Abstract

A Banach space $X$ is called subprojective if any of its infinite dimensional subspaces contains a further infinite dimensional subspace complemented in $X$. This paper is devoted to systematic study of subprojectivity. We examine the stability of subprojectivity of Banach spaces under various operations, such as direct or twisted sums, tensor products, and forming spaces of operators. Along the way, we obtain new classes of subprojective spaces.

Keywords: Banach space, complemented subspace, tensor product, space of operators.

1. Introduction and main results

Throughout this note, all Banach spaces are assumed to be infinite dimensional, and subspaces, infinite dimensional and closed, until specified otherwise. A Banach space $X$ is called subprojective if every subspace $Y \subseteq X$ contains a further subspace $Z \subseteq Y$, complemented in $X$. This notion was introduced in [41], in order to study the (pre)adjoints of strictly singular operators. Recall that an operator $T \in B(X, Y)$ is strictly singular ($T \in SS(X, Y)$) if $T$ is not an isomorphism on any subspace of $X$. In particular, it was shown that, if $Y$ is subprojective, and, for $T \in B(X, Y)$, $T^* \in SS(Y^*, X^*)$, then $T \in SS(X, Y)$.

Later, connections between subprojectivity and perturbation classes were discovered. More specifically, denote by $\Phi_+(X, Y)$ the set of upper semi-Fredholm operators – that is, operators with closed range, and finite dimensional kernel. If $\Phi_+(X, Y) \neq \emptyset$, we define the perturbation class

$$P\Phi_+(X, Y) = \{ S \in B(X, Y) : T + S \in \Phi_+(X, Y) \text{ whenever } T \in \Phi_+(X, Y) \}.$$
It is known that \( SS(X, Y) \subset P\Phi_+(X, Y) \). In general, this inclusion is proper. However, we get \( SS(X, Y) = P\Phi_+(X, Y) \) if \( Y \) is subprojective (see [1, Theorem 7.51] for this, and for similar connections to inessential operators).

Several classes of subprojective spaces are described in [16]. For instance, the spaces \( \ell^p \) \((1 \leq p < \infty)\) and \( c_0 \) are subprojective. Common examples of non-subprojective space are \( L^1 \) (since all Hilbertian subspaces of \( L^1 \) are not complemented), \( C(\Delta) \), where \( \Delta \) is the Cantor set, or \( \ell_\infty \) (for the same reason). The disc algebra is not subprojective, since by [43, III.E.3] it contains a copy of \( C(\Delta) \). By [41], \( L^p(0, 1) \) is subprojective if and only if \( 2 \leq p < \infty \). Consequently, the Hardy space \( H_p \) on the disc is subprojective for exactly the same values of \( p \). Indeed, \( H_p \) contains the disc algebra. For \( 1 < p < \infty \), \( H_p \) is isomorphic to \( L^p \). The space \( H_1 \) contains isomorphic copies of \( L_p \) for \( 1 < p \leq 2 \) [42, Section 3]. On the other hand, VMO is subprojective ([30], see also [37] for non-commutative generalizations).

In this paper we examine several aspects of subprojectivity. We start by collecting various facts needed to study subprojectivity (Section 2). Along the way, we prove that subprojectivity is stable under suitable direct sums (Proposition 2.2). However, subprojectivity is not a 3-space property (Proposition 2.8). Consequently, subprojectivity is not stable under the gap metric (Proposition 2.9). Considering the place of subprojective spaces in Gowers dichotomy, we observe that each subprojective space has a subspace with an unconditional basis. However, we exhibit a space with an unconditional basis, but with no subprojective subspaces (Proposition 2.11).

In Section 3, we investigate the subprojectivity of tensor products, and of spaces of operators. A general result on tensor products (Theorem 3.1) yields the subprojectivity on \( \ell(p) \otimes \ell(q) \) and \( \ell(p) \otimes \ell(q) \) for \( 1 \leq p, q < \infty \) (Corollary 3.3), as well as of \( K(L(p), L(q)) \) for \( 1 < p \leq 2 \leq q < \infty \) (Corollary 3.4). We also prove that the space \( B(X) \) is never subprojective (Theorem 3.10), and give an example of non-subprojective tensor product \( \ell_2 \otimes \ell_2 \) (Proposition 3.9).

