}\left(\frac{s}{s'},y,y'\right) f(s',y')\frac{ds'\,\mathrm{dvol}_{y'}}{s}.$$

As above, this is a b pseudo-differential operator (on the non-compact manifold $(\partial M)^+$) and induces maps, e.g., on Schwartz functions in the interior. It is convenient to think of a b metric of the form $\frac{dx^2}{x^2} + h(x,y)$ near the boundary as inducing the metric

$$\widetilde{g} = \frac{ds^2}{s^2} + h(0,y)$$

on $(\partial M)^+$, as this is a metric of bounded geometry and hence induces Sobolev spaces on which $N(P)$ acts. The normal operator is a homomorphism with respect to composition of operators. The analogue of the normal neighborhoods above are collar neighborhoods of the boundary, that is we can identify collar neighborhoods of the boundary in M with collar neighborhoods of the boundary in $(\partial M)^+$ and use these to relate the action of $N(P)$ with the action of P.

Lemma 4.24. *Let $a, s \in \mathbb{R}$.*
Let $\mathcal{U}$, $\mathcal{V}$, $\widetilde{\chi}$, χ be a 'normal setup' at $q \in \partial M$ with

$$\phi : \mathcal{V} \xrightarrow{\cong} \mathcal{V}^+$$

the associated normal fibration, and let $P \in \Psi_0^s(M)$. For every $\varepsilon > 0$ there is an neighborhood $\mathcal{W}_\varepsilon \subseteq \mathcal{U}$ of q such that

$$\left\|(\phi^{-1})^*\widetilde{\chi}N_q(P)\phi^*\chi u - \widetilde{\chi}P\chi u\right\|_{x^aL^2} \leq \varepsilon \left\|u\right\|_{x^aL^2}$$
$$\text{for all } u \in H_0^s(M,\Omega_0^{1/2}) \text{ with } \operatorname{supp} u \subseteq \mathcal{W}_\varepsilon.$$

Similarly, for $P \in \Psi_b^s(M)$ and every $\varepsilon > 0$ there is a collar neighborhood of the boundary,

$$\phi : C_\varepsilon(\partial M) \xrightarrow{\cong} [0,1) \times \partial M,$$

such that

$$\left\|(\phi^{-1})^*\widetilde{\chi}N(P)\phi^*\chi u - \widetilde{\chi}P\chi u\right\|_{x^aL^2} \leq \varepsilon \left\|u\right\|_{x^aL^2}$$
$$\text{for all } u \in H^s_b(M, \Omega_b^{1/2}) \text{ with } \operatorname{supp} u \subseteq C_\varepsilon(\partial M).$$

Proof. First assume that s is sufficiently negative so that the distributional kernel of P is a continuous function on M^2_0, then the distributional kernel of $(\phi^{-1})^*\widetilde{\chi}N_q(P)\phi^*\chi - \widetilde{\chi}P\chi$ is given by

$$\widetilde{\chi}(x,y)\chi(x',y')\left[\mathcal{K}_P\left(\frac{x}{x'}, \frac{y-y'}{x'}, 0, 0\right) - \mathcal{K}_P\left(\frac{x}{x'}, \frac{y-y'}{x'}, x', y'\right)\right],$$

and, since $\mathcal{K}_P$ is uniformly continuous, we can find a neighborhood $\mathcal{W}_\varepsilon \subseteq \mathcal{U}$ of q where this is uniformly bounded by ε. The same argument handles the general case since $\mathcal{K}_P(\cdot,\cdot,x',y')$ is smooth in x', y' and conormal in the other entries; i.e., since conormal distributions near a point $p \in \operatorname{diag}_0$ are smooth functions valued in conormal distributions (see §4.4.1). Alternately we can reduce the case of general s to the case of a continuous distributional kernel by composing with an invertible operator in $\Psi_0^{-N}(M)$ for large enough $N \in \mathbb{N}$, as these define bounded operators on $L^2_0(M, \Omega_0^{1/2})$.

The b-case follows by similar, simpler arguments. □

Having related the mapping properties of an operator with those of its normal operator, we can apply Anghel's criterion for Fredholmness to prove the following result.

Theorem 4.25. *Let $a, s \in \mathbb{R}$. The map induced by an operator $P \in \Psi^s_0(M)$,*

$$P : x^aH^t_0(M; \Omega_0^{1/2}) \to x^aH^{t-s}_0(M; \Omega_0^{1/2}),$$

is essentially injective if and only if for every $q \in \partial M$ the map

$$N_q(P) : x^aH^t_0(T_qM^+; \Omega_0^{1/2}) \to x^aH^{t-s}_0(T_qM^+; \Omega_0^{1/2})$$

is essentially injective.

Similarly, the map induced by $P \in \Psi^s_b(M)$,

$$P : x^aH^t_b(M; \Omega_b^{1/2}) \to x^aH^{t-s}_b(M; \Omega_b^{1/2}),$$

is essentially injective if and only if the map

$$N(P) : x^aH^t_b((\partial M)^+; \Omega_b^{1/2}) \to x^aH^{t-s}_b((\partial M)^+; \Omega_b^{1/2})$$

is essentially injective.

In either case, if the normal operator is essentially injective then it is in fact injective.

Proof. Without loss of generality we can assume $t = s$.

We start by showing that if the normal operator is essentially injective then it is injective. Indeed, assume that for some non-empty open set $\mathcal{V} \subseteq T_qM^+$, some $C > 0$, and every $u \in \mathcal{C}^\infty_c(\mathcal{V}, \Omega_0^{1/2})$ we have

$$\left\|N_q(P)u\right\|_{x^aL^2_0} \geq C\left\|u\right\|_{x^aL^2_0}, \tag{4.23}$$

then, for any $v \in \mathcal{C}_c^\infty(T_qM^+,\Omega_0^{1/2})$, we can use translation and dilation invariance of the L^2 norm and the operator $N_q(P)$ to show that the same estimate holds for $\|N_q(P)v\|_{x^aL^2}$. Since compactly supported smooth half-densities are dense in $x^aL^2(T_qM^+,\Omega_0^{1/2})$, this same estimate holds for any u in this larger space, and so $N_q(P)$ is injective (with a bounded left inverse) as an operator on $x^aL^2_0(T_qM^+;\Omega_0^{1/2})$. By Anghel's criterion this discussion applies to $N_q(P)$ whenever this is essentially injective as an unbounded operator on this space.

Next assume that P is essentially injective. By Anghel's criterion there is a compact set $K \subseteq M^\circ$ and a constant $C > 0$ such that every $u \in \mathcal{C}_c^\infty(M;\Omega_b^{1/2})$ with support in $M \setminus K$ satisfies

$$\|Pu\|_{x^aL^2_0} \geq C' \|u\|_{x^aL^2_0}. \tag{4.24}$$

Choose for each $q \in \partial M$ and $\varepsilon > 0$, let $\mathcal{V}_q$, $\mathcal{U}_q$, $\widetilde{\chi}_q$, χ_q be a 'normal setup' at q, and assume that $\mathcal{U}_q = \mathcal{W}_q(\varepsilon)$ in the notation of Lemma 4.24. If $\varepsilon < C'/2$, and $u \in \mathcal{C}_c^\infty(T_qM^+;\Omega_0^{1/2})$ has support in $\phi(\mathcal{W}_q(\varepsilon))$ then clearly (4.23) holds with $C = C'/2$. Thus P essentially injective implies $N_q(P)$ essentially injective (hence injective) for every $q \in \partial M$.

Finally assume that $N_q(P)$ is injective for every $q \in \partial M$ so that (4.23) holds (with the same C for every q). With the same notation as above, assume that each $\mathcal{U}_q$ has smooth boundary and that $\varepsilon < \frac{C}{2}$. Also choose finitely many $\{\mathcal{U}_{q_i}\}$ so that $\{\mathcal{U}_{q_i} \cap \partial M\}$ is an open cover of ∂M, and assume that $\sum \chi_{q_i} = 1$ in a neighborhood of the boundary. From Lemma 4.24 we know that, for each i

$$\|P\chi_{q_i}u\|_{x^aL^2_0} \geq \|\widetilde{\chi}_{q_i}P\chi_{q_i}u\|_{x^aL^2_0} \geq \frac{C}{2}\|u\|_{x^aL^2}, \text{ for all } u \in \mathcal{C}_c^\infty(\mathcal{U}_{q_i}).$$

Hence, if we denote $L^2_{0,\mathrm{Dir}}(\mathcal{U}_{q_i};\Omega_0^{1/2})$ the closure of $\mathcal{C}_c^\infty(\mathcal{U}_{q_i})$ in $L^2_0(M;\Omega_0^{1/2})$, the unbounded operator,

$$P\chi_{q_i} : x^aL^2_{0,\mathrm{Dir}}(\mathcal{U}_{q_i};\Omega_0^{1/2}) \to x^aL^2_0(M;\Omega_0^{1/2}),$$

has a left inverse, say G_{q_i}. Using these 'local inverses' we will show that P has a left parametrix and hence is essentially injective on $x^aH^s_0(M;\Omega_0^{1/2})$.

First use ellipticity of P to find a symbolic parametrix, $Q_\sigma \in \Psi_0^{-s}(M)$, such that

$$R_\sigma = \mathrm{Id} - Q_\sigma P \in \Psi_0^{-\infty}(M).$$

Define

$$G : x^aH^s_0(M,\Omega_0^{1/2}) \to x^aL^2_0(M,\Omega_0^{1/2}), \quad G(u) = \sum \widetilde{\chi}_{q_i}G_{q_i}\chi_{q_i}u$$

(where we are implicitly identifying $\mathcal{U}_q$ and $\mathcal{U}_q^+$), and $Q_N = Q_\sigma + R_\sigma G$, and note that

$$\begin{aligned}\operatorname{Id} - Q_N P &= R_\sigma - R_\sigma \sum \widetilde{\chi}_{q_i} G_{q_i} \chi_{q_i} P \\ &= R_\sigma - R_\sigma \sum \widetilde{\chi}_{q_i} G_{q_i} P \chi_{q_i} + R_\sigma \sum \widetilde{\chi}_{q_i} G_{q_i} [P, \chi_{q_i}] \\ &= R_\sigma (\operatorname{Id} - \sum \chi_{q_i}) + R_\sigma \sum \widetilde{\chi}_{q_i} G_{q_i} [P, \chi_{q_i}]\end{aligned}$$

is compact (by Lemma 4.12).

Similar simpler arguments work for b-operators. □

This gives a description of the normal operators as operators on non-compact spaces related to the interiors of the fibers of the front face. However in keeping with the general philosophy of these calculi we should be considering them acting on an appropriate compactification to a manifold with boundary.

Consider first the case of b-operators. We have essentially been using projective coordinates

$$\left(s = \frac{x}{x'}, y, x', y'\right)$$

to describe the normal operator acting on $\mathbb{R}_s^+ \times \partial M$. These coordinates are valid near the front face $\mathfrak{B}_{11}$ and the left face $\mathfrak{B}_{10}$ but away from the right face $\mathfrak{B}_{01}$; the corresponding coordinates near $\mathfrak{B}_{01}$ and away from $\mathfrak{B}_{10}$ would involve $t = \frac{1}{s}$. This suggests that we compactify $\mathbb{R}_s^+$ to an interval using $\frac{1}{s}$ as a boundary defining function for the point at infinity. This compactification is natural with respect to 'b-objects' on $\mathbb{R}_s^+ \times \partial M$. For instance, the metric $\widetilde{g}$ on $(\partial M)^+$ extends to this compactification $[0,1] \times \partial M$ as a b-metric, and indeed the normal operator of a b-operator on M defines a b-operator on $[0,1] \times \partial M$. In the sense described above, the action of a b-operator P on a weighted Sobolev space on M corresponds to the action of $N(P)$ on a weighted Sobolev space on $[0,1] \times \partial M$, only now this is:

$$\begin{gathered} P \text{ acting on } x^a H_b^k(M; \Omega_b^{1/2}) \\ \leftrightarrow N(P) \text{ acting on } \rho_0^a \rho_1^{-a} H_b^k([0,1] \times \partial M; \Omega_b^{1/2}), \end{gathered}$$

where ρ_0, ρ_1 are boundary defining functions for $\{0\} \times \partial M$ and $\{1\} \times \partial M$ respectively.

