AN OBSERVATION OF RANKIN ON HANKEL DETERMINANTS

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ABSTRACT. While studying the location of the zeros of the Eisenstein series $E_k(z)$, Rankin considered the determinants Δ_n of an associated Hankel matrix. He observed that the first few possess remarkable factorizations, and expressed the hope that a general theorem explaining these factorizations could be found. In this note we provide such a theorem by giving an explicit formula for Δ_n using work of Kaneko and Zagier on Atkin polynomials.

1. INTRODUCTION

The zeros of Eisenstein series were studied by R. Rankin in [Ran69], where he showed that for $k = 28, 30, 32, 34$ and 38 the zeros of E_k lie on the unit circle. Soon after R. Rankin's result, F.K.C. Rankin and Swinnerton-Dyer [RSD70] proved that the zeros of E_k lie on the unit circle for all even $k \geq 4$. In this note we confirm an observation made in [Ran69] about the determinants $\Delta_n = |H_n|$ of the Hankel matrix given by

$$
(1.1) \qquad H_n = \begin{pmatrix} g_0 & g_1 & \cdots & g_n \\ g_1 & g_2 & \cdots & g_{n+1} \\ g_2 & g_3 & \cdots & g_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_{n+1} & \cdots & g_{2n} \end{pmatrix},
$$

where the g_v are defined as follows. Let

$$
E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k}
$$

and let $\Delta(z)$ be the unique normalized weight 12 cusp form. Let j be the modular invariant defined by $j(z) := \frac{E_4^3(z)}{\Delta(z)}$ and with Fourier expansion

$$
j(z) = q^{-1} \sum_{n=0}^{\infty} a_n q^n,
$$

where $q = e^{2\pi i z}$. For a function F that is meromorphic on a fundamental domain $\mathcal{F}$, write $R(F)$ for the sum of the residues of F at points of $\mathcal{F}$ and

$$
j^\nu(z) = q^{-\nu} \sum_{n=0}^{\infty} a_n^{(\nu)} q^n.
$$

Then g_ν is defined by

$$(1.2) \qquad g_\nu := 2\pi i R(j^\nu E_2) = a_\nu^{(\nu)} - 24 \sum_{m=1}^{\nu} a_{\nu-m}^{(\nu)} \sigma(m),$$

where $\sigma(n) = \sum_{d|n} d$ and $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$. The first few values are

$$g_0 = 1, \quad g_1 = 720, \quad g_2 = 911520 \quad g_3 = 1301011200, \quad g_4 = 1958042030400.$$

This is sequence A030185 in [Slo]. Rankin gave the values up to Δ_{13} in a table by their prime factorization; the last of these is of size approximately $2.79 \cdot 10^{483}$ (the computations in Rankin's paper were made by Mr. Stephen Muir of the Atlas Computer Laboratory in Chilton, Didcot). Rankin went on to say that "... they possess remarkable factorizations; each of them is a highly composite number expressible as a product of powers of small primes. These results are given in §4 in the hope that they may stimulate someone to prove a general theorem about these determinants."

Here we prove such a theorem.

Theorem 1. *For $n \geq 1$, let H_n be as in (1.1) and $\Delta_n = |H_n|$. Then,*

$$\Delta_n = 2^{n^2+4n} \cdot 3^{n^2+2n} \cdot 5^n \cdot 7^n \cdot 13^n \cdot \prod_{r=2}^{n} \Big(\frac{(12r-13)(12r-7)(12r-5)(12r+1)}{(2r-1)^2(r-1)r} \Big)^{n-r+1}.$$

Note that the largest prime that can appear in the factorization of Δ_n is at most $12n+1$.

2. Proof of Theorem 1

Recall that if V is the space of polynomials in one variable over a field K, and $\phi : V \to K$ is a linear functional, then one can consider the scalar product on V defined by $(f, g) = \phi(fg)$. One can also consider the family (which for generic ϕ exists and is unique) of monic polynomials which are mutually orthogonal with respect to the scalar product.