Throughout Section 4, we work with \( C(K) \) spaces, with \( K \) compact metrizable. We begin by observing that \( C(K) \) is subprojective if and only if \( K \) is scattered. Then we prove that \( C(K, X) \) is subprojective if and only if both \( C(K) \) and \( X \) are (Theorem 4.1). Turning to spaces of operators, we show that, for \( K \) scattered, \( II_{\ell(p)}(C(K), \ell(q)) \) is subprojective (Proposition 4.4). We also study continuous fields on a scattered base space, proving that any scattered separable CCR \( C^* \)-algebra is subprojective (Corollary 4.7).

Section 5 shows that, in many cases, subprojectivity passes from a sequence space to the associated Schatten space (Proposition 5.1).

Proceeding to Banach lattices, in Section 6 we prove that \( p \)-disjointly homogeneous \( p \)-convex lattices \( (2 \leq p < \infty) \) are subprojective (Proposition 6.2).

In Section 7 (Proposition 7.1), we show that the lattice \( \overline{X(\ell_p)} \) is subprojective whenever \( X \) is. Consequently (Proposition 7.3), if \( X \) is a subprojective space with an unconditional basis and non-trivial cotype, then \( \text{Rad}(X) \) is subprojective.

Throughout the paper, we use the standard Banach space results and nota-
ation. By $B(X,Y)$ and $\mathcal{K}(X,Y)$ we denote the sets of linear bounded and compact operators, respectively, acting between Banach spaces $X$ and $Y$. $UB(X)$ refers to the closed unit ball of $X$. For $p \in [1, \infty]$, we denote by $p'$ the “adjoint” of $p$ (that is, $1/p + 1/p' = 1$).

2. General facts about subprojectivity

We start collecting general facts about subprojectivity by observing that subprojectivity passes to subspaces.

**Proposition 2.1.** Any subspace of a subprojective Banach space is subprojective.

In contrast to this, subprojectivity does not pass to quotients. Indeed, any separable Banach space is a quotient of $\ell_1$, which is subprojective. As noted in Section 1, there exist separable non-subprojective spaces.

However, subprojectivity does pass to direct sums.

**Proposition 2.2.** (a) Suppose $X$ and $Y$ are Banach spaces. Then the following are equivalent:

1. Both $X$ and $Y$ are subprojective.
2. $X \oplus Y$ is subprojective.

(b) Suppose $X_1, X_2, \ldots$ are Banach spaces, and $E$ is a space with a 1-unconditional basis. Then the following are equivalent:

1. The spaces $E, X_1, X_2, \ldots$ are subprojective.
2. $(\sum_n X_n)_E$ is subprojective.

In (b), we view $E$ as a space of all sequences, for which the norm $|\cdot|_E$ is finite. $(\sum_n X_n)_E$ refers to the space of all sequences $(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n$, endowed with the norm $|(x_n)_{n \in \mathbb{N}}| = |(|x_n|_{X_n})|_E$. Due to the 1-unconditionality (actually, 1-suppression unconditionality suffices), $(\sum_n X_n)_E$ is a Banach space.

In the proof of Proposition 2.2, and further in the paper, we will use two results stated below.

**Proposition 2.3.** Consider Banach spaces $X$ and $X'$, and $T \in B(X,X')$. Suppose $Y$ is a subspace of $X$, $T|_Y$ is an isomorphism, and $T(Y)$ is complemented in $X'$. Then $Y$ is complemented in $X$.

**Proof.** If $Q$ is a projection from $X'$ to $T(Y)$, then $T^{-1}QT$ is a projection from $X$ onto $Y$. $\blacksquare$

This immediately yields:
Corollary 2.4. Suppose $X$ and $X'$ are Banach spaces, and $X'$ is subprojective. Suppose, furthermore, that $Y$ is a subspace of $X$, and there exists $T \in B(X, X')$ so that $T|_Y$ is an isomorphism. Then $Y$ contains a subspace complemented in $X$.

The following version of “Principle of Small Perturbations” is folklore, and essentially contained in [5]. We include the proof for the sake of completeness.