It is also convenient to think of the normal operator in the 0-calculus as acting on a compact manifold with boundary. However, we will wait until the next section to describe this compactification and the resulting *reduced normal operator*.

4.8. **General coefficients and adjoints.**

We have been working with operators acting on half-densities for convenience. The theory is formally the same for operators acting on sections of vector bundles, with only notational differences. The distributional kernel of a pseudo-differential operator acting from sections of a bundle E to sections of a bundle F is a conormal distribution valued in the 'big' homomorphism bundle

$$\mathrm{Hom}(F,E) = \pi_L^* F \otimes \pi_R^* E'$$

defined so that the fiber over the point (ζ, ζ') consists of homomorphisms between E_ζ and $F_{\zeta'}$. One can either add the bundles into the theory from the beginning, or add them on after the constructions above by defining, e.g.,

$$A \in \Psi_b^{s,\mathcal{E}}(M; E, F) \iff$$

$$\mathcal{K}_A \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^s(M_b^2, \mathrm{diag}_b; \Omega_b^{1/2}) \otimes \mathcal{C}^\infty(M_b^2; \beta_b^* \,\mathrm{Hom}(F \otimes \Omega_b^{-1/2}, E \otimes \Omega_b^{1/2})),$$

and similarly $A \in \Psi_0^{s,\mathcal{E}}(M; E, F)$.

Adjoints of b-operators are again b-operators and likewise 0-operators. That is, for each value of $a \in \mathbb{R}$, the action of say $P \in \Psi_b^{s,\mathcal{E}}(M; \Omega_b^{1/2})$ on $x^a L_b^2(M; \Omega_b^{1/2})$ determines an abstract adjoint operator P_a^*, and the claim is that there is an element of the b calculus whose action on this space coincides with P_a^*. Indeed, if $a = 0$, the identity

$$(Pu, v)_{L_b^2} = (u, P_0^* v)_{L_b^2}, \text{ for every } u, v \in \mathcal{C}_c^\infty(M; \Omega_b^{1/2})$$

indicates that the distributional kernel of P_0^* is obtained from that of P by reflecting across the diagonal diag_b and taking complex conjugate. Thus

$$P \in \Psi_b^{s,(E_{10},E_{01},E_{11})}(M; F, G) \iff P_0^* \in \Psi_b^{s,(E_{01},E_{10},E_{11})}(M; G, F).$$

On the other hand, to find the distributional kernel of P_a^*, note that

$$\begin{aligned}(Pu, v)_{x^a L_b^2} &= (x^{-a} Pu, x^{-a} v)_{L_b^2} = (u, P_0^* x^{-2a} v)_{L_b^2} \\ &= (x^{-a} u, x^{-a}(x^{2a} P_0^* x^{-2a}) v)_{L_b^2} = (u, (x^{2a} P_0^* x^{-2a}) v)_{x^a L_b^2}\end{aligned}$$

hence

$$P \in \Psi_b^{s,(E_{10},E_{01},E_{11})}(M; F, G) \iff P_a^* \in \Psi_b^{s,(E_{01}+2a,E_{10}-2a,E_{11})}(M; G, F).$$

The same analysis holds *verbatim* for 0-operators.

5. Parametrices in the b and zero calculi

The symbolic calculus allows one to construct a 'parametrix' of any elliptic operator up to a smoothing error. This can be done very generally on manifolds with bounded geometry but is not enough for Fredholmness since smoothing errors are not generally compact. By restricting to b and 0 metrics there is a natural way of studying the behavior of an operator at infinity, namely via the normal operator, and Fredholm operators are precisely those elliptic operators whose normal operator is invertible. So far the analysis of an operator in the 'small' calculus does not require leaving the small calculus. In this section we show that by using the large calculus it is possible to construct an actual parametrix of a Fredholm b or 0-operator (i.e., one with a compact error). Moreover, as Fredholm operators on Hilbert spaces these operators have a distinguished parametrix, their generalized inverse, and we show that these too are elements of the corresponding large calculus.

5.1. **Analytic and meromorphic functions.**

We will often make use of vector-valued analytic and holomorphic functions. We briefly recall the definitions and refer to [Rudin, Functional Analysis] for a complete discussion.

Let $D \subseteq \mathbb{C}$ be a non-empty open subset and $\mathcal{B}$ a complex topological space. A function $F : D \to \mathcal{B}$ is said to be weakly holomorphic in D if for every continuous linear functional, $\Lambda \in \mathcal{B}^*$,

$$\Lambda \circ F : D \to \mathbb{C}$$

is a holomorphic function. Similarly, F is said to be strongly holomorphic in D if, for every $z \in D$, the limit

$$\lim_{z \to w} \frac{F(z) - F(w)}{z - w}$$

exists in $\mathcal{B}$. Clearly a strongly holomorphic function is weakly holomorphic. If $\mathcal{B}$ is a Frechet space, then every weakly holomorphic function is also strongly holomorphic, and moreover the Cauchy theorem holds: if γ is a closed path in D that is contractible within D, then

$$\int_\gamma F(z)\, dz = 0.$$

As usual, this implies that a holomorphic function has a power series expansion around each point in D.

A function F is meromorphic in D if at each point $\zeta \in D$ there is an integer $k \in \mathbb{N}_0$ and a neighborhood $\mathcal{U} \subseteq D$ of ζ such that $(z - \zeta)^k F(z)$ is holomorphic in $\mathcal{U}$. In which case the *order* of a point $\zeta \in D$ is defined to be the smallest k such that $(z - \zeta)^k F(z)$ is holomorphic in some neighborhood of ζ. The points with positive order are known as the *poles* of F. A meromorphic function has a Laurent series around each point $\zeta \in D$,

$$F(z) = \sum_{j=1}^{\text{order}(\zeta)} (\zeta - z)^{-j} A_j + F_0(z),$$

and A_1 is known as the *residue* at ζ.

If F is a meromorphic function into a space of linear operators, then there is a further measure of the size of a pole. We define the *rank* of a pole ζ to be

$$\text{Rank}(\zeta) = \sum_{j=1}^{\text{order}(\zeta)} \text{rank}\, A_j.$$

Notice that $\text{order}(\zeta) \leq \text{Rank}(\zeta)$.

We start out with a simple, but very useful, lemma.

Lemma 5.1 (Analytic Fredholm Theory). *Let $\mathcal{B}(\mathcal{H})$ the space of bounded operators on a Hilbert space $\mathcal{H}$, $D \subseteq \mathbb{C}$ be connected open subset of $\mathbb{C}$, and let $T : D \to \mathcal{B}(\mathcal{H})$ be an operator-valued analytic function taking values in the compact operators on $\mathcal{H}$. If there is a single point in D where $\mathrm{Id} - T(z)$ is invertible, then there is a discrete set $S \subseteq D$ such that $\mathrm{Id} - T(z)$ is invertible on $D \setminus S$. In which case the function*

$$z \mapsto (\mathrm{Id} - T(z))^{-1}$$

is analytic on $D \setminus S$ and extends to a meromorphic function on D with finite rank residues.

Proof. We will show that each point $z_0 \in D$ has a neighborhood $\mathcal{U}$ of one of two types: either $\mathrm{Id} - T(z)$ fails to be invertible at a most a discrete set of points in $\mathcal{U}$ or $\mathrm{Id} - T(z)$ fails to be invertible at every point in $\mathcal{U}$. It follows that the neighborhoods of the latter type form a closed and open set, hence consist of either D or the empty set. Since by assumption there is a point in D where $\mathrm{Id} - T(z)$ is invertible, all points will have a neighborhood of the former type.

First, if $\mathrm{Id} - T(z_0)$ is invertible, then $\mathrm{Id} - T(z)$ is invertible in a neighborhood of z_0, and

$$\frac{(\mathrm{Id} - T(z))^{-1} - (\mathrm{Id} - T(z_0))^{-1}}{z - z_0} = \frac{(\mathrm{Id} - T(z))^{-1}[T(z) - T(z_0)](\mathrm{Id} - T(z_0))^{-1}}{z - z_0}$$

shows that $z \mapsto (\mathrm{Id} - T(z))^{-1}$ is analytic near z_0.

If $\mathrm{Id} - T(z_0)$ is not invertible, then let Π_0 be the orthogonal projection onto the (necessarily finite dimensional) null space of $\mathrm{Id} - T(z_0)$ and let $\Pi_0' = \mathrm{Id} - \Pi_0$. The operators $\mathrm{Id} - T(z) + \Pi_0$ are invertible at z_0, hence in a neighborhood of z_0 and the inverse is analytic in this neighborhood. Notice that

$$S(z) = (\mathrm{Id} - T(z) + \Pi_0)^{-1} \implies \mathrm{Id} - T(z) = (\mathrm{Id} - \Pi_0 S(z))(\mathrm{Id} - T(z) + \Pi_0),$$

so $\operatorname{Id}-T(z)$ will be invertible if and only if $\operatorname{Id}-\Pi_0 S(z)$ is invertible. On the other hand, solving $(\operatorname{Id}-\Pi_0 S(z))u = f$ reduces, after decomposing $u = \Pi_0 u + \Pi_0' u$, $f = \Pi_0 f + \Pi_0' f$ and using the orthogonality of the ranges of Π_0 and Π_0', to solving

$$\Pi_0(\operatorname{Id}-S(z))\Pi_0 u = \Pi_0 f - \Pi_0 S(z)\Pi_0' f.$$

Hence near z_0 the invertibility of $\operatorname{Id}-T(z)$ is equivalent to the invertibility of the family of finite dimensional matrices $\Pi_0[\operatorname{Id}-S(z)]\Pi_0$ on $\operatorname{null}(\operatorname{Id}-T(z_0))$, i.e., to having

$$d(z) = \det\left[\Pi_0(\operatorname{Id}-S(z))\Pi_0\right] \neq 0.$$

Since $\Pi_0(\operatorname{Id}-S(z))\Pi_0$ is analytic, so is its determinant. Hence either $d(z)$ is identically zero in a neighborhood of z_0, or its zeros form a discrete subset of this neighborhood. Furthermore, in the latter case, the inverse is given by the 'adjugate matrix' multiplied by $\frac{1}{d(z)}$, so it has a meromorphic continuation to this neighborhood with finite-rank residues. □

5.2. **The Mellin transform.**

One of the most useful tools for working with polyhomogeneous expansions is the Mellin transform.

Recall that this is defined, e.g., for $u \in \mathcal{C}_c^\infty(\mathbb{R}^+, \frac{dx}{x})$ by

$$\mathcal{M}u(\zeta) = \int_0^\infty x^{i\zeta} u(x) \frac{dx}{x},$$

and is simply the Fourier transform after a change of variables, so it extends to an isomorphism between weighted spaces

$$x^a L^2(\mathbb{R}^+, \tfrac{dx}{x}) \to L^2\left(\{\eta = a\}; d\xi\right) \tag{5.1}$$

where $\eta = \operatorname{Im}\zeta$ and $\xi = \operatorname{Re}\zeta$. The inverse of the Mellin transform as a map (5.1) is given by

$$\mathcal{M}^{-1}(v)(x) = \frac{1}{2\pi}\int_{\eta=a} v(\zeta)x^{-i\zeta}\, d\xi.$$

Using the inclusions $x^a L^2 \subset x^b L^2$ whenever $b < a$ one can see that the Mellin transform of a function in $x^a L^2(\mathbb{R}^+, \frac{dx}{x})$ is holomorphic in the half-plane $\{\eta < a\}$.

