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Some notions of transitivity for operator spaces

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Abstract. The famous Mazur Rotation Problem asks whether any separable transitive Banach space (that is, a Banach space where any point on the unit sphere can be mapped into any other point on the unit sphere by a surjective isometry) is necessarily isometric to a Hilbert space. In spite of enormous progress since the 1930's, the problem remains open. In this paper we investigate related non-commutative phenomena. We show that the only completely uniquely maximal (or matrix convex transitive) operator space is a one-dimensional one. Relaxing the conditions somewhat, we show that any matrix-level convex transitive finite dimensional space has to be completely isometric to Pisier's space OH, of corresponding dimension. Finally, we equip ℓ^2 with an operator space structure which is (i) completely almost transitive, and (ii) homogeneous, but not 1-homogeneous.

1. Introduction

In this paper, we investigate non-commutative analogues of the famous Mazur Rotation Problem. To state the problem (dating back to the 1930s) for Banach spaces, denote by $\mathfrak{I}(Z)$ the group of surjective isometries of a normed space Z. We use the notation $\mathbf{B}(Z)$ and $\mathbf{S}(Z)$ for the closed unit ball and the unit sphere of Z, respectively.

Definition 1.1. The space Z is called:

(1) *transitive* if, for any $z \in \mathbf{S}(Z)$, $\mathfrak{I}(Z)z = \mathbf{S}(Z)$.
(2) *almost transitive* if, for any $z \in \mathbf{S}(Z)$, $\overline{\mathfrak{I}(Z)z} = \mathbf{S}(Z)$.
(3) *convex transitive* if, for any $z \in \mathbf{S}(Z)$, $\overline{\operatorname{conv} \mathfrak{I}(Z)z} = \mathbf{B}(Z)$.
(4) *maximal* if no equivalent renorming of Z can increase the group of its surjective isometries (that is, if Z' is an equivalent renorming of Z, then $\mathfrak{I}(Z) = \mathfrak{I}(Z')$).
(5) *uniquely maximal* if Z is maximal and, whenever Z' is an equivalent renorming of Z, then the equality $\mathfrak{I}(Z) = \mathfrak{I}(Z')$ holds if and only if the norm of Z' is a scalar multiple of the norm on Z.

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It is known (see e.g. [2]) that

$$(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (5) \Rightarrow (4). \tag{1.1}$$

Clearly Hilbert spaces are transitive. Moreover, one can show that $L^p(\mu)$ is transitive whenever μ is a measure without atoms and not σ-finite (hence the space $L^p(\mu)$ is not separable). S. Mazur conjectured that any separable transitive Banach space is isometric to a Hilbert space, and proved that conjecture in the finite dimensional case. Since then, transitivity of Banach spaces has been studied intensively.

For an introduction to Mazur Rotation Problem and related questions, the reader is referred to e.g. [2], [10, Chapter 12], or [15, Chapter 9]. For the recent progress, see [4], [8], and [7]. The last two papers also note the deep connections between transitivity problems and representations of groups. Furthermore, [8] contains a list of open questions.

We study some phenomena related to transitivity in the operator space context. We use the standard operator space facts and notation, which can be found in e.g. [5], [11], or [14]. Throughout the paper, we use $M_n(E)$ for the space of E-valued $n \times n$ matrices (E is an operator space). The sequence of norms $(\|\cdot\|_n)$ (sometimes we omit the lower index n, and simply use the notation $\|\cdot\|$) on the spaces $M_n(E)$ (satisfying Ruan's axioms) is referred to as an *operator space structure* (*o.s.s.* for short). For $U \in B(E, F)$, denote by U_n the *n-th amplification* of U (taking $M_n(E)$ to $M_n(F)$), then $\|U\|_{cb} = \sup_n \|U\|_n$.

Imitating Definition 1.1(4,5), we say:

Definition 1.2. An operator space E is *completely maximal* if there is no equivalent operator space structure on E that strictly enlarges the group of surjective complete isometries on E. More specifically, denote by $\mathfrak{G}(E)$ the group of surjective complete isometries of E ($T \in \mathfrak{G}(E)$ iff $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$). Then E is maximal if $\mathfrak{G}(E') \subset \mathfrak{G}(E)$ for any equivalent o.s.s. E'. If E is completely maximal and the only equivalent o.s.s.'s E' with $\mathfrak{G}(E) = \mathfrak{G}(E')$ are scalar multiples of the original o.s.s., we say that E is *completely uniquely maximal*.

In the Banach space case, unique maximality is equivalent to convex transitivity as shown by Cowie [3]. In our context we prove:

Theorem 1.3. *For an operator space E, the following are equivalent:*

(1) E is completely uniquely maximal.
(2) E is absolutely matrix convex transitive.
(3) $E = \mathbb{C}$.

Absolute matrix convex transitivity is a non-commutative counterpart of convex transitivity (Definition 1.1(3)), based on the Effros-Webster concept of absolute matrix convexity [6]; see Subsection 2.1 for the precise definition. Theorem 1.3 will be established in Subsection 2.2.

Note that, in contrast to Theorem 1.3, the class of uniquely maximal Banach spaces is quite wide. For instance (see [2, Example 2.32]), the spaces $L^\infty(0,1)$ and $C(\Delta)$ (Δ is the Cantor discontinuum) are uniquely maximal.

We can also introduce a different non-commutative version of transitivity.

Definition 1.4. we say that an operator space E is *matrix-level transitive* (*matrix-level almost transitive*, *matrix-level convex transitive*) if, for every $n \in \mathbb{N}$, the space $S^2_n[E]$ is transitive (resp. almost transitive, convex transitive).

By [12], for m finite or infinite, and $n \in \mathbb{N}$, $S_n^2[OH_m]$ is completely isometric to $OH_{n^2 m}$, hence OH_m is matrix-level transitive. For finite dimensional spaces, the converse is true.