Proposition 2.5. Suppose $(x_k)$ is a seminormalized basic sequence in a Banach space $X$, and $(y_k)$ is a sequence so that $\lim_k |x_k-y_k| = 0$. Suppose, furthermore, that every subspace of $\text{span}[y_k : k \in \mathbb{N}]$ contains a subspace complemented in $X$. Then $\text{span}[x_k : k \in \mathbb{N}]$ contains a subspace complemented in $X$.

Proof. Replacing $x_k$ by $x_k/|x_k|$, we can assume that $(x_k)$ is normalized. Denote the biorthogonal functionals by $x_k^*$, and set $K = \sup_k |x_k^*|$. Passing to a subsequence, we can assume that $\sum_k |x_k - y_k| < 1/(2K)$. Define the operator $U \in B(X)$ by setting $Ux = \sum_k x_k^*(x)(y_k-x_k)$. Clearly $|U| < 1/2$, and therefore, $V = I_X + U$ is invertible. Furthermore, $Vx_k = y_k$. If $Q$ is a projection from $X$ onto a subspace $W \subset \text{span}[y_k : k \in \mathbb{N}]$, then $P = V^{-1}QV$ is a projection from $X$ onto a subspace $Z \subset \text{span}[x_k : k \in \mathbb{N}]$.

Remark 2.6. Note that, in the proof above, the kernels and the ranges of the projections $Q$ and $P$ are isomorphic, via the action of $V$.

Proof of Proposition 2.2. Due to Proposition 2.1, in both (a) and (b), only the implication $(2) \Rightarrow (1)$ needs to be established.

(a) Throughout the proof, $P_X$ and $P_Y$ stand for the coordinate projections from $X \oplus Y$ onto $X$ and $Y$, respectively. We have to show that any subspace $E$ of $X \oplus Y$ contains a further subspace $G$, complemented in $X \oplus Y$.

Show first that $E$ contains a subspace $F$ so that either $P_X|_F$ or $P_Y|_F$ is an isomorphism. Indeed, suppose $P_X|_F$ is not an isomorphism, for any such $F$. Then $P_X|_F$ is strictly singular, hence there exists a subspace $F \subset E$, so that $P_X|_F$ has norm less than 1/2. But $P_X + P_Y = I_{X \oplus Y}$, hence, by the triangle inequality, $\|P_Yf\| \geq \|f\| - \|P_Xf\| \geq \|f\|/2$ for any $f \in F$. Consequently, $P_Y|_F$ is an isomorphism.

Thus, by passing to a subspace, and relabeling if necessary, we can assume that $E$ contains a subspace $F$, so that $P_X|_F$ is an isomorphism. By Corollary 2.4, $F$ contains a subspace $G$, complemented in $X$.

Set $F' = P_X(F)$, and let $V$ be the inverse of $P_X : F \rightarrow F'$. By the subprojectivity of $X$, $F'$ contains a subspace $G'$, complemented in $X$ via a projection $Q$. Then $P = VQ P_X$ gives a projection onto $G = V(G') \subset F$.

(b) Here, we denote by $P_n$ the coordinate projection from $X = (\sum_k X_k)\ell$ onto $X_n$. Furthermore, we set $Q_n = \sum_{k=1}^n P_k$, and $Q_n^* = I - Q_n$. We have to show that any subspace $Y \subset X$ contains a subspace $Y_0$, complemented in $X$. To this end, consider two cases.

(i) For some $n$, and some subspace $Z \subset Y$, $Q_n|_Z$ is an isomorphism. By part (a), $X_1 \oplus \ldots \oplus X_n = Q_n(X)$ is subprojective. Apply Corollary 2.4 to obtain $Y_0$. 