More generally one can define the Mellin transform for sections of an trivial orientable bundle over a manifold. The result will depend on the choice of trivialization, but can be invariantly defined as a section of an appropriate density bundle. We will mostly work with the Mellin transform in s in the product $\mathbb{R}_s^+ \times X$ or (one of) its compactification $[0,1] \times X$. The Mellin transform of a smooth function on $[0,1]\times X$ will be a family of smooth functions on X.

If u is a smooth function on $[0,1] \times X$ that vanishes to infinite order at both $\{0\}\times X$ and $\{1\}\times X$, then standard properties of the Fourier transform (namely, the Paley-Wiener theorems) show that $\mathcal{M}(u)$ is an entire function valued in $\mathcal{C}^\infty(X)$ satisfying, for any $N, j, k \in \mathbb{N}_0$ and multi-index α,

$$\sup_{|\operatorname{Im}\zeta|\leq N} \left|(1+|\zeta|)^k \partial_\zeta^j D_y^\alpha \mathcal{M}(u)\right| < \infty \tag{5.2}$$

and conversely.

Although we will not need it, it is useful to have a similar characterization when u has non-trivial asymptotics. So let X be a manifold with corners, let E be an index set and let $\mathcal{E}$ be the family of index sets on $[0,1] \times X$ that assigns E to $\{0\} \times X$, ∞ to $\{1\} \times X$, and $\mathbb{N}_0$ to all other boundary hypersurfaces. For any constant $C \in \mathbb{R}$, define the truncated index set

$$E^+(C) = \{(s,p) : \operatorname{Re} s > C\},$$

and let $\mathcal{E}^+(C)$ denote the index family on $[0,1] \times X$ obtained as above but with $E^+(C)$ instead of E. The spaces of meromorphic functions associated to these via the Mellin transform are defined as follows. Let $\mathcal{M}^E(X)$ be the space of all meromorphic functions $f : \mathbb{C} \to \mathcal{C}^\infty(X)$ satisfying

i) if $f \in \mathcal{M}^{\mathcal{E}}(X)$ and f has a pole of order k at $\zeta \in \mathbb{C}$ then $(-i\zeta, k-1) \in E$, and
ii) for large enough $N \in \mathbb{N}$ there is a constant $C_N > 0$ such that

$$\|f(\zeta)\|_{C^N(\partial M)} \leq C_N(1+|\zeta|)^{-N}, \text{ in the region } |\operatorname{Im}\zeta| \leq N, |\operatorname{Re}\zeta| \geq C_N.$$

Proposition 5.2. *The Mellin transform induces an isomorphism between* $\mathcal{A}^{\mathcal{E}}([0,1]\times X)$ *and* $\mathcal{M}^E(X)$. *If* $f \in \mathcal{M}^E(X)$ *and* C *is a real number satisfying*

$$C \notin \{\operatorname{Re} s : (s,p) \in E\}$$

then the inverse Mellin transform corresponding to integration along $\operatorname{Im}\zeta = C$ *sends* f *to* $\mathcal{A}^{\mathcal{E}^+(C)}(X)$.

Remark 5.3. The proof of this proposition (Proposition 5.27 in [Melrose: APS Book]) is very similar to the proof of Proposition 5.5 below.

5.3. **Parametrices in the b-calculus.**

We know that elliptic elements of the small b calculus induce Fredholm maps on weighted Sobolev spaces when their normal operator is invertible. In this section we will show that this is always the case off of a discrete set of weights and construct a parametrix for those cases. We will follow [Melrose, APS Book, Chapter 5] closely.

Our aim is to construct a parametrix with a negligible error. We start out with the symbolic parametrix from the symbolic calculus. Thus if $P \in \Psi_b^s(M)$ is elliptic there is an operator $Q_\sigma \in \Psi_b^{-s}(M)$ such that

$$\operatorname{Id} - Q_\sigma P,\ \operatorname{Id} - PQ_\sigma \in \Psi_b^{-\infty}(M).$$

Since these errors are not compact, we want to improve the construction and 'solve away' the error at the front face. That is, if $R = \operatorname{Id} - PQ_\sigma$, we want to find Q_1 in the b-calculus such that, for some index family $\mathcal{E}$ with $\operatorname{Re}\mathcal{E} > 0$,

$$\mathrm{Id} - P(Q_\sigma + Q_1) \in \Psi_b^{-\infty,\mathcal{E}}(M) \iff N(P)N(Q_\sigma) = N(R).$$

So we want to choose Q_1 so that $N(Q_1) = N(P)^{-1}N(R)$ and we need to study the inverse of the normal operator.

To analyze the normal operator we want to take advantage of its $\mathbb{R}^+$ invariance by conjugating by the Mellin transform. Notice the effect of the Mellin transform on b-vector fields,

$$\mathcal{M}(x\partial_x u)(\zeta) = -i\zeta\mathcal{M}(u)(\zeta),$$
$$\mathcal{M}(Tu)(\zeta) = T\mathcal{M}(u)(\zeta) \text{ for all } T \in \mathrm{Diff}(\partial M).$$

For a b-operator we refer to the Mellin transform (in $s = \frac{x}{x'}$) of the normal operator, $\mathcal{M}(N(P))(\zeta)$, as **the indicial family** of P and denote it $I(P; -i\zeta)$[25]. The indicial family is a one parameter family of operators on ∂M.

An equivalent definition of $I(P;\zeta)$ acting on a b-half density f on ∂M is to choose an extension $\widetilde{f}$ of f to M and then set

$$I(P;\zeta)f = [x^{-\zeta}P(x^\zeta\widetilde{f})]\big|_{\partial M}. \tag{5.3}$$

This shows that the indicial family depends slightly on the choice of bdf; if $\widehat{x} = e^\omega x$ is another bdf, the indicial families satisfy

$$I_{\widehat{x}}(P;\zeta) = e^{-\omega}I_x(P;\zeta)e^\omega.$$

Also, this shows that the indicial family is intimately related to the action of P on polyhomogeneous conormal half-densities. Indeed, for any $\zeta \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, we have

$$P(x^\zeta(\log x)^\ell f) = x^\zeta(\log x)^\ell I(P;\zeta)f\big|_{\partial M} + \begin{cases} \mathcal{O}(x^{\zeta+1}) & \text{if } \ell = 0 \\ \mathcal{O}(x^\zeta(\log x)^{\ell-1}) & \text{else} \end{cases}$$

We define the **boundary spectrum of** P also known as the set of **indicial roots of** P to be

$$\mathrm{spec}_b(P) = \left\{\zeta \in \mathbb{C} : \exists f \in \mathcal{C}^\infty(\partial M; \Omega_b^{1/2}) \setminus \{0\}, I(P;\zeta)f = 0\right\},$$

and the refined set of indicial roots to be

$$\mathrm{Spec}_b(P) = \{(\zeta,\ell) \in \mathbb{C}\times\mathbb{N}_0 : \\ \exists f \in \mathcal{C}^\infty(M;\Omega_b^{1/2}), f\big|_{\partial M} \neq 0, P(x^\zeta(\log x)^\ell f) = \mathcal{O}(x^{\zeta+1})\}.$$

Clearly if a polyhomogeneous conormal half-density $f \in \mathcal{A}^F_{\mathrm{phg}}(M;\Omega_b^{1/2})$ is in the null space of P, we must have $\mathrm{lead}(F) \subseteq \mathrm{Spec}_b(P)$ and the leading coefficients must be in the null space of $I(P;\cdot)$.

[25]This definition differs from [Melrose: APS Book] by the factor of i, but is consistent with [Mazzeo: Edge]

Proposition 5.4. *If* $s > 0$ *and* $P \in \Psi_b^s(M)$ *is elliptic then* $\operatorname{spec}_b(P)$ *satisfies*

$$(\lambda_j) \subseteq \operatorname{spec}_b(P),\ |\lambda_j| \to \infty \implies |\operatorname{Re}\lambda_j| \to \infty. \tag{5.4}$$

The inverse of the indicial family of P *is an analytic function*

$$\mathbb{C} \setminus \operatorname{spec}_b(P) \ni \zeta \mapsto I(P;\zeta)^{-1} \in \Psi^s(\partial M)$$

and extends to a meromorphic function on the complex plane with poles of finite order and rank. Furthermore, the residues at the points of $\operatorname{spec}_b(P)$ *are finite rank smoothing operators.*

Proof. Let Q_σ be a symbolic parametrix for P, so that $R = \operatorname{Id} - PQ_\sigma \in \Psi_b^{-\infty}(M)$. Clearly, if $\operatorname{Id} - I(R;\zeta)$ is invertible we have

$$I(P;\zeta)^{-1} = I(Q_\sigma;\zeta)[\operatorname{Id} - I(R;\zeta)]^{-1} \tag{5.5}$$

and so it suffices to analyze the operators on the right.

Directly from (5.2), the indicial family of R is an entire family of smoothing operators on the boundary with kernel rapidly decreasing as $|\operatorname{Im}\zeta| \to \infty$ in any region where $|\operatorname{Re}\zeta|$ is bounded. In particular, as a family of operators

$$I(R;\zeta) : L^2(\partial X;\Omega^{1/2}) \to L^2(\partial X;\Omega^{1/2}),$$

the operator norm satisfies that for any $\ell, N \in \mathbb{N}$ there is a constant $C_{N,\ell}$ such that

$$\|I(R;\zeta)\| \leq C_{N,\ell}(1 + |\zeta|)^{-\ell}, \text{ whenever } |\operatorname{Re}\zeta| \leq N.$$

So if we denote by $\mathcal{U} \subseteq \mathbb{C}$ the region where this operator norm is less than one then we know that, in any region where $|\operatorname{Re}\zeta|$ is bounded, taking $|\zeta|$ large enough guarantees that $\zeta \in \mathcal{U}$. On $\mathcal{U}$, $\operatorname{Id} - I(R;\zeta)$ is invertible as a map from $L^2(\partial X;\Omega^{1/2})$ to itself, so we can apply analytic Fredholm theory (Lemma 5.1) and conclude that $(\operatorname{Id} - I(R;\zeta))^{-1}$ defines a meromorphic function on the complex plane with poles on a discrete set and finite rank residues. On $\mathcal{U}$ we can write the inverse using a Neumann series, thus

$$(\operatorname{Id} - I(R;\zeta))^{-1} = \operatorname{Id} - S(\zeta)$$
$$\implies S(\zeta) = I(R;\zeta) + I(R;\zeta)^2 + I(R;\zeta)S(\zeta)I(R;\zeta),$$

and by unique continuation the meromorphic functions on either side of the equality coincide everywhere. In particular, we see that $S(\zeta)$ takes values in the smoothing operators on the boundary and satisfies the same type of rapid decay estimates as $I(R;\zeta)$. This shows that the poles of $\operatorname{Id} - S(\zeta)$ satisfy (5.4) and hence, by the next paragraph, so do the elements of $\operatorname{spec}_b(P)$.