Theorem 1.5. *For a finite dimensional operator space E, the following are equivalent.*

1. *(1) E is matrix-level transitive.*
2. *(2) E is matrix-level almost transitive.*
3. *(3) E is matrix-level convex transitive.*
4. *(4) For every n, $S_n^2[E]$ is isometric to a Hilbert space.*
5. *(5) E is completely isometric to $OH_{\dim E}$.*

This theorem will be proved in Section 3.

Question 1.6. It is well known that $L^p(0,1)$ $(1 \leq p < \infty)$ is almost transitive. Can we equip this space (for $p \neq 2$) with a suitable operator space structure, and make it matrix-level almost transitive? What are the transitivity properties of the canonical o.s.s. of $L^p(0,1)$?

Further, we can adjust Definition 1.1(2) to the non-commutative setting by considering complete isometries instead of isometries.

Definition 1.7. An operator space E is *completely almost transitive* if, for any $x \in \mathbf{S}(E)$, $\overline{\mathfrak{G}(E)x} = \mathbf{S}(E)$ (equivalently, there exists $x \in \mathbf{S}(E)$ so that $\overline{\mathfrak{G}(E)x} = \mathbf{S}(E)$).

An operator space E is called *c-homogeneous* ($c \geq 1$) if any $T \in B(E)$ is completely bounded, with $\|T\|_{cb} \leq c\|T\|$. We say that E is *homogeneous* if it is c-homogeneous for some c. Clearly any 1-homogeneous 1-Hilbertian space E is completely almost transitive — in fact, we have $\mathfrak{G}(E)x = \mathbf{S}(E)$ for any $x \in \mathbf{S}(E)$. Furthermore, for $p \in [1,2) \cup (2,\infty)$, the space $L^p(0,1)$ is completely almost transitive. To establish this, observe that the surjective isometries on the space in question (described in e.g. [9, Chapter 3] and [10, Section 12.4]) are, in fact, complete isometries. However, $L^p(0,1)$ is not homogeneous.

We show that ℓ^2 can be equipped with a c-homogeneous (but not 1-homogeneous) completely almost transitive o.s.s. H. A corresponding Banach space question (whether ℓ^2 has a non-trivial equivalent almost transitive renorming) is still open.

Theorem 1.8. *There exists a homogeneous completely almost transitive operator space H, isometric to ℓ^2 on the Banach space level, which is not 1-homogeneous.*

This space is constructed in Section 4.

Remark 1.9. As noted in (1.1), in the Banach space case almost transitivity implies convex transitivity which is equivalent to unique maximality, which, in turn, is stronger than maximality. Analogous implications do not hold for operator spaces. To be more specific, we say that Z' is a *c.b.-equivalent renorming* of an operator space Z iff they are identified as vector spaces, and the identity map $Z \to Z'$ is a complete isomorphism. The construction of H in Theorem 1.8 shows that H is c-completely isomorphic to the minimal operator space $\mathrm{MIN}(L^2(0,1))$. In other words, $\mathrm{MIN}(L^2(0,1))$ is a c.b.-equivalent renorming of H. The space $\mathrm{MIN}(L^2(0,1))$ is 1-homogeneous, hence this renorming strictly enlarges the group of surjective complete isometries (in the proof of Theorem 1.8, we explicitly construct a unitary which is not a complete isometry on H). Thus, albeit H is completely almost transitive, it fails a non-commutative version of maximality.

Question 1.10. In there a finite dimensional analogue of Theorem 1.8? That is, does there exist a finite dimensional completely almost transitive operator space H, which is not 1-homogeneous? Such an H would almost transitive as a Banach space, hence it must be isometric (as a Banach space) to $\ell^2_{\dim H}$. The proof of Theorem 1.8 uses a subgroup $G \subset \mathfrak{I}(L^2(0,1))$ which acts almost transitively on $\mathbf{S}(L^2(0,1))$ (that is, $\overline{Gx} = \mathbf{S}(L^2(0,1))$ for every $x \in \mathbf{S}(L^2(0,1))$), yet $\overline{G}$ is a strict subset of $\mathfrak{I}(L^2(0,1))$. Thus, we can start with: Does the unitary group of ℓ^2_n have a closed strict subgroup which acts transitively (or almost transitively) on $\mathbf{S}(\ell^2_n)$?

Question 1.11. Our construction (used to prove Theorem 1.8) gives a c-homogeneous space, for c close to 1. How large can c be? Does ℓ^2 admit a non-homogeneous completely almost transitive operator space structure?

2. Absolute matrix convex transitivity

2.1. Matrix convexity. The contents of this section are from [6]. A set $K = (K_n)_n$ of matrices over E consists of subsets $K_n \subseteq M_n(E)$ for all $n \in \mathbb{N}$. A subset $A \subset M_m(E)$ can be considered as a set of matrices over E by setting $K_n = A$ if $n = m$, $K_n = \emptyset$ otherwise. We say that a set of matrices $K = (K_n)_n$ over E is *absolutely matrix convex* if:

(i) For all $x \in K_n$ and $y \in K_m$, $x \oplus y \in K_{m+n}$.
(ii) For all $x \in K_n$, $\alpha \in \mathbf{B}(M_{m,n})$, $\beta \in \mathbf{B}(M_{n,m})$, $\alpha x \beta \in K_m$.

By way of example, Ruan's axioms show that for any operator space E the set of matrices $(\mathbf{B}(M_n(E)))_n$ is absolutely matrix convex.

If $K = (K_n)_n$ is a set of matrices over E, we define the *absolutely matrix convex hull* of K (abs-mat-conv(K)) as the smallest absolutely matrix convex set of matrices over E that contains K. Such a smallest set exists, since an intersection of absolutely matrix convex sets is again absolutely matrix convex. The following nice characterization of the absolutely matrix convex hull of a set of matrices is due to B.E. Johnson [6, Lemma 3.2].