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(ii) For every \( n, Q_n|_Y \) is not an isomorphism - that is, for every \( n \in \mathbb{N} \), and every \( \varepsilon > 0 \), there exists a norm one \( y \in Y \) so that \( |Q_n y| < \varepsilon \). Therefore, for every sequence of positive numbers \( (\varepsilon_i) \), we can find \( 0 = N_0 < N_1 < N_2 < \ldots \), and a sequence of norm one vectors \( y_i \in Y \), so that, for every \( i \), \( |Q_{N+i} y_i| < \varepsilon_i \). By a small perturbation principle, we can assume that \( Y \) contains norm one vectors \( (y'_i) \) so that \( Q_{N_i} y'_i = Q_{N_i+1} y'_i = 0 \) for every \( i \). Write 
\[ y'_i = (z_j)_{j=0}^{N_i+1}, \]
with \( z_j \in X_j \). Then \( Z = \text{span}\{0, \ldots, 0, z_j, 0, \ldots\} : j \in \mathbb{N} \} \) (\( z_j \) is in \( j \)-th position) is complemented in \( X \). Indeed, if \( z_j \neq 0 \), find \( z^*_j \in X^*_j \) so that \( |z^*_j| = |z_j|^{-1} \), and \( \langle z^*_j, z_j \rangle = 1 \). If \( z_j = 0 \), set \( z^*_j = 0 \). For \( x = (x_j)_{j \in \mathbb{N}} \in X \), define \( Rx = (\langle z^*_j, x_j \rangle z_j)_{j \in \mathbb{N}} \). It is easy to see that \( R \) is a projection onto \( Z \), and \( |R| \) does not exceed the unconditionality constant of \( E \).

Now note that \( J : Z \to \mathcal{E} : (\alpha_1 z_1, \alpha_2 z_2, \ldots) \mapsto (\alpha_1 |z_1|, \alpha_2 |z_2|, \ldots) \) is an isometry. Let \( Y^r = \text{span}\{y'_i : i \in \mathbb{N}\} \), and \( Y_E = J(Y^r) \). By the subprojectivity of \( \mathcal{E}, Y_E \) contains a subspace \( W \), which is complemented in \( \mathcal{E} \) via a projection \( R_1 \). Then \( J^{-1} R_1 J R \) is a projection from \( X \) onto \( Y_0 = J^{-1}(W) \subset X \).

**Remark 2.7.** From the last proposition it follows the (strong) \( p \)-sum of subprojective Banach spaces is subprojective. On the other hand, the infinite weak sum of subprojective spaces need not be subprojective.

Recall that if \( X \) is a Banach space, then
\[
\ell_{p}^{weak}(X) = \{ x = (x_n)_{n=1}^{\infty} \in X \times X \times X \ldots : \sup_{x^* \in X^*} (\sum_{n=1}^{\infty} |x^*(x_n)|^p)^{\frac{1}{p}} < \infty \}.
\]

It is known that \( \ell_{p}^{weak}(X) \) is isomorphic to \( B(\ell_p, X) \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \), see [8, Theorem 2.2]. We show that, for \( X = \ell_r, (r \geq p') \), \( B(\ell_p, X) \) contains a copy of \( \ell_X \), and therefore, is not subprojective. To this end, denote by \( (e_i) \) and \( (f_i) \) the canonical bases in \( \ell_r \) and \( \ell_p \) respectively. For \( \alpha = (\alpha_i) \in \ell_X \), define \( B(\ell_p, X) \ni U \alpha : e_i \mapsto \alpha_i f_i \). Clearly, \( U \) is an isomorphism.

Note that the situation is different for \( r < p' \). Then, by Pitt’s Theorem, \( B(\ell_p, \ell_r) = K(\ell_p, \ell_r) \). In the next section we prove that the latter space is subprojective.

Next we show that subprojectivity is not a 3-space property.

**Proposition 2.8.** For \( 1 < p < \infty \) there exists a non-subprojective Banach space \( Z_p \), containing a subspace \( X_p \), so that \( X_p \) and \( Z_p/X_p \) are isomorphic to \( \ell_p \).

**Proof.** [22, Section 6] gives us a short exact sequence
\[
0 \longrightarrow \ell_p \overset{j_p}{\longrightarrow} Z_p \overset{q_p}{\longrightarrow} \ell_p \longrightarrow 0,
\]
where the injection \( j_p \) is strictly cosingular, and the quotient map \( q_p \) is strictly singular. By [22, Theorem 6.2], \( Z_p \) is not isomorphic to \( \ell_p \). By [22, Theorem 6.5], any non-strictly singular operator on \( Z_p \) fixes a copy of \( Z_p \). Consequently, \( j_p(\ell_p) \) contains no complemented subspaces (by [27, Theorem 2.a.3], any complemented subspace of \( \ell_p \) is isomorphic to \( \ell_p \)).
It is easy to see that subprojectivity is stable under isomorphisms. However, it is not stable under a rougher measure of “closeness” of Banach spaces – the gap measure. If \( Y \) and \( Z \) are subspaces of a Banach space \( X \), we define the gap (or opening)

\[
\Theta_X(Y, Z) = \max \left\{ \sup_{y \in Y, \|y\| = 1} \text{dist}(y, Z), \sup_{z \in Z, \|z\| = 1} \text{dist}(z, Y) \right\}.
\]

We refer the reader to the comprehensive survey [33] for more information. Here, we note that \( \Theta_X \) satisfies a “weak triangle inequality”, hence it can be viewed as a measure of closeness of subspaces. The following shows that subprojectivity is not stable under \( \Theta_X \).