Atkin [KZ98, page 3] defined a sequence of polynomials $A_n(j) \in \mathbb{Q}[j]$, one for each degree n, as the orthogonal polynomials with respect to a scalar product. The particular scalar product used by Atkin is defined in several equivalent ways in [KZ98, Proposition 3], one of them being

$$(f, g) := \text{ constant term of } fgE_2 \text{ as a Laurent series in } q.$$

Then from the definition (1.2) we see that

$$g_\nu = (j^\nu, 1).$$

Proof of Theorem 1. A recursion for the Atkin polynomials A_n is given by ([KZ98, equation (18)]):

$$(2.1) \qquad A_{n+1}(j) = (j - (\lambda_{2n} + \lambda_{2n+1})) A_n(j) - \lambda_{2n-1} \lambda_{2n} A_{n-1}(j),$$

where the numbers λ_n are defined by the continued fraction expansion

$$(2.2) \qquad \sum_{k=0}^{\infty} g_k x^k = \frac{g_0}{1 - \frac{\lambda_1 x}{1 - \frac{\lambda_2 x}{1 - \cdots}}}.$$

By [Kra05, Theorem 29], we can use the recurrence in (2.1) to give a formula for Δ_n in terms of the λ_n:

$$(2.3) \qquad \Delta_n = \det_{0 \leq i, r \leq n}(g_{i+r}) = \prod_{r=1}^{n}(\lambda_{2r-1}\lambda_{2r})^{n-r+1}.$$

Equation (19) of [KZ98] gives an explicit formula for the λ_n:

$$(2.4) \qquad \lambda_1 = 720, \qquad \lambda_n = 12\left(6 + \frac{(-1)^n}{n-1}\right)\left(6 + \frac{(-1)^n}{n}\right) \quad \text{for } n > 1.$$

For $r > 1$ this gives

$$\lambda_{2r-1}\lambda_{2r} = 12\left(6 + \frac{(-1)^{2r-1}}{(2r-1)-1}\right)\left(6 + \frac{(-1)^{2r-1}}{2r-1}\right)12\left(6 + \frac{(-1)^{2r}}{(2r)-1}\right)\left(6 + \frac{(-1)^{2r}}{2r}\right)$$
$$= \frac{36(12r-13)(12r-7)(12r-5)(12r+1)}{(2r-1)^2(r-1)r}.$$

Plugging this formula into equation (2.3) and simplifying yields the result. □

Let $\nu_p(m)$ be the highest power of p that divides a non-zero integer m. From Theorem 1 one can obtain $\nu_p(\Delta_n)$ for any prime p. In the case $p = 2$ it has a simple expression.

Corollary 2. *We have*

$$\nu_2(\Delta_n) = 4n - s_2(n) + 2\sum_{r=1}^{n} s_2(r),$$

where $s_2(r)$ is the sum of the digits of r in base 2.

Proof. From Theorem 1 we see that

$$\nu_2(\Delta_n) = n^2 + 4n - \nu_2\left(\prod_{r=2}^{n}\left((r-1)r\right)^{n-r+1}\right),$$

so we only need to show that $\nu_2\left(\prod_{r=2}^{n}\left((r-1)r\right)^{n-r+1}\right) = n^2 - 2\sum_{r=1}^{n}s_2(r) + s_2(n)$. Using the fact that $\nu_2(r) = 1 + s_2(r-1) - s_2(r)$ we obtain

$$\nu_2\left(\prod_{r=2}^{n}\left((r-1)r\right)^{n-r+1}\right) = \sum_{r=2}^{n}(n-r+1)(\nu_2(r) + \nu_2(r-1))$$
$$= 2\sum_{r=2}^{n}(n-r+1) - \sum_{r=2}^{n}(n-r+1)s_2(r) + \sum_{r=1}^{n-2}(n-r-1)s_2(r)$$
$$= n^2 - s_2(n) - 2s_2(n-1) + \sum_{r=1}^{n-2}s_2(r)(n-r-1-(n-r+1))$$
$$= n^2 - 2\sum_{r=1}^{n}s_2(r) + s_2(n). \qquad \square$$

We point out that the sequence $(\sum_{r=1}^{n}s_2(r))_n$ is sequence A000788 in [Slo].

As seen in [KZ98], there are many ways to approach the Atkin polynomials. In this spirit, we briefly explain another way in which one could obtain a closed formula for Δ_n. From Section 4 of [Jun03] we have

$$(2.5) \qquad \Delta_n = ||A_0(j)||^2 \cdot ||A_1(j)||^2 \cdots ||A_n(j)||^2 = \prod_{i=0}^{n}(A_i, A_i).$$

For $n \geq 1$, ([KZ98, Proposition 6])

$$(2.6) \qquad (A_n, A_n) = -12^{6n+1}\frac{(-1/12)_n(5/12)_n(7/12)_n(13/12)_n}{(2n-1)!(2n)!},$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $(A_0, A_0) = (1,1) = 1$. Thus,

$$(2.7) \qquad \Delta_n = \prod_{i=1}^{n} -12^{6i+1}\frac{(-1/12)_i(5/12)_i(7/12)_i(13/12)_i}{(2i-1)!(2i)!}.$$

Acknowledgements

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