Lemma 2.1. *Given a set K of matrices over E,*

$$\text{abs-mat-conv}(K) = \Big\{ \sum_{i=1}^n \alpha_i x_i \beta_i \ : \ x_i \in K_{k_i}, \alpha_i \in M_{j,k_i}, \beta_i \in M_{k_i,j} \\ \text{such that } \sum_{i=1}^n \alpha_i \alpha_i^* \leq \mathbb{1}_j, \sum_{i=1}^n \beta_i^* \beta_i \leq \mathbb{1}_j \Big\},$$

where $\mathbb{1}_j$ denotes the $j \times j$ identity matrix.

Note that, if X is a Banach space, then

$$\text{abs-mat-conv}(\mathbf{B}(X)) = \big(\mathbf{B}(M_n(\text{MAX}(X)))\big)_n. \tag{2.1}$$

The following definition is similar to that of convex transitivity (Definition 1.1(3)).

Definition 2.2. An operator space E is *absolutely matrix convex transitive* if for every $m \in \mathbb{N}$ and every $x \in \mathbf{S}(M_m(E))$

$$\overline{\text{abs-mat-conv}}\{U_m x \ : \ U \in \mathfrak{G}(E)\} = \big(\mathbf{B}(M_n(E))\big)_n.$$

In the proof of Theorem 1.3, we use non-commutative separation results (in the Hahn-Banach mould). For $f = (f_{ij}) \in M_n(E^*)$ and $x = (x_{k\ell}) \in M_m(E)$, write $\langle f, x\rangle = \big(f_{ij}(x_{kl})\big)_{ijkl} \in M_{mn}$. Then [6, Proposition 4.1] and its proof yield:

Theorem 2.3. *(1) Let K be a closed absolutely convex set of matrices over E and $x_0 \in M_n(E) \setminus K_n$ for some $n \in \mathbb{N}$. Then there exists $f \in M_n(E^*)$ such that for all $m \in \mathbb{N}$ and all $x \in K_m$,*

$$\|\langle f, x\rangle\|_{M_{mn}} \leq 1 \quad \text{but} \quad \|\langle f, x_0\rangle\|_{M_{n^2}} > 1.$$

(2) Let K be a weak closed absolutely convex set of matrices over E^* and $x_0 \in M_n(E^*) \setminus K_n$ for some $n \in \mathbb{N}$. Then there exists $f \in M_n(E)$ such that for all $m \in \mathbb{N}$ and all $x \in K_m$,*

$$\|\langle f, x\rangle\|_{M_{mn}} \leq 1 \quad \text{but} \quad \|\langle f, x_0\rangle\|_{M_{n^2}} > 1.$$

The following result is, essentially, [6, Corollary 4.2],

Theorem 2.4. *(1) If K is a set of matrices over E, then for any $f \in M_n(E^*)$,*

$$\sup\big\{\|\langle f, x\rangle\|_{M_{mn}} : x \in K\big\} = \sup\big\{\|\langle f, x\rangle\|_{M_{mn}} : x \in \overline{\text{abs-mat-conv}}(K)\big\}.$$

(2) If K is a set of matrices over E^*, then for any $x \in M_n(E)$,

$$\sup\big\{\|\langle f, x\rangle\|_{M_{mn}} : f \in K\big\} = \sup\big\{\|\langle f, x\rangle\|_{M_{mn}} : f \in \overline{\text{abs-mat-conv}}^{w^*}(K)\big\}.$$

2.2. Proof of Theorem 1.3. For convenience, the proof is broken into a number of steps. We use some ideas from [3]. Let us start with one implication.

Proof of Theorem 1.3 (2) $\Rightarrow$ (1). Suppose that E with the o.s.s. $\|\cdot\|$ is absolutely matrix convex transitive, and let $|\cdot|$ be an equivalent o.s.s. in E such that

$$\mathfrak{G}(E, \|\cdot\|) \subseteq \mathfrak{G}(E, |\cdot|).$$

Fix $m \in \mathbb{N}$, and consider $x \in \mathbf{S}(M_m(E))$. By absolute matrix convex transitivity,

$$\overline{\text{abs-mat-conv}}\{U_m x \;:\; U \in \mathfrak{G}(E)\} = \big(\mathbf{B}(M_n(E))\big)_n$$

Now let $y \in \mathbf{S}(M_m(E))$. Given $\varepsilon > 0$, by Lemma 2.1 there exist $U^i \in \mathfrak{G}(E, \|\cdot\|)$, $\alpha_i \in M_m$, $\beta_i \in M_m$ $(1 \leq i \leq s)$ such that

$$\left\|y - \sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right\|_m < \varepsilon \quad \text{and} \quad \sum_{i=1}^{s} \alpha_i\alpha_i^* \leq \mathbb{1}_m, \sum_{i=1}^{s} \beta_i^*\beta_i \leq \mathbb{1}_m.$$

Therefore, if K is the constant of equivalence between $\|\cdot\|$ and $|\cdot|$, by the triangle inequality

$$\begin{aligned} |y|_m &\leq \left|y - \sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right|_m + \left|\sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right|_m \\ &\leq K\left\|y - \sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right\|_m + \left|\sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right|_m \\ &\leq K\varepsilon + \left|\sum_{i=1}^{s} \alpha_i (U^i_m x)\beta_i\right|_m. \end{aligned}$$