Thus we only need to analyze $I(Q_\sigma;\zeta)$. We can assume, by adding a smoothing operator to Q_σ, that the kernel of Q_σ is supported in a small neighborhood of the diagonal. Since we have already analyzed the indicial families of smoothing operators, there is no loss of generality in assuming this has been done. In a coordinate chart, the distributional kernel of the indicial family of Q_σ is given by

$$I(Q_\sigma;\zeta) = \frac{1}{(2\pi)^n}\int e^{i(y-y')\eta} b(0,y',\zeta,\eta)\, d\eta \tag{5.6}$$

where the amplitude b is given by

$$b(r,y',\zeta,\eta) = \int e^{i\zeta \log s + i\eta(y-y')} \mathcal{K}_{Q_\sigma}(r,s,y,y') \frac{ds}{s} dy'.$$

The representation (5.6) shows that $I(Q_\sigma;\zeta)$ is an entire family of operators in $\Psi^{-s}(\partial M)$.

It follows from (5.5) that $I(P;\zeta)^{-1}$ is meromorphic and that its residues are finite rank smoothing operators. □

Having described the inverse of the indicial family, we can apply the inverse of the Mellin transform and find an inverse of the normal operator. Recall that inverting the Mellin transform requires choosing a constant C and integrating along $\{\operatorname{Im}\zeta = C\}$, which is not a problem for $I(P;-i\zeta)^{-1}$ as long as we avoid its poles, i.e., $\operatorname{spec}_b(P)$. It will also be useful to keep track of the poles 'above' $\operatorname{Re}\zeta = C$ and those 'below' this line, so we define

$$\Sigma^+(C) = \{(\gamma,p) \in \operatorname{Spec}_b(P) : \operatorname{Re}\gamma > C\},$$
$$\Sigma^-(C) = \{(-\gamma,p) \in \operatorname{Spec}_b(P) : \operatorname{Re}\gamma < C\}.$$

(The change from γ to $-\gamma$ in $\Sigma^-(C)$ is related to the fact that we use boundary defining functions s and $1/s$ for zero and infinity in $\overline{\mathbb{R}^+}$.) These are index sets and we replace them with their smallest smooth index supersets without changing the notation.

Proposition 5.5. *If $s > 0$ and $P \in \Psi^s_b(M)$ is elliptic then the normal operator of P induces an isomorphism*

$$N(P) : x^a H^t_b(\partial M^+;\Omega_b^{1/2}) \to x^a H^{t-s}_b(\partial M^+;\Omega_b^{1/2}),$$

and hence P is Fredholm on $x^a H^t_b(M;\Omega_b^{1/2})$, if and only if $a \notin \{\operatorname{Re}\zeta : \zeta \in \operatorname{spec}_b(P)\}$.

In which case there exists $Q \in \Psi_b^{-s,(\Sigma^+(a),\Sigma^-(a),\mathbb{N}_0)}(M)$ such that

$$\begin{gathered}\sigma(Q) = \sigma(P)^{-1},\\ N(Q) = N(P)^{-1} \text{ (as maps on } x^a H^t_b(\partial M^+;\Omega_b^{1/2}) \text{).}\end{gathered} \tag{5.7}$$

Any such Q is a parametrix for P as an operator on $x^a H^t_b(M;\Omega_b^{1/2})$ for any $t \in \mathbb{R}$.

Proof. From the proof of the previous proposition, we know that the inverse of the indicial family of P has the form

$$I(P;\zeta)^{-1} = I(Q_\sigma;\zeta)(\operatorname{Id} - S(\zeta)) = I(Q_\sigma;\zeta) + S_1(\zeta),$$

and so we want to find an element of the large calculus Q_1 such that $I(Q_1;\zeta) = S_1(\zeta)$. If there is such an element then the kernel of its normal operator would satisfy $\mathcal{M}(\mathcal{K}_{N(Q_1)})(\zeta) = \mathcal{K}_{S_1}(-i\zeta)$, so we will define a putative kernel, $\mathcal{K}_N$, by taking the inverse Mellin transform of $S_1(-i\zeta)$.

Recall that $\mathcal{K}_{S_1}(\zeta)$ is a meromorphic family valued in $\mathcal{C}^\infty((\partial M)^2)$ with poles only at $\operatorname{spec}_b(P)$ and with rapid decay as $|\operatorname{Im}\zeta| \to \infty$. Thus we can take the inverse Mellin transform of the kernels $\mathcal{K}_{S_1}(-i\zeta)$ by integrating along the line $\operatorname{Im}\zeta = a$, (recall the notation $\operatorname{Re}\zeta = \xi$, $\operatorname{Im}\zeta = \eta$)

$$\mathcal{K}_N(s,y,y') = \frac{1}{2\pi}\int_{\eta=a} s^{-i\zeta}\mathcal{K}_{S_1}(-i\zeta,y,y')\,d\xi,$$

and we need to establish that this is polyhomogeneous.

Directly estimating the integral near $s=0$ we find

$$\left|D_s^p D_{y,y'}^\delta \mathcal{K}_N(s,y,y')\right| \le C|s|^{a-p}$$

and in particular the kernel vanishes at the boundary to order a. By integrating along $\operatorname{Im}\zeta = N+\varepsilon$ instead of integrating along $\operatorname{Im}\zeta = a$ we can arrange for the kernel to vanish to arbitrarily large order at the boundary. We can compare the integral along these two lines using Cauchy's formula and the rapid vanishing of the integrand, and find that the value of the integral changes by the residues at the poles of $\mathcal{K}_{S_1}(-i\zeta)$. Thus we find that, near $s=0$,

$$\mathcal{K}_N(s,y,y') - \sum_{\substack{(\alpha,p)\in\operatorname{Spec}_b(P)\\ \operatorname{Re}\alpha\in[a,N]}} s^\alpha(\log s)^p A_{\alpha,p}(y,y') \in \dot{\mathcal{C}}^N(\mathbb{R}^+\times(\partial M)^2),$$

where $\dot{\mathcal{C}}^N$ denotes functions in $\mathcal{C}^N$ that vanish to order N at the boundary. This is precisely what is needed to show that $\mathcal{K}_N$ is polyhomogeneous with index set $E^+(a)$ at $s=0$.

After a change of variable, $\zeta \mapsto -\zeta$ and $s \mapsto \frac{1}{s}$, the same argument can be applied to find the expansion at $s=\infty$. Since the faces $s=0$ and $s=\infty$ correspond to $\mathfrak{B}_{11}\cap\mathfrak{B}_{10}$ and $\mathfrak{B}_{11}\cap\mathfrak{B}_{01}$ respectively, we have shown that $\mathcal{K}_N$ is a polyhomogeneous conormal function on the front face of M_b^2. So there is an element of the large calculus, $Q_1 \in \Psi_b^{-\infty,(\Sigma^+(a),\Sigma^-(a),\mathbb{N}_0)}(M)$, such that $N(Q_1) = \mathcal{K}_N$ and hence $I(Q_1;\zeta) = S_1(\zeta)$. It follows that $Q = Q_\sigma + Q_1$ is an operator of the form required in the proposition.

It follows directly from the formula for composition (Theorem 4.20) that any Q' that satisfies (5.7) also satisfies

$$\operatorname{Id} - PQ',\ \operatorname{Id} - QP' \in \Psi_b^{-1,(\Sigma^+(a),\Sigma^-(a),\mathbb{N}_0+1)}(M)$$

and hence, by the mapping properties (Theorem 4.21), Q' is a parametrix for P as an operator on $x^a H_b^t(M;\Omega_b^{1/2})$. We point out that for $Q = Q_\sigma + Q_1$ the residue operator is in a better space:

$$\operatorname{Id} - PQ,\ \operatorname{Id} - QP \in \Psi_b^{-\infty,(\Sigma^+(a),\Sigma^-(a),\mathbb{N}_0+1)}(M).$$

□

We next improve the parametrix obtained in this proposition to one where the error term vanishes to infinite order at the front face. Similarly to the construction on a closed manifold, we would like to go about this by considering the Neumann series for $PQ = \mathrm{Id} - R$, i.e., $\mathrm{Id} + R + R^2 + \dots$. On the one hand, this requires that we first improve the error so that it vanishes to infinite order at a side face, since otherwise the series will not converge even asymptotically. On the other hand, taking powers of R will mean taking extended unions of the corresponding index sets, so we define for every $C \in \mathbb{R}$

$$\widehat{\Sigma}^+(C) = \overline{\bigcup_{k \in \mathbb{N}_0}} [\Sigma^+(C) + k], \quad \widehat{\Sigma}^-(C) = \overline{\bigcup_{k \in \mathbb{N}_0}} [\Sigma^-(C) + k],$$

These index sets are large enough to include all of the 'accidental multiplicities' that occur in constructing the refined parametrix.

The following lemma is, for b-differential operators, the fact that one can solve a regular singular equation by adding powers of logarithms when the right hand side decays like an indicial root.

Lemma 5.6. *Let $P \in \Psi_b^s(M)$ be elliptic operator of positive order $s > 0$ and let $f \in \mathcal{A}^E_{\mathrm{phg}}(M; \Omega_b^{1/2})$. For any $C > \operatorname{Re} E$, there is solution of the problem*

$$Pu - f \in \dot{\mathcal{C}}^\infty(M; \Omega_b^{1/2}), \quad u \in \mathcal{A}_{\mathrm{phg}}^{E \overline{\cup} \widehat{\Sigma}^+(C)}(M; \Omega_b^{1/2}).$$

Proof. Fix a product decomposition of a collar neighborhood of the boundary $[0,1)_x \times \partial M$. From the definition of the indicial operator it is clear that we can prescribe the coefficient of $x^s (\log x)^p$ if $(s,p) \notin \mathrm{Spec}_b(P)$, so we only need to consider $(s,p) \in \mathrm{Spec}_b(P)$.

Assume $s \in \mathrm{spec}_b(P)$. Since $I(P;\zeta)^{-1}$ is meromorphic we can find a small path $\gamma(z)$ around s containing no other pole of $I(P;\zeta)^{-1}$. Let $v \in \mathcal{C}^\infty(\partial M)$, replacing $I(P;\zeta)^{-1}$ with its Laurent expansion at s we find that

$$\frac{1}{2\pi} \oint_{\gamma(z)} \frac{x^{-i\zeta} I(P;\zeta)^{-1}}{\zeta - s} v(y)\, d\zeta = \sum_{j=0}^{\mathrm{ord}(s)} x^s (\log x)^j u_{s,j}, \tag{5.8}$$

with $u_{s,j} \in \mathcal{C}^\infty(\partial M)$. Let u be a half-density supported in the collar neighborhood with expansion at the front face given by the (5.8), then from (5.3)

$$P(u) = x^s v + \sum_{j=0}^{\mathrm{ord}(s)} x^{s+1} (\log x)^j v_{s,j},$$

and since v is arbitrary, this means we can solve $Pu = x^s v + \mathcal{O}(x^{s+1})$. To solve $Pu = x^s (\log x)^p v$ note that $x^s (\log x)^p = (\partial_s)^p x^s$, and we can proceed as before. □

With this lemma in hand, we can complete the construction of a parametrix. We will need one more index set,

$$\widehat{\Sigma}(a) = \{(0,0)\} \cup \Big[(\mathbb{N}_0 + 1) \overline{\cup} (\widehat{\Sigma}^+(a) + \widehat{\Sigma}^-(a)) \Big].$$

Proposition 5.7. *Let $s > 0$, $P \in \Psi_b^s(M)$ be elliptic and let*

$$a \in \mathbb{R} \setminus \{\operatorname{Re}\zeta : \zeta \in \operatorname{spec}_b(P)\}.$$

There are elements of the large calculus

$$Q_\ell, Q_r \in \Psi_b^{-s,(\widehat{\Sigma}^+(a),\widehat{\Sigma}^-(a),\widehat{E}(a))}(M)$$

such that

$$\operatorname{Id} - Q_\ell P \in \Psi_b^{-\infty,(\widehat{\Sigma}^+(a),\infty,\infty)}(M), \tag{5.9}$$

$$\operatorname{Id} - P Q_r \in \Psi_b^{-\infty,(\infty,\widehat{\Sigma}^-(a),\infty)}(M). \tag{5.10}$$

Proof. We start with the parametrix $Q_\sigma + Q_1$ from the proof of the previous proposition. As stated above, our first concern is to improve the error term at the left face, $\mathfrak{B}_{01}$. So we seek Q_2 such that $\operatorname{Id} - P(Q_\sigma + Q_1 + Q_2)$ vanishes to infinite order at $\mathfrak{B}_{10}$, i.e., such that PQ_2 has the same polyhomogeneous expansion at $\mathfrak{B}_{10}$ as $R_1 = \operatorname{Id} - P(Q_\sigma + Q_1)$.