**Proposition 2.9.** There exists a Banach space \( X \) with a subprojective subspace \( Y \) so that, for every \( \varepsilon > 0 \), \( X \) contains a non-subprojective space \( Z \) with \( \Theta_X(Y, Z) \leq \varepsilon \).

**Proof.** Our \( Y \) will be isomorphic to \( \ell_p \), where \( p \in (1, \infty) \) is fixed. By Proposition 2.8, there exists a non-subprojective Banach space \( W \), containing a subspace \( W_0 \), so that both \( W_0 \) and \( W' = W/W_0 \) are isomorphic to \( \ell_p \). Denote the quotient map \( W \to W' \) by \( q \). Consider \( X = W \oplus_1 W' \) and \( Y = W_0 \oplus_1 W' \subset E \). Furthermore, for \( \varepsilon > 0 \), define \( Z_\varepsilon = \{ \varepsilon w \oplus_1 qw : w \in W \} \). Clearly, \( Y \) is isomorphic to \( \ell_p \oplus \ell_p \cong \ell_p \), hence subprojective, while \( Z_\varepsilon \) is isomorphic to \( W \), hence not subprojective. By [33, Lemma 5.9], \( \Theta_X(Y, Z_\varepsilon) \leq \varepsilon \).

Looking at subprojectivity through the lens of Gowers dichotomy and observing that a subprojective Banach space does not contain hereditarily indecomposable subspaces, we immediately obtain the following.

**Proposition 2.10.** Every subprojective space has a subspace with an unconditional basis.

The converse to the above proposition is false.

**Proposition 2.11.** There exists a Banach space with an unconditional basis, without subprojective subspaces.

**Proof.** In [18, Section 5], T. Gowers and B. Maurey construct a Banach space \( X \) with a 1-unconditional basis, so that any operator on \( X \) is a strictly singular perturbation of a diagonal operator. We prove that \( X \) has no subprojective subspaces. In doing so, we are re-using the notation of that paper. In particular, for \( n \in \mathbb{N} \) and \( x \in X \), we define \( [x]_{(n)} \) as the supremum of \( \sum_{i=1}^{n} |x_i|_1 \), where \( x_1, \ldots, x_n \) are successive vectors so that \( x = \sum_i x_i \). By [18, Lemma 4], for every block subspace \( Y \) in \( X \), every \( c > 1 \), and every \( n \in \mathbb{N} \), there exists \( y \in Y \) so that \( 1 = |y| \leq |y|_{(n)} < c \). This technical result can be used to establish a remarkable property of \( X \): suppose \( Y \) is a subspace of \( X \), with a normalized block basis \( (y_k) \). Then any zero-diagonal (relative to the basis \( (y_k) \)) operator on \( Y \) is strictly singular. Consequently, any \( T \in B(Y) \) can be written as \( T = \Lambda + S \), where \( \Lambda \) is diagonal, and \( S \) is zero-diagonal, hence strictly singular. This result is proved.
in [18] for $Y = X$, but an inspection (involving Lemma 27 and Corollary 28 of the cited paper) yields the generalization described above.

Suppose, for the sake of contradiction, that $X$ contains a subprojective subspace $Y$. A small perturbation argument shows we can assume $Y$ to be a block subspace. Blocking further, we can assume that $Y$ is spanned by a block basis $(y_j)$, so that $1 = \|y_j\| \leq \|y_j\|_{(j)} < 1 + 2^{-j}$. We achieve the desired contradiction by showing that no subspace of $Z = \text{span}\{y_1 + y_2, y_3 + y_4, \ldots\}$ is complemented in $Y$.