By the hypothesis on $|\cdot|$, $|U_m^i x|_m = |x|_m$ for each $1 \le i \le s$. By the absolute matrix convexity of the set of unit balls $(\mathbf{B}(M_n(E, |\cdot|)))_n$,

$$\left| \sum_{i=1}^{s} \alpha_i (U_m^i x) \beta_i \right|_m \le |x|_m.$$

Letting $\varepsilon \to 0$, we get that $|y|_m \le |x|_m$. Similarly, $|x|_m \le |y|_m$ (because both x and y were chosen arbitrarily in the unit sphere of $M_m(E)$). Therefore, for each $m \in \mathbb{N}$ there exists a constant $r_m > 0$ such that

$$\{x \in M_m(E) \ : \ \|x\|_m = 1\} \subseteq \{y \in M_m(E) \ : \ |y|_m = r_m\}.$$

By the compatibility conditions in the definition of o.s.s. (specifically, the fact that the canonical inclusion $(M_n(E), \|\cdot\|_n) \to (M_{n+1}(E), \|\cdot\|_{n+1})$ is an isometry), we conclude that all the r_m's are the same constant $r > 0$. Hence for every $n \in \mathbb{N}$ and every $x \in M_n(E)$ we have $|x|_n = r \|x\|_n$, so $(E, \|\cdot\|)$ is completely uniquely maximal. □

We split the proof of Proof of Theorem 1.3 (1) ⇒ (2) into two parts. For an operator space $(E, \|\cdot\|)$, we denote its dual space by $(E^*, \|\cdot\|)$. Recall that given a map $U : E \to F$ and $n \in \mathbb{N}$, we denote by U_n the amplification of U from $M_n(E)$ to $M_n(F)$.

Lemma 2.5. *If $(E, \|\cdot\|)$ be a completely uniquely maximal operator space, then, for any $x \in M_m(E)$ and $f \in M_n(E^*)$,*

$$\|x\|_m \|f\|_n = \sup_{U \in \mathfrak{G}(E)} \|\langle f, U_m x\rangle\|_{M_{mn}} = \sup_{U \in \mathfrak{G}(E)} \|\langle U_n^* f, x\rangle\|_{M_{mn}}.$$

Proof. Fix $n \in \mathbb{N}$ and $f = (f_{ij}) \in M_n(E^*)$. By homogeneity, we may assume that $\|f\|_n > 1$. For each $m \in \mathbb{N}$ and $x \in M_m(E)$ define

$$|x|_m = \|x\|_m \vee \sup\left\{ \|\langle f, U_m x\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|) \right\}.$$

Let us now prove that $|\cdot|$ is an o.s.s. on E equivalent to $\|\cdot\|$:

(a) Let $\lambda \in \mathbb{C}$ and $x, y \in M_m(E)$. Clearly, $|\lambda x|_m = |\lambda| \cdot |x|_m$, and $|x|_m = 0$ implies $\|x\|_m = 0$ and therefore $x = 0$. As for the triangle inequality, note that

$$\|x + y\|_m \le \|x\|_m + \|y\|_m \le |x|_m + |y|_m,$$

and

$$\begin{aligned} \sup\left\{ \|\langle f, U_m(x+y)\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\} \\ \le \sup\left\{ \|\langle f, U_m x\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\} + \\ + \sup\left\{ \|\langle f, U_m y\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\} \le |x|_m + |y_m|, \end{aligned}$$

and thus

$$|x + y|_m \le |x|_m + |y_m|.$$

Therefore, each $|\cdot|_m$ is a norm on $M_m(E)$.

(b) Let $x \in M_m(E)$, $y \in M_{m'}(E)$. Then

$$|x \oplus y|_{m+m'} = \|x \oplus y\|_{m+m'} \vee \sup\left\{ \|\langle f, U_{m+m'}(x \oplus y)\rangle\|_{M_{(m+m')n}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}$$
$$= \|x\|_m \vee \|y\|_{m'} \vee \sup\left\{ \|\langle f, (U_m x \oplus U_{m'} y)\rangle\|_{M_{(m+m')n}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}$$
$$= \|x\|_m \vee \|y\|_{m'} \vee \sup\left\{ \|\langle f, U_m x\rangle \oplus \langle f, U_{m'} y)\rangle\|_{M_{(m+m')n}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}$$
$$= \|x\|_m \vee \|y\|_{m'} \vee \sup\left\{ \|\langle f, U_m x\rangle\|_{M_{mn}} \vee \|\langle f, U_{m'} y)\rangle\|_{M_{m'n}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}$$
$$= |x|_m \vee |y|_{m'}$$

Thus, $|\cdot|$ satisfies the first of Ruan's axioms.

(c) Now let $x \in M_m(E)$, $\alpha, \beta \in M_m$. Since $\|\cdot\|$ is an o.s.s.,

$$\|\alpha x \beta\|_m \leq \|\alpha\|_{M_m} \|x\|_m \|\beta\|_{M_m}.$$

On the other hand, for every $y \in M_m(E)$,

$$\|\langle f, \alpha y \beta\rangle\|_{M_{mn}} = \|\alpha(\langle f, y\rangle)\beta\|_{M_m[M_n]} \leq \|\alpha\|_{M_m} \|\langle f, y\rangle\|_{M_{mn}} \|\beta\|_{M_m},$$

from where it follows that $|\cdot|$ satisfies the second of Ruan's axioms.