From the composition formula we know that

$$P \circ \Psi_b^{-\infty,\mathcal{E}}(M) \subseteq \Psi_b^{-\infty,\mathcal{E}}(M).$$

Let $\widetilde{Q} \in \Psi_b^{-\infty,\mathcal{E}}(M)$, $(s,p) \in \operatorname{lead}(E_{10})$, and let $A_{s,p}$ be the coefficient of $\rho_{10}^s (\log \rho_{10})^p$ in the expansion of $\mathcal{K}_{\widetilde{Q}}$ at $\mathfrak{B}_{10}$. From the proof of the composition formula, it is easy to see that if the corresponding coefficient in the expansion of $\mathcal{K}_{P\widetilde{Q}}$ is $I(P;\zeta)A_{s,p}$.

In the construction of Q_1 in the Proposition 5.5 we specified that $N(Q_1) = \mathcal{K}_N$ but nothing more. Since $\mathcal{K}_N$ was constructed to invert the indicial family of P, simply demanding that Q_1 be obtained by extending $\mathcal{K}_N$ into M_b^2 without changing the asymptotics at the left face, $\mathfrak{B}_{10}$, improves the error at this face to

$$R_1 = \operatorname{Id} - P(Q_\sigma + Q_1) \in \Psi_b^{-\infty,(\Sigma'(a),\Sigma^-(a),\mathbb{N}_0+1)}(M),$$
$$\text{where } \Sigma'(a) = \Sigma^+(a) \setminus \operatorname{lead}(\Sigma^+(a)).$$

Proceeding iteratively using Lemma 5.6 we can find an element of the large calculus supported away from the right face with index set $\Sigma'(a)\overline{\cup}\widehat{\Sigma}^+(a) \subseteq \widehat{\Sigma}^+(a)$ at the left face,

$$Q_2 \in \Psi_b^{-\infty,(\widehat{\Sigma}^+(a),\infty,\mathbb{N}_0+1)}(M), \text{ such that}$$
$$R_1 - PQ_2 \in \Psi_b^{-\infty,(\infty,\Sigma^-(a),\mathbb{N}_0+1)}(M).$$

Let $R_2 = \mathrm{Id} - P(Q_\sigma + Q_1 + Q_2)$ $(= R_1 - PQ_2)$ and note that

$R_2^\ell \in \Psi_b^{-\infty,(\infty,\Sigma_\ell^-(a),\mathbb{N}_0+\ell)}$,

$$\text{where } \Sigma_\ell^-(a) = \Sigma^-(a)\overline{\cup}(\Sigma^-(a)+1)\overline{\cup}\cdots\overline{\cup}(\Sigma^-(a)+\ell-1) \subseteq \widehat{\Sigma}^-(a),$$

hence the asymptotic sum $S \sim \sum R_2^\ell$ satisfies

$$S \in \Psi_b^{-\infty,(\infty,\widehat{\Sigma}^-(a),\mathbb{N}_0+1)}(M),$$
$$R' = \mathrm{Id} - (\mathrm{Id} - R_2)(\mathrm{Id} + S) \in \Psi_b^{-\infty,(\infty,\widehat{\Sigma}^-(a),\infty)}(M).$$

This gives us a right parametrix

$$Q_r = (Q_\sigma + Q_1 + Q_2)(\mathrm{Id} + S) \in \Psi_b^{-s,(\widehat{\Sigma}^+(a),\widehat{\Sigma}^-(a),\widehat{\Sigma}(a))}(M)$$

with error term $R_r = \mathrm{Id} - PQ$ in the space we wanted.

To get a left parametrix we will apply the same construction to an adjoint of P. Let P' denote the transpose of P with respect to a b-density. Notice that

$$(x^{-z}Px^z u, v) = (u, x^z P' x^{-z} v) \text{ implies}$$
$$(\zeta,\ell) \in \mathrm{Spec}_b(P') \iff (-\zeta,\ell) \in \mathrm{Spec}_b(P).$$

Thus

$$a \in \mathbb{R} \setminus \{\mathrm{Re}\,\zeta : \zeta \in \mathrm{spec}_b(P)\} \iff -a \in \mathbb{R} \setminus \{\mathrm{Re}\,\zeta : \zeta \in \mathrm{spec}_b(P')\},$$

and with self-explanitory notation

$$\begin{aligned}\Sigma_P^+(a) &= \{(\gamma,\ell) \in \mathrm{Spec}_b(P) : \mathrm{Re}\,\gamma > C\} \\ &= \left\{(-\gamma,\ell) \in \mathrm{Spec}_b(P') : \mathrm{Re}\,\gamma < C\right\} = \Sigma_{P'}^-(a), \text{ and } \Sigma_P^-(a) = \Sigma_{P'}^+(a)\end{aligned}$$

so we can apply the construction above to find a right parametrix for P':

$$Q_r' \in \Psi_b^{-s,(\widehat{\Sigma}_{P'}^+(a),\widehat{\Sigma}_{P'}^-(a),\widehat{\Sigma}(a))}(M) = \Psi_b^{-s,(\widehat{\Sigma}_P^-(a),\widehat{\Sigma}_P^+(a),\widehat{\Sigma}(a))}(M)$$
$$\text{with } R_r' \in \Psi_b^{-\infty,(\infty,\widehat{\Sigma}_{P'}^-(a),\infty)}(M) = \Psi_b^{-\infty,(\infty,\widehat{\Sigma}_P^+(a),\infty)}(M).$$

Again taking transpose we end up with a left parametrix for P

$$Q_\ell = (Q_r')' \in \Psi_b^{-s,(\widehat{\Sigma}_P^+(a),\widehat{\Sigma}_P^-(a),\widehat{\Sigma}(a))}(M)$$
$$\text{with } R_\ell = (R_r')' \in \Psi_b^{-\infty,(\widehat{\Sigma}_{P'}^-(a),\infty,\infty)}(M) = \Psi_b^{-\infty,(\widehat{\Sigma}_P^+(a),\infty,\infty)}(M).$$

□

The operators Q_ℓ and Q_r are respectively left and right parametrices of P as an operator on $x^a H_b^t(M;\Omega_b^{1/2})$. Since the error terms are very residual, and very residual operators form a bi-ideal (Proposition 4.22) we can now show that the generalized inverse of P is an element of the large calculus. Define one more index set

$$\Sigma(a) = \Sigma^+(a)\overline{\cup}\Sigma^-(a).$$

Theorem 5.8. *Let $s > 0$, $P \in \Psi_b^s(M)$ be elliptic and let*

$$a \in \mathbb{R} \setminus \{\operatorname{Re}\zeta : \zeta \in \operatorname{spec}_b(P)\}$$

so that, as a map

$$x^a H_b^t(M; \Omega_b^{1/2}) \to x^a H_b^{t-s}(M; \Omega_b^{1/2}),$$

P is Fredholm.

There is an element of the large b-calculus,

$$G \in \Psi_b^{-s,(\Sigma(a),\Sigma(a),\mathbb{N}_0)}(M) + \Psi_b^{-\infty,(\Sigma(a),\Sigma(a),\infty)}(M),$$

that, for any $t \in \mathbb{R}$ coincides with the generalized inverse of P as a map

$$x^a H_b^{t-s}(M; \Omega_b^{1/2}) \to x^a H_b^t(M; \Omega_b^{1/2}). \tag{5.11}$$

Thus the error terms

$$\mathcal{P}_\ell = \operatorname{Id} - GP \in \Psi_b^{-\infty,(\Sigma^+(a),\Sigma^+(a)-2a,\infty)}(M),$$
$$\mathcal{P}_r = \operatorname{Id} - PG \in \Psi_b^{-\infty,(\Sigma^-(a)+2a,\Sigma^-(a),\infty)}(M)$$

project $x^a H_b^*(M)$ orthogonally onto the null space and cokernel of P respectively.

Proof. Since P has closed range there is no problem defining G as an abstract operator, say from $x^a H_b^s(M; \Omega_b^{1/2})$ to $x^a L_b^2(M; \Omega_b^{1/2})$ such that

$$\mathcal{P}_\ell = \operatorname{Id} - GP, \quad \mathcal{P}_r = \operatorname{Id} - PG$$

are respectively the orthogonal projections onto $\operatorname{null}(P)$ and off of $\operatorname{Ran}(P)$.

An element of the null space f satisfies $R_\ell f = f$ hence from Proposition 4.22 f is polyhomogeneous with index set $\widehat{\Sigma}^+(a)$ at the boundary. In fact once we know that f is polyhomogeneous (say with index set F) and in the null space of P we can improve the index set to $\Sigma^+(a)$ since we must have $\operatorname{lead}(F) \subseteq \operatorname{Spec}_b(P)$.

The null space of P is finite dimensional (since P is Fredholm) and hence the projection of $x^a L^2$ onto the null space has integral kernel

$$\mathcal{P}_\ell(x, y, x', y') = \sum f_i(x, y)(x')^{-2a} f_i(x', y') \in \Psi_b^{-\infty,(\Sigma^+(a),\Sigma^+(a)-2a,\infty)}(M).$$

Since the cokernel is the kernel of the adjoint, identical reasoning yields

$$\mathcal{P}_r(x, y, x', y') \in \Psi_b^{-\infty,(\Sigma^-(a)+2a,\Sigma^-(a),\infty)}(M).$$

Recall that the generalized inverse of a Fredholm operator is related to any pair of left and right parametrices via

$$G - Q_\ell = R_\ell Q_r - R_\ell \mathcal{P}_\ell Q_r - Q_\ell \mathcal{P}_r + R_\ell G R_r \tag{5.12}$$

(as shown in (2.4)), and from Proposition 4.22 the right-hand-side is a very residual operator, so we can realize G as an element of the large calculus, $G \in \Psi_b^{-s,\mathcal{G}}(M)$. The equation for G gives us expressions for the index family $\mathcal{G}$, but we can obtain better index sets by analyzing G directly.