Suppose $P$ is an infinite rank projection from $Y$ onto a subspace of $Z$. Write $P = \Lambda + S$, where $S$ is a strictly singular operator with zeroes on the main diagonal, and $\Lambda = (\lambda_j)_{j=1}^{\infty}$ is a diagonal operator (that is, $\Lambda y_j = \lambda_j y_j$ for any $j$). As $\sup_j |y_j|_{(j)} < \infty$, by [18, Section 5] we have $\lim_j S y_j = 0$. Note that $(\Lambda + S)^2 = \Lambda + S$, hence $\text{diag}(\lambda_j^2 - \lambda_j) = \Lambda^2 - \Lambda = S - \Lambda S - S \Lambda - S^2$ is strictly singular, or equivalently, $\lim_j \lambda_j(1 - \lambda_j) = 0$. Therefore, there exists a $0 \to 1$ sequence $(X_j)$ so that $X' - \Lambda$ is compact (equivalently, $\lim_j (\lambda_j - X_j) = 0$), where $X' = \text{diag}(X_j)$ is a diagonal projection. Then $P = X' + S'$, where $S' = S + (\Lambda - X')$ is strictly singular, and satisfies $\lim_j S'y_j = 0$. The projection $P$ is not strictly singular (since it is of infinite rank), hence $X' = P - S'$ is not strictly singular. Consequently, the set $J = \{j \in \mathbb{N} : X_j = 1\}$ is infinite.

Now note that, for any $j$, $|P y_j - y_j| \geq 1/2$. Indeed, $P y_j \in Z$, hence we can write $P y_j = \sum_k \alpha_k(y_{2k-1} + y_{2k})$. Let $\ell = \lfloor j/2 \rfloor$. By the 1-unconditionality of our basis, $|y_j - P y_j| \geq |y_j - \alpha \ell(y_{2\ell-1} + y_{2\ell})| \geq \max\{1 - \alpha \ell, |\alpha \ell|\} \geq 1/2$. For $j \in J$, $S'y_j = P y_j - y_j$, hence $|S'y_j| \geq 1/2$, which contradicts $\lim_j |S'y_j| = 0$.

\[\text{Remark 2.12.}\] The preceding statement provides an example of an atomic order continuous Banach lattice without subprojective subspaces. By [29, Section 2.4], any non-order continuous Banach lattice contains a subprojective subspace $c_0$. Also, if a Banach lattice is non-atomic order continuous with an unconditional basis, then it contains a subprojective subspace $\ell_2$ (i.e. [23, Theorem 2.3]).

Finally, one might ask whether, in the definition of subprojectivity, the projections from $X$ onto $Z$ can be uniformly bounded. More precisely, we call a Banach space $X$ \textit{uniformly subprojective} (with constant $C$) if, for every subspace $Y \subseteq X$, there exists a subspace $Z \subseteq Y$ and a projection $P : X \to Z$ with $\|P\| \leq C$. The proof of [16, Proposition 2.4] essentially shows that the following spaces are uniformly subprojective: (i) $\ell_p$ ($1 \leq p < \infty$) and $c_0$; (ii) the Lorentz sequence spaces $l_{p,u}$; (iii) the Schreier space; (iv) the Tsirelson space; (v) the James space. Additionally, $L_p(0,1)$ is uniformly subprojective for $2 \leq p < \infty$. This can be proved by combining Kadets-Pelczynski dichotomy with the results of [2] about the existence of “nicely complemented” copies of $\ell_2$. Moreover, any $c_0$-saturated separable space is uniformly subprojective, since any isomorphic copy of $c_0$ contains a $\lambda$-isomorphic copy of $c_0$, for any $\lambda > 1$ [27, Proposition 2.1.3]. By Sobczyk’s Theorem, a $\lambda$-isomorphic copy of $c_0$ is $2\lambda$-complemented in every separable superspace. In particular, if $K$ is a countable metric space, then $C(K)$ is uniformly subprojective [11, Theorem 12.30].
However, in general, subprojectivity need not be uniform. Indeed, suppose $2 < p_1 < p_2 < \ldots < \infty$, and $\lim_n p_n = \infty$. By Proposition 2.2(b), $X = (\sum_n L_{p_n}(0,1))_2$ is subprojective. The span of independent Gaussian random variables in $L_p$ (which we denote by $G_p$) is isometric to $\ell_2$. Therefore, by [17, Corollary 5.7], any projection from $L_p$ onto its subspace $G_p$ has norm at least $c_0 \sqrt{p}$, where $c_0$ is a universal constant. Thus, $X$ is not uniformly subprojective.