Now consider $V \in \mathfrak{G}(E, \|\cdot\|)$. Then for $m \in \mathbb{N}$ and $x \in M_m(E)$,

$$|V_m x|_m = \|V_m x\|_m \vee \sup\left\{ \|\langle f, (UV)_m x\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}$$
$$= \|x\|_m \vee \sup\left\{ \|\langle f, U_m x\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\} = |x|_m.$$

Thus $\mathfrak{G}(E, |\cdot|) \subset \mathfrak{G}(E, \|\cdot\|)$, so by complete unique maximality there exists a positive constant r such that for all $m \in \mathbb{N}$ and $x \in M_m(E)$

$$r\|x\|_m = |x|_m = \|x\|_m \vee \sup\left\{ \|\langle f, U_m x\rangle\|_{mn} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}.$$

Since $\|f\|_n > 1$, there exist $x_0 \in M_n(E)$ such that $\|x_0\|_n = 1$ and $\|\langle f, x_0\rangle\|_{n^2} > 1$ (see [5, Section 3.2], or [6, Section 4]). Thus

$$r = r\|x_0\|_n = |x_0|_n \geq \|\langle f, x_0\rangle\|_{n^2} > 1,$$

and therefore we have that in fact for all $m \in \mathbb{N}$ and $x \in M_m(E)$,

$$r\|x\|_m = \sup\left\{ \|\langle f, U_m x\rangle\|_{mn} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}. \tag{2.2}$$

Now for every $U \in \mathfrak{G}(E, \|\cdot\|)$,

$$\|\langle f, U_m x\rangle\|_{M_{mn}} \leq \|f\|_n \|U_m x\|_m = \|f\|_n \|x\|_m,$$

hence (taking x with $\|x\|_m = 1$), we conclude that $r \leq \|f\|_n$. Given $\varepsilon > 0$, there exist $y \in M_n(E)$ with $\|y\|_n = 1$ and such that

$$\|\langle f, y\rangle\|_{M_{n^2}} \geq \|f\|_n - \varepsilon.$$

Since the identity map is a complete isometry, we have

$$r = r\|y\|_n = \sup\left\{ \|\langle f, U_n y\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\} \geq \|\langle f, y\rangle\|_{n^2} \geq \|f\|_n - \varepsilon,$$

and thus $r \geq \|f\|_n$. Therefore, for every $m \in \mathbb{N}$ and $x \in M_m(E)$ (2.2) yields

$$\|f\|_n \|x\|_m = \sup\left\{ \|\langle f, U_m x\rangle\|_{M_{mn}} \ : \ U \in \mathfrak{G}(E, \|\cdot\|)\right\}.$$

To finish the proof, note that $\langle f, U_m x\rangle = \langle U_n^* f, x\rangle$. $\square$

Proof of Theorem 1.3 (1) $\Rightarrow$ (2). Suppose, for the sake of contradiction, that there is a completely uniquely maximal operator space E which is not absolutely matrix convex transitive. Then there exist $m \in \mathbb{N}$ and $x \in M_m(E)$ such that

$$K = \overline{\text{abs-mat-conv}}\{U_m x \ : \ U \in \mathfrak{G}(E)\} \subsetneq \{y \in M_m(E) \ : \ \|y\|_m \leq 1\}.$$

Consider $z \in \{y \in M_m(E) \ : \ \|y\|_m \leq 1\} \setminus K$. By Theorem 2.3, there exists $f \in M_m(E^*)$ such that for all $x' \in K$, $\|\langle f, z\rangle\| > 1 \geq \|\langle f, x'\rangle\|$. In particular, $\|\langle f, z\rangle\| > 1 \geq \|\langle f, U_m x\rangle\|$ for any $U \in \mathfrak{G}(E)$. This contradicts Lemma 2.5. $\square$

To complete the proof of Theorem 1.3, we need:

Lemma 2.6. *Let $(E, \|\cdot\|)$ be a completely uniquely maximal operator space, and let $(X, \|\cdot\|_1)$ be its underlying Banach space. Then $E = \text{MAX}(X, \|\cdot\|_1)$, and moreover $(X, \|\cdot\|_1)$ is uniquely maximal.*

Proof. To show that $E = \text{MAX}(X, \|\cdot\|_1)$, apply the definition of absolute matrix convex transitivity to $x \in \mathbf{S}(E)$. By (2.1) we obtain

$$\big(\mathbf{B}(M_n(E))\big)_n = \overline{\text{abs-mat-conv}}\{Ux : U \in \mathfrak{G}(E)\} \subseteq \overline{\text{abs-mat-conv}}\,\mathbf{B}(X) = \big(\mathbf{B}(M_n(\text{MAX}(X)))\big)_n,$$

Thus, $E = \text{MAX}(X)$. To show the unique maximality of X, suppose $|\cdot|_1$ is an equivalent norm on X, such that $\mathfrak{I}(X, \|\cdot\|_1) \subseteq \mathfrak{I}(X, |\cdot|_1)$ (here, as in Section 1, $\mathfrak{I}(Z)$ denotes the group of surjective linear isometries of a normed space Z). Maximal spaces are 1-homogeneous, hence $\mathfrak{G}(E, \|\cdot\|) = \mathfrak{I}(X, \|\cdot\|_1)$, and $\mathfrak{G}(\text{MAX}(X, |\cdot|_1)) = \mathfrak{I}(X, |\cdot|_1)$. However,

$$\begin{aligned}\|Id : E \to \text{MAX}(X, |\cdot|_1)\|_{cb} &= \|Id : \text{MAX}(X, \|\cdot\|_1) \to \text{MAX}(X, |\cdot|_1)\|_{cb} \\ &= \|Id : (X, \|\cdot\|_1) \to (X, |\cdot|_1)\| < \infty,\end{aligned}$$

and the same holds for the converse. As $(E, \|\cdot\|)$ is completely uniquely maximal, $\text{MAX}(X, \|\cdot\|_1)$ and $\text{MAX}(X, |\cdot|_1)$ are the same up to a constant, and the same is true for $\|\cdot\|_1$ and $|\cdot|_1$. $\square$

Proof of Theorem 1.3. (3) $\Rightarrow$ (1) is easy, while (1) $\Leftrightarrow$ (2) has been shown above. It remains to show that (1) (or (2)) implies (3). So, suppose E is a completely uniquely maximal (equivalently, absolutely matrix convex transitive) operator space.