In this paragraph we will abbreviate $\mathcal{A}^*_{\mathrm{phg}}(M;\Omega_b^{1/2})$ by $\mathcal{A}^*_{\mathrm{phg}}$, and assume $u \in \mathcal{A}^F_{\mathrm{phg}}(M;\Omega_b^{1/2})$. The equation $PGu = (\mathrm{Id} - \mathcal{P}_r)u$ shows that $PGu \in \mathcal{A}^{F\overline{\cup}\Sigma^-(a)}_{\mathrm{phg}}$. On the other hand we always have $Gu \in \mathcal{A}^{(F+G_{11})\overline{\cup}G_{10}}_{\mathrm{phg}}$ and, since $\mathrm{Re}\, F$ can be taken arbitrarily large, these together imply that G_{10} coincides with $\Sigma^-(a)$ up to terms killed off by P. Since elements of the null space of P are polyhomogeneous with index set $\Sigma^+(a)$, we have $G_{10} \subseteq \Sigma^+(a)\overline{\cup}\Sigma^-(a)$. The same analysis for the adjoint of G using the equation $P^*G^* = \mathrm{Id} - P_\ell$ shows that we can take $G_{10} = \Sigma^+(a)\overline{\cup}\Sigma^-(a)$. Since PG determines G up to elements in the null space of P, we can conclude that

$$G \in \Psi_b^{-s,(\Sigma(a),\Sigma(a),\mathbb{N}_0)}(M) + \Psi_b^{-\infty,(\Sigma(a),\Sigma(a),\infty)}(M),$$

as required. □

As a consequence we note that if

$$u \in x^a H^s_b(M;\Omega_b^{1/2}) \text{ and } Pu \in \mathcal{A}^*_{\mathrm{phg}}(M;\Omega_b^{1/2})$$

then we must have $u \in \mathcal{A}^{\Sigma^+(a)}_{\mathrm{phg}}(M,\Omega_b^{1/2})$. Indeed, we can choose $c \leq a$ such that $C \notin \{\mathrm{Re}\,\zeta : \zeta \in \mathrm{spec}_b(P)\}$ and apply a left parametrix to Pu to see that u is polyhomogeneous; the actual index set follows from the definition of $\mathrm{Spec}_b(P)$.

Also notice that the null space of P on $x^a L^2_b(M,\Omega_b^{1/2})$ can only increase as a decreases and decrease as a increases. Together with the previous paragraph this implies that we can take a large enough so that elements of the null space of P in $x^a L^2_b(M,\Omega_b^{1/2})$ must vanish to infinite order at the boundary, and elements in $x^{-a}L^2_b(M,\Omega_b^{1/2})$ in the cokernel of P must vanish to infinite order at the boundary. Sometimes a unique continuation property holds and we can conclude that P is respectively injective or surjective - this is the case for instance if M is one-dimensional.

5.4. **Interlude: the scattering calculus.**

For later use, it will be useful to discuss another pseudo-differential calculus.

If M is the interior of a manifold with boundary, a metric g on M is called a **scattering metric** if, near the boundary, it takes the form

$$g_{\mathrm{sc}} = \frac{dx^2}{x^4} + \frac{h(x,y)}{dx^2}.$$

The vector fields of bounded pointwise length with respect to a scattering metric form the space of *scattering vector fields*,

$$\mathcal{V}_{\mathrm{sc}} = \left\{ V \in \Gamma(TM) : Vx = \mathcal{O}(x^2), V\big|_{\partial M} = 0 \right\}.$$

This space is a Lie algebra with respect to the usual vector field Lie bracket, and it is the space of sections of a vector bundle over $\overline{M}$, ${}^{\mathrm{sc}}TM$, the *scattering tangent bundle*. The scattering tangent bundle is a non-standard extension of TM from the interior to $\overline{M}$, and scattering metrics extend from TM to ${}^{\mathrm{sc}}TM$ non-degenerately.

The motivating example of a scattering metric is the metric induced by the Euclidean metric on the radial compactification of Euclidean space. That is, if RC is the identification of $\mathbb{R}^m$ with the interior of the half-sphere $\mathbb{S}^m_+$,

$$RC(\zeta) = \left(\frac{1}{\sqrt{1+|\zeta|^2}}, \frac{\zeta}{\sqrt{1+|\zeta|^2}} \right),$$

then the Euclidean metric on $\mathbb{R}^m$ induces a scattering metric on $\mathbb{S}^m_+$.

Scattering metrics have bounded geometry, so the discussion of the uniform calculus applies and can be used to define Sobolev spaces, construct symbolic parametrices, and establish mapping properties. The kernels of uniform operators associated to a scattering metric, as distributions on $\overline{M}^2$, degenerate at the corner $(\partial M)^2$ and, just as for b and 0 metrics, it is convenient to find an alternate compactification of M^2. The 'correct' compactification in this case is the scattering double space[26],

$$M^2_{\mathrm{sc}} = [[M^2; (\partial M)^2], \partial(\mathrm{diag}_b)].$$

And one can proceed as before to define the small scattering calculus of pseudo-differential operators to be those distributions on M^2_{sc} that are conormal with respect to the closure of diag_M (which is known as $\mathrm{diag}_{\mathrm{sc}}$). One difference with the b- and 0-calculi is that the study of scattering differential operators does not require a large calculus. For instance, if a scattering pseudo-differential operator is Fredholm then it has a parametrix in the small scattering calculus (see [Melrose, *op. cit.*] or [Mazzeo-Melrose, Pseudodifferential operators on manifolds with fibered boundaries]).

The normal operator in the scattering calculus is a family of operators parametrized by ∂M. For each point on $q \in \partial M$ we have a translation invariant pseudo-differential operator on $T_q M \cong \mathbb{R}^m$. Conjugating by Fourier transform reduces this to a symbol and so the product of normal operators is commutative (when acting on scalars)[27]

Lemma 5.9. *Elements in the null space of a fully elliptic scattering operator vanish to infinite order at the boundary.*

5.5. **Parametrices in the 0-calculus.**

In this section we use the construction of a parametrix in the b-calculus to construct a parametrix for Fredholm operators in the small 0-calculus. As before, our aim is to show that the generalized inverse of such an operator is an element of the large 0-calculus.

In §5.3 we analyzed the normal operator in the b-calculus by means of its indicial family. An element of the 0-calculus, P, has an indicial family as well, defined using its action on polyhomogeneous expansions by (5.3). It is easy to see that the indicial family of P, as a family of operators on ∂M,

[26] See [Melrose, Spectral and scattering theory for the Laplacian on asymptotically euclidean spaces]

[27] This is a reflection of the fact that $[\mathcal{V}_{\mathrm{sc}}, \mathcal{V}_{\mathrm{sc}}] = x\mathcal{V}_{\mathrm{sc}}$.

is given by multiplication by an element of $\mathcal{C}^\infty(\partial M)$. The indical roots or boundary spectrum of a 0-operator are defined using its indicial family just as in §5.3. There is a unique family of dilation invariant operators on $\mathbb{R}^+$ parametrized by ∂M whose Mellin transform is the indicial family, this is known as the indicial operator of P. In the b-calculus the indicial operator coincides with the normal operator, but this is not true for an element of the 0-calculus.

For instance, for a 0-differential operator

$$P = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha, \tag{5.13}$$

and each point $q \in \partial M$, the normal operator is a $\mathbb{R}^{m-1}_u \rtimes \mathbb{R}^+_s$-invariant operator on T_qM^+ given by

$$N_q(P) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(q)(s\partial_s)^j(s\partial_u)^\alpha,$$

while the indicial operator of P is the operator on $\mathbb{R}^+_s$ given by

$$I_q(P) = \sum_{j\leq k} a_{j,\alpha}(q)(s\partial_s)^j,$$

and the indicial family is the entire family of operators on ∂M

$$I(P;\zeta) = \sum_{j\leq k} a_{j,\alpha}\big|_{\partial M}(\zeta)^j. \tag{5.14}$$

As we have anticipated, the analysis of the normal operator of an element of the zero calculus proceeds by conjugating by the Fourier transform in the translation-invariant directions and the taking advantage of the remaining dilation invariance. Consider a 0-differential operator P as in (5.13). Conjugating $N_q(P)$ by the Fourier transform yields

$$\widehat{N}(P)(q,\eta) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(q)(s\partial_s)^j(si\eta)^\alpha,$$

then dilation invariance suggests introducing new coordinates (for $\eta \neq 0$),

$$t = s|\eta|, \quad \widehat{\eta} = \frac{\eta}{|\eta|}$$

which takes $\widehat{N}(P)$ to

$$\widetilde{N}(P)(q,\widehat{\eta}) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(q)(t\partial_t)^j(ti\widehat{\eta})^\alpha.$$

This is a family of b-operators on $\mathbb{R}^+_s$ depending smoothly on the parameters $(q,\widehat{\eta}) \in S^*\partial M$. Notice that the indicial family of $\widetilde{N}(P)$ is precisely (5.14) and does not depend on the fiber variables $\widehat{\eta}$. Compactifying $\mathbb{R}^+$ by introducing $r = \frac{1}{t}$ takes $\widetilde{N}(P)$ to the **reduced normal operator**, which near $r = 0$ has the form

$$\mathcal{N}(P)(q, \widehat{\eta}) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(q)(r\partial_r)^j r^{-|\alpha|}(i\widehat{\eta})^\alpha$$
$$= r^{-k} \sum_{j+|\alpha|\leq k} \widetilde{a}_{j,\alpha}(q)(r^2\partial_r)^j (i\widehat{\eta})^\alpha.$$

where $\widetilde{a}$ are suitably modified coefficients. Thus, at $r = 0$, the reduced normal operator is a weighted element of the scattering calculus.

Why does this help? Consider the composition of two invariant operators on the group $\mathbb{R}^{m-1} \rtimes \mathbb{R}^+$,

$$\mathcal{K}_{AB}(s, u) = \int \mathcal{K}_A\left((s, u) * (t, v)^{-1}\right) \mathcal{K}_B(t, v) \frac{dt\ dv}{t^m}$$
$$= \int \mathcal{K}_A(\frac{s}{t}, \frac{u-v}{t}) \mathcal{K}_B(t, v) \frac{dt\ dv}{t^m}$$
$$\implies \mathcal{F}(\mathcal{K}_{AB})(s, \eta) = \int e^{-iu\cdot\eta} \mathcal{K}_A\left(\frac{s}{t}, \frac{u-v}{t}\right) \mathcal{K}_B(t, v) \frac{du\ dt\ dv}{t^m}$$
$$= \int e^{-i\widetilde{u}\cdot(t\eta) - iv\cdot\eta} \mathcal{K}_A\left(\frac{s}{t}, \widetilde{u}\right) \mathcal{K}_B(t, v) \frac{d\widetilde{u}\ dt\ dv}{t}$$
$$= \int \mathcal{F}(\mathcal{K}_A)\left(\frac{s}{t}, t\eta\right) \mathcal{F}(\mathcal{K}_B)(t, \eta) \frac{dt}{t}$$

This is *almost*, for each η, the composition of two b-operators. The problem is that $\mathcal{F}(\mathcal{K}_A)$ depends on $t\eta$ instead of η. We get around this by introducing polar coordinates around the zero section in $T^*\partial M$, i.e., replacing η with $R\widehat{\eta}$, since then from

$$\mathcal{F}(\mathcal{K}_{AB})(s, R\widehat{\eta}) = \int \mathcal{F}(\mathcal{K}_A)\left(\frac{s}{t}, tR\widehat{\eta}\right) \mathcal{F}(\mathcal{K}_B)(t, R\widehat{\eta}) \frac{dt}{t}$$

we see that the dependence of $\mathcal{F}(\mathcal{K}_A)$ on $t\widehat{\eta}$ is now lower-order in R and the operators compose as b-operators near $R = 0$.