As in the proof of (1) $\Rightarrow$ (2), we use the separation argument to show that, for any $f \in M_n(E^*)$ of norm 1,

$$\overline{\text{abs-mat-conv}}^{w^*}\Big(U_n^* f : U \in \mathfrak{G}(E)\Big) = \cup_{r\in\mathbb{N}} \mathbf{B}(M_r(E^*)). \tag{2.3}$$

Indeed, clearly $\text{abs-mat-conv}\Big(U_n^* f \ : \ U \in \mathfrak{G}(E)\Big) \subset \cup_{r\in\mathbb{N}} \mathbf{B}(M_r(E^*))$. The right hand side is weak* closed, hence

$$\overline{\text{abs-mat-conv}}^{w^*}\Big(U_n^* f : U \in \mathfrak{G}(E)\Big) \subset \cup_{r\in\mathbb{N}} \mathbf{B}(M_r(E^*)).$$

Suppose, for the sake of contradiction, that the opposite inclusion does not hold. Pick $g \in \mathbf{B}(M_m(E^*)) \setminus \overline{\text{abs-mat-conv}}^{w^*}\big(U_n^* f : U \in \mathfrak{G}(E)\big)$. Use Theorem 2.3 to find

$x \in M_m(E)$ so that

$$\|\langle g, x\rangle\|_{M_{m^2}} > 1 \geq \sup\Big\{\|\langle g, h\rangle\|_{M_{mn}} : h \in \overline{\text{abs-mat-conv}}^{w^*}\big(U_n^* f : U \in \mathfrak{G}(E)\big)\Big\}$$
$$= \sup_{U \in \mathfrak{G}(E)} \|\langle U_n^* f, x\rangle\|_{M_{mn}}.$$

(the last equality follows from Theorem 2.4). Lemma 2.5 now yields:

$$\|x\|_m \geq \|\langle g, x\rangle\|_{M_{m^2}} > 1 \geq \|x\|_m \|f\|_n = \|x\|_m,$$

a contradiction.

Now suppose that, in (2.3), $n = 1$ – that is, $f \in \mathbf{S}(E^*)$. Then

$$\overline{\text{abs-mat-conv}}^{w^*}\mathbf{S}(E^*) \supset \overline{\text{abs-mat-conv}}^{w^*}\Big(U^* f : U \in \mathfrak{G}(E)\Big) = \cup_{r \in \mathbb{N}} \mathbf{B}(M_r(E^*)).$$

Consequently, for any $x \in M_m(E)$, Theorem 2.4 yields:

$$\|x\|_m = \sup_{h \in \mathbf{B}(M_m(E^*))} \|\langle h, x\rangle\|_{M_{m^2}} \leq \sup_{h \in \overline{\text{abs-mat-conv}}^{w^*}\mathbf{S}(E)} \|\langle x, h\rangle\|_{M_{mn}}$$
$$= \sup_{g \in \mathbf{S}(E)} \|\langle x, g\rangle\|_{M_m} = \|x\|_{M_m(\text{MIN}(X))},$$

where X is the underlying Banach space of E. On the other hand, Lemma 2.6 shows that $\|x\|_m = \|x\|_{M_m(\text{MAX}(X))}$. Consequently, $Id : \text{MIN}(X) \to \text{MAX}(X)$ is a complete isometry. By [11, Chapter 14], X is finite dimensional.

Lemma 2.6 also shows that X is convex transitive (or equivalently, uniquely maximal). By [15, Proposition 9.6.1] (or [2, Corollary 2.42]), $X = \ell_n^2$ for some n. By [11, Theorem 14.3], $\text{MIN}(\ell_n^2) \neq \text{MAX}(\ell_n^2)$ for $n > 1$. □

3. Matrix-level transitivity: proof of Theorem 1.5

It is well known (see e.g. [2]) that

$$\text{transitivity } \Rightarrow \text{almost transitivity } \Rightarrow \text{convex transitivity}.$$

By [2, Corollary 2.42] (see also [15, Chapter 9]), a finite dimensional Banach space is convex transitive iff it is isometric to a Hilbert space. This proves the equivalence of (1) - (4). By [12], (5) ⇒ (4). The lemma below gives (4) ⇒ (5).

Lemma 3.1. *Suppose E is an m-dimensional operator space, where m can be finite or infinite, and $S_n^2[E]$ is isometric to a Hilbert space for any n. Then E is completely isometric to OH_m.*

A completely isomorphic version of this lemma was established in [1].

Proof. Note first that, for any $a_1, \ldots, a_s$ in a Hilbert space H,

$$\mathbb{E}_\varepsilon \|\sum_{j=1}^s \varepsilon_j a_j\|^2 = \sum_{j=1}^s \|a_j\|^2$$

((ε_j) are independent Bernoulli random variables). Consequently, if (ε_i) and (ε'_j) are independent Bernoulli random variables, and $(a_{ij})_{i,j=1}^n \subset H$, then

$$\mathbb{E}_{\varepsilon,\varepsilon'} \|\sum_{i,j=1}^s \varepsilon_i \varepsilon'_j a_{ij}\|^2 = \sum_{i,j=1}^s \|a_{ij}\|^2. \tag{3.1}$$

Denote by $e_{ij} \in S_n^2$ the matrix units, and consider $\sum_{i,j} e_{ij} \otimes x_{ij} \in S_n^2[E]$. Note, for every $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_1', \ldots, \varepsilon_n'$ of absolute value 1, we have

$$\left\| \sum_{i,j} e_{ij} \otimes x_{ij} \right\| = \left\| \sum_{i,j} \varepsilon_i \varepsilon_j' e_{ij} \otimes x_{ij} \right\|. \tag{3.2}$$