To analyze the behavior as $R \to \infty$, it is useful to represent the kernel of an operator $A \in \Psi_0^k(M)$ (for simplicity with $k \in \mathbb{Z}$) using its full symbol, a. For instance, consider the coordinates on M_0^2 near the front face

$$\tau = \frac{x - x'}{x + x'}, \quad \rho = x + x', \quad U = \frac{y - y'}{x + x'}, \quad y',$$

for which a bdf for $\mathfrak{B}_{11}$ is ρ, and bdfs for $\mathfrak{B}_{10}$ and $\mathfrak{B}_{01}$ are given by $\tau = -1$ and $\tau = 1$ respectively. If A is supported in a coordinate chart near the front face, then (up to factors of $2\pi i$) its distributional kernel is given in terms of its full symbol by

$$\mathcal{K}_A(\tau, \rho, U, y') = \left[\int_{\mathbb{R}_\xi \times \mathbb{R}_\eta^{m-1}} e^{i(\tau\xi + U\cdot\eta)} a(\rho, y', \xi, \eta)\ d\xi d\eta\right] \left|\frac{d\tau d\rho dU dy'}{(1-\tau^2)^m \rho^m}\right|^{1/2}$$

hence the distributional kernel of its normal operator is[28]

$$\mathcal{K}_{N_q(A)}(\tau, U) = \left[\int_{\mathbb{R}_\xi \times \mathbb{R}^{m-1}_\eta} e^{i(\tau\xi + U\cdot\eta)} a(q,\xi,\eta)\, d\xi d\eta \right] \left| \frac{d\tau dU}{(1-\tau^2)^m} \right|^{1/2}$$

and its Fourier transform yields

$$\mathcal{K}_{\widehat{N}(A)(q,\eta)}(\tau) = \left[\int_{\mathbb{R}_\xi} e^{i\tau\xi} a(q,\xi,\eta)\, d\xi \right] \left| \frac{d\tau}{(1-\tau^2)^m} \right|^{1/2}.$$

To get the reduced normal operator of P we introduce polar coordinates around the zero section of $T^*\partial M$ and we treat the radial function R as a new variable including introducing the appropriate b-half density,

$$\mathcal{K}_{\widetilde{N}(A)(q,\widehat{\eta})}(\tau, R) = \left[\int_{\mathbb{R}_\xi} e^{i\tau\xi} a(q,\xi,R\widehat{\eta})\, d\xi \right] \left| \frac{d\tau dR}{(1-\tau^2)^m R} \right|^{1/2}, \tag{5.15}$$

Finally, we compactify $\mathbb{R}^+_R$ by introducing $S = \frac{1}{R}$. Note that the resulting kernel is singular at $\{S = 0\}$, so we blow-up the point $(\tau = 0, S = 0)$ by introducing the new coordinate $T = \tau/S$. The resulting operator is referred to as the **reduced normal operator of** P, $\mathcal{N}(P)$. Near the new front face and the diagonal, its kernel takes the form

$$\begin{aligned}\mathcal{K}_{\mathcal{N}(A)(y,\widehat{\eta})}(T,S) &= \left[\int_{\mathbb{R}_\xi} e^{iST\xi} a\left(q,\xi,\frac{\widehat{\eta}}{S}\right) d\xi \right] \left| \frac{dTdS}{(1-(ST)^2)^m} \right|^{1/2} \\ &\xrightarrow{\widehat{\xi}=S\xi} \left[\int_{\mathbb{R}_\xi} e^{iT\widehat{\xi}} a\left(q,\frac{\widehat{\xi}}{S},\frac{\widehat{\eta}}{S}\right) d\widehat{\xi} \right] \left| \frac{dTdS}{(1-(ST)^2)^m S^2} \right|^{1/2}\end{aligned}$$

and, since the symbol has an expansion in homogeneous terms, this is easily seen to be S^{-k} times a sc operator.

The reduced normal operator was first considered in [Lauter, Pseudodifferential analysis on conformally compact manifolds][29]. As anticipated, it is a homomorphism into a family of operators parametrized by the cosphere bundle of the boundary, acting on half-densities on the unit interval – which is really a compactification of the inward-pointing normal bundle trivialized by the choice of bdf,

$$\mathcal{N} : \Psi^k_0(M) \longrightarrow \mathcal{C}^\infty(S^*\partial M, \rho_{\mathrm{sc}}^{-k}\Psi^k_{\mathrm{b,sc}}(\overline{N^+M}/\partial M)). \tag{5.16}$$

[28] Notice that the canonical isomorphism $\Omega_0(M_0^2)\big|_{\mathfrak{B}_{11}} \cong \Omega_0(\mathfrak{B}_{11})$ is given by

$$\left| \frac{d\tau d\rho dU dy'}{(1-\tau^2)^m \rho^m} \right| \mapsto \left| \frac{d\tau dU dy'}{(1-\tau^2)^m} \right|$$

and restricting to the fiber over q removes the dy' factor.

[29] In [Mazzeo: Edge] the normal operator is analyzed without compactifying $\mathbb{R}^+_R$. Instead of a family of b, sc operators one obtains a family of b-operators with a 'Bessel structure at infinity'.

As such, $\mathcal{N}(P)$ in turn has three model operators. First the interior symbol, $\sigma_{\mathrm{b,sc}}(\mathcal{N}(P))$, given at the point $(u;\omega) \in S^*\mathcal{I}$ $(= \mathcal{I} \times \{\pm 1\})$ by

$$\sigma_{\mathrm{b,sc}}\left(\mathcal{N}(P)(q,\widehat{\eta})\right)(u,\omega) = \sigma_0(P)(0,q;\omega,0) \tag{5.17}$$

as can be checked from (5.15) using the Taylor expansion of $a\,(r,y,\xi,\eta)$ in η. Next, the b-indicial family of $\mathcal{N}(P)$ which coincides with that of P. The kernel of the indicial operator is obtained from that of $\mathcal{N}(P)$ by restricting to $R = 0$,

$$\left[\int_{\mathbb{R}_\xi} e^{i\tau\xi} a(q,\xi,0)d\xi\right] \left|\frac{d\tau}{(1-\tau^2)^m}\right|^{1/2},$$

and Mellin transform in $s = (1+\tau)/(1-\tau)$ produces the b-indicial family. We point out that the b-indicial operator/family depends only on the parameters in ∂M and not on the fiber variables in $S^*\partial M$. Finally there is a 'suspended family' obtained by restricting the kernel of $\mathcal{N}(P)$ to the sc face, i.e., setting $S = \frac{1}{R}$ equal to zero. Near this face the kernel has the form (up to a density factor)

$$S^{-k} \int_{\mathbb{R}_\xi} e^{iT\widehat{\xi}}\, \sigma_0(A)(q,\widehat{\xi},\widehat{\eta})d\widehat{\xi}$$

and the cusp suspended family is given by taking the Fourier transform, thus

$$I_{\mathrm{sc}}\left(\mathcal{N}(A)(q,\widehat{\eta})\right)(\xi) = \sigma_0(A)(q,\xi,\widehat{\eta}).$$

One of the main difficulties in dealing with the reduced normal operator is that as a map (5.16) it is not surjective. For instance, the b-indicial family is independent of the fiber variable in $S^*\partial M$, and more generally (5.15) shows that the reduced normal operator is smooth as a function of $R\widehat{\eta}$ and not just R and $\widehat{\eta}$ separately. Indeed it is obvious from the construction that in order for a family of operators to be in the image of (5.16) it needs to be smooth in $\eta = R\widehat{\eta}$ and we need to be able to take the inverse Fourier transform in η and end up with a distribution conormal to the origin. Thus the proof of the following proposition is mostly concerned with making sure the constructions stay in the image of the reduced normal operator.

Proposition 5.10. *Let $P \in \Psi_0^s(M)$ be an elliptic operator of positive order such that $\mathrm{spec}_b(P)$ is a discrete set, choose*

$$a \in \mathbb{R} \setminus \{\mathrm{Re}\,\zeta : \zeta \in \mathrm{spec}_b(P)\},$$

and assume that $\dim \mathrm{null}_{x^a L^2_0} \mathcal{N}(P)(q,\widehat{\eta})$ is independent of $(q,\widehat{\eta}) \in S^*\partial M$, then there is an element of the large 0-calculus

$$Q_1 \in \Psi_0^{-s,(\Sigma(a),\Sigma(a),\mathbb{N}_0)}(M)$$

such that $\mathcal{N}(Q_1)(q,\widehat{\eta})$ is, for every $(q,\widehat{\eta}) \in S^*\partial M$, the generalized inverse of $\mathcal{N}(P)(q,\widehat{\eta})$ as an operator on $\rho_b^a\rho_{\mathrm{sc}}^t H^{t'}_{\mathrm{b,sc}}([0,1])$.

Remark 5.11. While it is always true that $\operatorname{spec}_b(\mathcal{N}(P)(q,\widehat{\eta}))$ is discrete, and $\operatorname{spec}_b(P) = \bigcup_{(q,\widehat{\eta})} \operatorname{spec}_b(\mathcal{N}(P)(q,\widehat{\eta}))$, it is not necessarily true that $\operatorname{spec}_b(P)$ is discrete.

Proof. Showing that the generalized inverse is in the b, sc-calculus is easy given the material covered in previous sections, the difficulty lies in showing that the generalized inverse is in the image of the reduced normal operator.

Consider for example an element $T \in \Psi_0^{-\infty,(F_{10},F_{01},\mathbb{N}_0)}(M)$. The restriction of the kernel of T to the *interior* of the front face is a smooth function which extends to a polyhomogeneous function on the fibrewise radial compactification of the front face. For instance, in the local coordinates near the front face

$$\tau = \frac{x-x'}{x+x'}, \quad U = \frac{y-y'}{x+x'}, \quad r = x+x', \quad y',$$

the kernel of $N(T)$ is a smooth function in $U \in \mathbb{R}$ and $\tau \in (-1,1)$ with polyhomogeneous expansions as $\tau \to \pm 1$ and $U \to \pm\infty$. The index sets at $\tau = -1$ and $\tau = 1$ are F_{10} and F_{01} respectively, while as $|U| \to \infty$ we approach the corner at the intersection of $\mathfrak{B}_{10}$ and $\mathfrak{B}_{01}$ so the index set is $F_{10} + F_{01}$ and the expansion is determined by the expansions at $\tau = \pm 1$. Applying Fourier transform in U yields a distribution in η conormal with respect to $\{\eta = 0\}$ with a homogeneous expansion of order $-\operatorname{Re}(F_{10} + F_{01})$. The reduced normal operator is obtained by replacing η with $R\widehat{\eta}$ and by the analysis above we know that we have rapid decay as $R \to \infty$. It thus seems that there is not an obvious characterization of the range of the reduced normal operator, even for smoothing operators[30]. However this does show that a simple sufficient condition for an element in $\mathcal{C}^\infty(S^*\partial M, \Psi_{\mathrm{b,sc}}^{-\infty,\mathcal{E}}(\overline{N^+M}/\partial M))$ to be in the image of the reduced normal operator is that its distributional kernel be a Schwartz function of $\eta = R\widehat{\eta}$.

Our plan is to review the construction of the parametrix in the b-calculus staying in the image of the reduced normal operator at each step. First, let Q_σ be a symbolic parametrix of P so that

$$R_\sigma = \operatorname{Id} - PQ_\sigma \in \Psi_0^{-\infty}(M).$$

The reduced normal operator of R_σ vanishes to infinite order at the scattering end and so can be thought of as an element of

$$\mathcal{N}(R_\sigma) \in \mathcal{C}^\infty(S^*\partial M, \rho_1^\infty \Psi_b^{-\infty}(\overline{N^+M}/\partial M)).$$

To improve this parametrix we invert the indicial family of the operators $\mathcal{N}(\operatorname{Id} - R_\sigma)$. As a family of b-operators for each (q,η) the indical family of $\mathcal{N}(\operatorname{Id} - R_\sigma)$ (which in fact only depends on q) is invertible off of a discrete set which, by our assumption on a, does not intersect $\{\operatorname{Im}\zeta = a\}$. Its inverse $[\operatorname{Id} - S(\zeta)](q) = I[\mathcal{N}(\operatorname{Id} - R_\sigma)(q,\eta)]^{-1}$ is a smooth family of meromorphic functions with values in the 'smoothing operators' on $\{0\}$. We can define a polyhomogeneous function $\mathcal{K}_N(\tau, q)$ on the b-front face by taking the inverse

[30]On the other hand, if $F_{10} = F_{01} = \infty$, this is much simpler.