Indeed, we can identify $S_n^2[E]$ with $OH_n \otimes_h E \otimes_h OH_n$, and $\sum_{i,j} e_{ij} \otimes x_{ij}$ – with $\sum_{i,j} \theta_i \otimes x_{ij} \otimes \theta_j$ ($\theta_1, \ldots, \theta_n$ is an orthonormal basis in OH_n). Define an isometry $U_\varepsilon : OH_n \to OH_n : \theta_i \mapsto \varepsilon_i \theta_i$. The operator $U_{\varepsilon'}$ is defined similarly. Then

$$\sum_{i,j} \varepsilon_i \varepsilon_j' e_{ij} \otimes x_{ij} = (U_\varepsilon \otimes I_E \otimes U_{\varepsilon'}) \sum_{i,j} e_{ij} \otimes x_{ij}.$$

Due to the properties of the Haagerup tensor product, and due to the 1-homogeneity of OH, we have $\|U_\varepsilon \otimes I_E \otimes U_{\varepsilon'}\| \le \|U_\varepsilon\| \|U_{\varepsilon'}\| = 1$, hence

$$\left\| \sum_{i,j} e_{ij} \otimes x_{ij} \right\| \le \left\| \sum_{i,j} \varepsilon_i \varepsilon_j' e_{ij} \otimes x_{ij} \right\|.$$

To prove the converse inequality, note that

$$\sum_{i,j} e_{ij} \otimes x_{ij} = (U_{\overline{\varepsilon}} \otimes I_E \otimes U_{\overline{\varepsilon'}}) \sum_{i,j} \varepsilon_i \varepsilon_j' e_{ij} \otimes x_{ij}.$$

By (3.1) and (3.2),

$$\left\| \sum_{i,j} e_{ij} \otimes x_{ij} \right\|^2 = \sum_{i,j} \|e_{ij} \otimes x_{ij}\|^2 = \sum_{i,j} \|x_{ij}\|^2. \tag{3.3}$$

Now note that E is isometric to ℓ_m^2 (here $m = \dim E$). Find an orthonormal basis $u_1, \ldots, u_m \in E$, and consider the formal identity $T : E \to OH_m : u_i \mapsto \theta_i$, where $\theta_1, \ldots, \theta_m$ is an orthonormal basis in OH_m. By (3.3), $I_{S_n^2} \otimes T$ is an isometry for every n. By [13], T is a complete isometry. □

4. Theorem 1.8: the construction

Imitating the group of surjective isometries on $L^p(0,1)$ for $p \neq 2$ (see [9, Chapter 3] and [10, Section 12.4]), we denote by $\mathcal{F}$ the set of all functions $h \in L^2(0,1)$ so that (i) $\|h\|_2 = 1$, and (ii) $\overline{\text{supp} h} = [0,1]$. Clearly $\overline{\mathcal{F}} = \mathbf{S}(L^2(0,1))$. For any $h \in \mathcal{F}$ consider the weighted compositions operator $U_h \in B(L^2(0,1))$, defined by $[U_h f](t) = h(t) f(k_h(t))$, where $k_h(t) = \int_0^t |h(s)|^2 \, ds$. It is easy to check that U_h is a unitary, and these operators form a group. In fact, $U_{h_1} U_{h_2} = U_h$, where $h(t) = h_1(t) h_2(k_{h_1}(t))$, and $U_h^{-1} = U_g$ for $g(t) = 1/h(k_h^{-1}(t))$. Also, $U_h \mathbf{1} = h$, which shows that the orbit of $\mathbf{1}$ under the action of $\{U_h : h \in \mathcal{F}\}$ is dense in $\mathbf{S}(L^2(0,1))$ ($\mathbf{1}(t) = 1$ for every t). Consequently, for any $\xi \in L^2(0,1)$, $\overline{\{U_h \xi : h \in \mathcal{F}\}} = \mathbf{S}(L^2(0,1))$. Indeed, we have to show that, for any $\eta \in \mathbf{S}(L^2(0,1))$, and any $\varepsilon > 0$, there exists $h \in \mathcal{F}$ so that $\|U_h \xi - \eta\| < \varepsilon$. Find $\eta' \in \mathbf{S}(L^2(0,1))$ so that $\overline{\text{supp} \eta'} = [0,1]$, and $\|\eta - \eta'\| < \varepsilon/2$. By the above, there exists $h_1 \in \mathcal{F}$ so that $U_{h_1} \eta' = \mathbf{1}$. Moreover, $\|U_{h_2} \mathbf{1} - \xi\| < \varepsilon/2$, for some $h_2 \in \mathcal{F}$. Find $h \in \mathcal{F}$ so that $U_h = U_{h_2} U_{h_1}$, then

$$\|U_h \eta - \xi\| \le \|U_h \eta' - \xi\| + \|\eta - \eta'\| \le \varepsilon.$$

Fix $n = 100$ and $\varepsilon = 1/n$. Consider the norm one mutually orthogonal functions $e_0 = \mathbf{1}$ and $e_1 = \sqrt{n}(\mathbf{1}_{(0,1/(2n))} - \mathbf{1}_{(1/(2n),1/n)})$, and the contraction

$$T : L^2(0,1) \to \ell_2^2 : \xi \mapsto \Big(\frac{\langle \xi, e_0\rangle}{\sqrt{2}}, \langle \xi, e_1\rangle\Big).$$

Define the o.s.s. H on $L^2(0,1)$ by setting, for $x \in M_k \otimes L^2(0,1)$,

$$\|x\|_{M_k(H)} = \max\big\{\|x\|_{M_k(\mathrm{MIN}(L^2(0,1)))}, \sup_{h\in\mathcal{F}} \|(I_{M_k} \otimes TU_h)(x)\|_{M_k(R_2)}\big\}$$

(here, R_m is the m-dimensional row space). It is easy to check that the above definition satisfies Ruan's axioms. Moreover, U_h is a complete isometry for any $h \in \mathcal{F}$, hence the space H is completely almost transitive. We show that H is not 1-homogeneous. To this end, consider the unitary $V : H \to H$, with $Ve_0 = e_1$, $Ve_1 = e_0$, and $V\xi = \xi$ whenever $\xi \in \mathrm{span}[e_0, e_1]^\perp$. We prove that V is not a complete isometry.

In fact, let $x_0 = a_0 \otimes e_0 + a_1 \otimes e_1$, where $a_0, a_1 \in R_2$ are given by $a_0 = (1/\sqrt{2}, 0)$, $a_1 = (0, 1)$. Identifying $R_2 \otimes R_2$ with R_4, we see that $(I_{R_2} \otimes T)(x_0) = (1/2, 0, 0, 1)$, hence $\|x_0\|^2_{R_2(H)} \geq 5/4$. We shall show that, for $\sigma = 10^{-5}$, $\|(I_{R_2} \otimes V)x_0\|^2_{R_2(H)} \leq 5/4 - \sigma$. It suffices to prove that, for any $h \in \mathcal{F}$,

$$\|(I_{R_2} \otimes TU_hV)x_0\|^2_{R_2(R_2)} \leq 5/4 - \sigma. \tag{4.1}$$

Fix $h \in \mathcal{F}$, and set $g = U_he_1$. Then

$$(I_{R_2} \otimes V)x_0 = a_1 \otimes e_0 + a_0 \otimes e_1,\ (I_{R_2} \otimes U_hV)x_0 = a_1 \otimes h + a_0 \otimes g,$$

$$\begin{aligned}(I_{R_2} \otimes TU_hV)x_0 &= \Big(\frac{1}{\sqrt{2}}\big(a_1\langle h, e_0\rangle + a_0\langle g, e_0\rangle\big), a_1\langle h, e_1\rangle + a_0\langle g, e_1\rangle\Big)\\ &= \Big(\frac{1}{2}\langle g, e_0\rangle, \frac{1}{\sqrt{2}}\langle g, e_0\rangle, \frac{1}{\sqrt{2}}\langle g, e_1\rangle, \langle h, e_1\rangle\Big),\end{aligned}$$

hence

$$\|(I_{R_2} \otimes TU_hV)x_0\|^2_{R_2(R_2)} = v_{01} + \frac{v_{00} + v_{11}}{2} + \frac{v_{10}}{4}, \tag{4.2}$$

where, for $i = 0, 1$, $v_{0i} = |\langle h, e_i\rangle|^2$, and $v_{1i} = |\langle g, e_i\rangle|^2$. By the Pythagorean theorem, $v_{00} + v_{01} \leq 1$, $v_{00} + v_{10} \leq 1$, $v_{11} + v_{01} \leq 1$, and $v_{11} + v_{10} \leq 1$.

Now let $c = 1 - v_{01}$. By (4.2),

$$\begin{aligned}\|(I_{R_2} \otimes TU_hV)x_0\|^2_{R_2(R_2)} &= \frac{v_{01} + v_{00}}{2} + \frac{v_{11} + v_{10}}{4} + \frac{v_{01} + v_{11}}{4} + \frac{v_{01}}{4}\\ &\leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1-c}{4} = \frac{5}{4} - \frac{c}{4}.\end{aligned}$$

Thus, it remains to prove (4.1) for $c < 4\sigma$.

So suppose $v_{01} = 1 - c$, with $c < 4\sigma$, and show that $v_{10} < 4/n^2$. To this end, use the notation $k = k_h$ – that is, $k(t) = \int_0^t |h(s)|^2\, ds$. Let $\omega = \langle h, e_1\rangle / |\langle h, e_1\rangle|$, then

$$\||h| - e_1\|^2 \leq \|h - \omega e_1\|^2 = 2 - 2|\langle h, e_1\rangle| = 2c < 8\sigma < \varepsilon^2.$$

Set

$$\phi(t) = \int_0^t |e_1(s)|^2\, ds = \begin{cases} nt & 0 \leq t \leq 1/n \\ 1 & t > 1/n \end{cases}.$$

For any t,

$$\begin{aligned}|\sqrt{\phi(t)} - \sqrt{k(t)}| &= \Big| \|e_1 \mathbf{1}_{(0,t)}\| - \|h\mathbf{1}_{(0,t)}\| \Big| \\ &\le \|(e_1 - |h|)\mathbf{1}_{(0,t)}\| \le \|e_1 - |h|\| < \varepsilon,\end{aligned}$$

hence in particular, $k(t) \ge (\sqrt{\phi(t)} - \varepsilon)^2$ whenever $\sqrt{nt} > \varepsilon$ ($t > \varepsilon^2/n = 1/n^3$).

Recall that $g = U_h e_1$, hence $g(t) = h(t)e_1(k(t))$. Let $t_0 = k^{-1}(1/n)$. Then

$$\frac{1}{n} = k(t_0) \ge \left(\sqrt{nt_0} - \varepsilon\right)^2,$$

and consequently,

$$t_0 \le \Big(\frac{\varepsilon}{\sqrt{n}} + \frac{1}{n}\Big)^2 < \frac{4}{n^2}.$$

Then $e_1(k(t)) = 0$ for $t > t_0$, hence, by Hölder Inequality,

$$\begin{aligned} v_{10} &= \Big| \int_0^1 h(t)e_1(k(t))\,dt \Big|^2 = \Big| \int_0^{t_0} h(t)e_1(k(t))\,dt \Big|^2 \\ &\le t_0 \int_0^{t_0} |h(t)|^2 |e_1(k(t))|^2\,dt = t_0 < \frac{4}{n^2}.\end{aligned}$$

By (4.2),

$$\|(I_{R_2} \otimes TU_hV)x_0\|^2_{R_2(R_2)} = \frac{v_{01} + v_{00}}{2} + \frac{v_{01} + v_{11}}{2} + \frac{v_{10}}{4} \le 1 + \frac{1}{n^2} < \frac{5}{4} - \sigma.$$

This establishes (4.1) for $c < 4\sigma$.

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