Mellin transform of $I[\mathcal{N}(Q_\sigma)](\zeta)S(\zeta)$ along the line $\{\operatorname{Im}\zeta = a\}$. Let χ be a Schwartz function on $\mathbb{R}^{m-1}$ that is identically equal to one in a neighborhood of the origin and let $\widetilde{Q}_1$ be the operator in the large b-calculus with kernel $\chi(\eta)\mathcal{K}_N$. Notice that since the kernel of $\widetilde{Q}_1$ is smooth in η with rapid decay as $|\eta| \to \infty$ and polyhomogeneous in τ there is an element

$$Q_1 \in \Psi_0^{-\infty,(\Sigma^+(a),\Sigma^-(a),\mathbb{N}_0)}(M), \quad \mathcal{N}(Q_1)(q,\widehat{\eta}) = \widetilde{Q}_1(q).$$

Thus $R_1 = \operatorname{Id} - P(Q_\sigma + Q_1) \in \Psi_0^{-\infty,(\Sigma'(a),\Sigma^-(a),\mathbb{N}_0)}(M)$ has reduced normal operator

$$\mathcal{N}(R_1) \in \mathcal{C}^\infty(S^*\partial M, \Psi_b^{-s,(\Sigma'(a),\Sigma^-(a),\mathbb{N}_0+1)}(\overline{N^+M}/\partial M)),$$

where $\Sigma'(a) = \Sigma^+(a) \setminus \operatorname{lead}\Sigma^+(a)$.

The next step in the construction of the parametrix is to solve away the expansion of $\mathcal{N}(R_1)$ at the left face. Recall that this comes down to solving $\mathcal{N}(P)u - f \in \dot{\mathcal{C}}^\infty(I)$ uniformly and is accomplished, e.g., via the formula

$$\frac{1}{2\pi}\oint_{\gamma(z)} \frac{x^{-i\zeta}I(P;\zeta)^{-1}}{\zeta - s} v(y)\, d\zeta.$$

Since $I(P;\zeta)$ is depends on q but not on ζ, we can construct $\widetilde{Q}_2$ in the large b-calculus so that $P\widetilde{Q}_2$ and R_1 have the same expansion at the left face. The dependence of $\widetilde{Q}_2$ on η is the same as that of $\mathcal{N}(R_1)$ and hence there is an element of the large 0-calculus

$$Q_2 \in \Psi_0^{-\infty,(\widehat{\Sigma}^+(a),\infty,\mathbb{N}_0)}(M),$$
$$\mathcal{N}(Q_2)(q,\widehat{\eta}) = \widetilde{Q}_2(q,\widehat{\eta}) \in \Psi_b^{-\infty,(\widehat{\Sigma}^+(a),\infty,\mathbb{N}_0+1)}(M)$$

The rest of the construction involves asymptotically summing the Neumann series of $\operatorname{Id} - P(Q_\sigma + Q_1 + Q_2)$ and can be done inside the image of the reduced normal operator. Taking adjoints, we conclude that it is possible to construct left and right parametrices, $\widetilde{Q}_\ell$, $\widetilde{Q}_r$ for the family of operators $\mathcal{N}(P)(q,\widehat{\eta})$ in the image of the reduced normal operator whose error terms $\widetilde{R}_\ell$, $\widetilde{R}_r$ are very residual.

It follows that if $(q,\widehat{\eta}) \in S^*\partial M$ and $u \in \rho_0^a\rho_1^b H^t_{\mathrm{b,sc}}([0,1],\Omega^{1/2}_{\mathrm{b,sc}})$ (for any $b,t \in \mathbb{R}$) is in the null space of $\mathcal{N}(P)(q,\widehat{\eta})$ then u vanishes to infinite order at $\{1\}$. Indeed, any such element satisfies $\widetilde{R}_\ell u = u$ hence is polyhomogeneous and vanishes at $\{1\}$ at least as quickly as $\widetilde{R}_\ell$, i.e., to infinite order. Since $\widetilde{R}_r$ is very residual and vanishes to infinite order at $\{1\}$, we can apply the same argument to elements of the cokernel of $\mathcal{N}(P)(q,\widehat{\eta})$. Since each $\mathcal{N}(P)(q,\widehat{\eta})$ is Fredholm, we can construct the orthogonal projections onto its kernel and cokernel using a finite basis of elements in the respective spaces and sp cpnclude that the corresponding integral kernels are Schwartz functions of η. (That they are smooth in $(q,\widehat{\eta})$ follows from the smoothness of the bundles $\operatorname{null}\mathcal{N}(P)$ and $\operatorname{coker}\mathcal{N}(P)$.) As explained above this means that these projections are in the image of the reduced normal operator.

Finally, using formula (5.12):

$$G - \widetilde{Q}_\ell = \widetilde{R}_\ell \widetilde{Q}_r - \widetilde{R}_\ell \widetilde{\mathcal{P}}_\ell \widetilde{Q}_r - \widetilde{Q}_\ell \widetilde{\mathcal{P}}_r + \widetilde{R}_\ell G \widetilde{R}_r,$$

the generalized inverse is also in the image of the reduced normal operator. The index sets for these operators follow just as in the b-case. □

As explained at the end of section 5.3, for each (y, η) we can find a large enough weight so that $\mathcal{N}(P)(q,\eta)$ is injective, and a large enough negative weight so that $\mathcal{N}(P)(q,\eta)$ is surjective. Denote by $\overline{\delta}(q,\eta)$ the infimum of the 'injective weights' and by $\underline{\delta}(q,\eta)$ the supremum of the 'surjective weights'. Notice that these are necessarily elements of $\operatorname{spec}_b(\mathcal{N}(P)(q,\eta))$. As $\underline{\delta}$ and $\overline{\delta}$ are everywhere finite semi-continuous functions on $S^*\partial M$ we can define

$$\overline{\delta}(P) = \min_{(q,\eta)} \overline{\delta}(q,\eta), \quad \underline{\delta}(P) = \max_{(q,\eta)} \underline{\delta}(q,\eta).$$

Let $a \notin \{\operatorname{Re}\zeta : \zeta \in \operatorname{spec}_b(P)\}$, the upshot is that if $a < \underline{\delta}(P)$ then P is essentially surjective on $x^a H_0^t(M)$, while if $a > \overline{\delta}(P)$ then P is essentially injective on $x^a H_0^t(M)$ (directly from Theorem 4.25). If $\overline{\delta}(P) < a < \underline{\delta}(P)$ then P is Fredholm on $x^a H_0^t(M)$, but in general *we do not have* $\overline{\delta}(P) < \underline{\delta}(P)$. So it is entirely possible that there are no weights for which P is Fredholm.

As expected, for semi-Fredholm weights we can construct a parametrix in the large 0-calculus.

Theorem 5.12. *Let $P \in \Psi_0^s(M)$ be an elliptic operator of positive order such that $\operatorname{spec}_b(P)$ is a discrete set, and choose*

$$a \in \mathbb{R} \setminus \{\operatorname{Re}\zeta : \zeta \in \operatorname{spec}_b(P)\}$$

such that either $a < \underline{\delta}(P)$ or $a > \overline{\delta}(P)$. There are elements of the large 0-calculus

$$G, \mathcal{P}_\ell, \mathcal{P}_r \in \Psi_0^{-s,(\Sigma(a),\Sigma(a),\mathbb{N}_0)}(M) + \Psi_0^{-\infty,(\Sigma(a),\Sigma(a),\infty)}(M)$$

such that $\mathcal{P}_\ell = \operatorname{Id} - GP$, $\mathcal{P}_r = \operatorname{Id} - PG$ and G acts as the generalized inverse of P as a map

$$P : x^a H_0^t(M, \Omega_0^{1/2}) \to x^a H_0^{t-s}(M, \Omega_0^{1/2})$$

for any $t \in \mathbb{R}$. If $a < \underline{\delta}(P)$ then $\mathcal{P}_r$ is very residual,

$$\mathcal{P}_r \in \Psi_0^{-\infty,(\Sigma^-(a)+2a,\Sigma^-(a),\infty)}(M)$$

and if $a > \overline{\delta}(P)$ then $\mathcal{P}_\ell$ is very residual,

$$\mathcal{P}_\ell \in \Psi_0^{-\infty,(\Sigma^+(a),\Sigma^+(a)-2a,\infty)}(M).$$

Proof. As usual we construct a good parametrix and use it to show that the generalized inverse is in the large calculus.

First consider the simpler case where $\mathcal{N}(P)(q,\eta)$ is actually invertible on $\rho_b^a \rho_{\mathrm{sc}}^t H^{t'}_{\mathrm{b,sc}}([0,1])$. Using Proposition 5.10 we can find

$$Q_1 \in \Psi_0^{-s,(\Sigma(a),\Sigma(a),\mathbb{N})}(M) \text{ s.t.}$$
$$\mathcal{N}(P)(q,\eta)\mathcal{N}(Q_1)(q,\eta) = \operatorname{Id}.$$

The error term $R_1 = \operatorname{Id} - PQ_1$ still has a cononormal singularity at the diagonal, so we find

$$Q_2 \in \Psi_0^{-s,(\infty,\infty,\mathbb{N}_0+1)}(M) \text{ s.t.}$$
$$R_2 = \operatorname{Id} - P(Q_1+Q_2) \in \Psi_0^{-\infty,(\Sigma(a),\Sigma(a),\mathbb{N}_0+1)}(M).$$

From this point one can proceed as in the b-calculus: first use the indicial operator to get an error that vanishes to infinite order at the left face and then asymptotically sum its Neumann series to find

$$Q_r \in \Psi_0^{-s,(\widehat{\Sigma}(a),\widehat{\Sigma}(a),\widehat{\Sigma}(a))}(M) \text{ s.t.}$$
$$\operatorname{Id} - PQ_r \in \Psi_0^{-\infty,(\widehat{\Sigma}(a),\infty,\infty)}(M)$$

and then take transpose to find a left parametrix in the large calculus with very residual residue. This implies that the generalized inverse is in the large calculus and we can determine its index sets just as in the b-calculus,

$$G \in \Psi_0^{-s,(\Sigma(a),\Sigma(a),\mathbb{N}_0)}(M) + \Psi_0^{-\infty,(\Sigma(a),\Sigma(a),\infty)}(M)$$
$$\mathcal{P}_\ell = \operatorname{Id} - GP \in \Psi_0^{-\infty,(\Sigma^+(a),\Sigma^+(a)-2a,\infty)}(M)$$
$$\mathcal{P}_r = \operatorname{Id} - PG \in \Psi_0^{-\infty,(\Sigma^-(a)+2a,\Sigma^-(a),\infty)}(M).$$

Next consider the semi-Fredholm case. Let P^* denote the formal adjoint of P as an operator on $x^a L_0^2(M,\Omega_0^{1/2})$, then P^*P and PP^* are elliptic and essentially self-adjoint. If P is essentially injective then P^*P is Fredholm and if we denote its generalized inverse by G' then the generalized inverse of P is $G'P^*$; similarly if P is essentially surjective then PP^* is Fredholm and if its generalized inverse is G'' then the generalized inverse of P is P^*G''. $\square$