3. Subprojectivity of tensor products and spaces of operators

Throughout the paper, $X \otimes Y$ stands for the algebraic tensor product of $X$ and $Y$. A tensor norm $|\cdot|_{\otimes}$ assigns, to each pair of Banach spaces $X$ and $Y$, a norm on $X \otimes Y$, in such a way that:

1. For any Banach spaces $X'$ and $Y'$, any $T \in B(X,X')$ and $S \in B(Y,Y')$, and any $x \in X \otimes Y$, we have $|(T \otimes S)x|_{\otimes} \leq |T||S||x|_{\otimes}$.

2. For any $x \in X \otimes Y$, $|x|_{\otimes} \leq |z|_{\otimes} \leq |x|_{\otimes}$ (the left and right hand sides refer to the injective and projective tensor norms, respectively).

The tensor product $X \otimes Y$ is the completion of $X \otimes Y$ in the norm $|\cdot|_{\otimes}$. In particular, $\otimes$ and $\hat{\otimes}$ stand for the injective and projective tensor products. For more information about tensor products, the reader is referred to [7, Chapter 12], or to [9].

Suppose $X_1$, $X_2$, \ldots, $X_k$ are Banach spaces with unconditional FDD, implemented by finite rank projections $(P_{1,n})$, $(P_{2,n})$, \ldots, $(P_{k,n})$, respectively. That is, $P_{i,n}P_{i,m} = 0$ unless $n = m$, $\lim_n \sum_{n=1}^{N} P_{i,n} = I_{X_i}$, point-norm, and $\sup_{N,\pm} |\sum_{n=1}^{N} \pm P_{i,n}| < \infty$ (this quantity is sometimes referred to as the unconditional FDD constant of $X_i$). Let $E_{in} = \text{ran}(P_{i,n})$.

We say that a sequence $(w_j)_{j=1}^{N} \subset X_1 \otimes X_2 \otimes \ldots \otimes X_k$ is block-diagonal if there exists a sequence $0 = N_1 < N_2 < \ldots$ so that

$$w_j \in \left( \sum_{n=N_j+1}^{N_{j+1}} E_{1,n} \right) \otimes \left( \sum_{n=N_j+1}^{N_{j+1}} E_{2,n} \right) \otimes \ldots \otimes \left( \sum_{n=N_j+1}^{N_{j+1}} E_{k,n} \right).$$

Suppose $E$ is an unconditional sequence space, and $\hat{\otimes}$ is a tensor product of Banach spaces. The Banach space $X_1 \hat{\otimes} X_2 \hat{\otimes} \ldots \hat{\otimes} X_k$ is said to satisfy the $E$-estimate if there exists a constant $C \geq 1$ so that, for any block diagonal sequence $(w_j)_{j\in \mathbb{N}}$ in $X_1 \hat{\otimes} X_2 \hat{\otimes} \ldots \hat{\otimes} X_k$, we have

$$C^{-1} |(\{w_j\})_{j\in \mathbb{N}}|_E \leq |\sum_j w_j| \leq C |(\{w_j\})_{j\in \mathbb{N}}|_E$$

(3.1)

**Theorem 3.1.** Suppose $X_1$, $X_2$, \ldots, $X_k$ are subprojective Banach spaces with unconditional FDD, and $\hat{\otimes}$ is a tensor product. Suppose, furthermore, that for any finite increasing sequence $i = [1 \leq i_1 < \ldots < i_k \leq k]$, there exists an unconditional sequence space $E_i$, so that $X_{i_1} \hat{\otimes} X_{i_2} \hat{\otimes} \ldots \hat{\otimes} X_{i_k}$ satisfies the $E_i$-estimate. Then $X_1 \hat{\otimes} X_2 \hat{\otimes} \ldots \hat{\otimes} X_k$ is subprojective.
Note that a tensor product need not be associative. We presume the “natural” location of brackets: $X_1 \widehat{\otimes} X_2 \widehat{\otimes} X_3 = (X_1 \widehat{\otimes} X_2) \widehat{\otimes} X_3$, etc. A different position of brackets is equivalent to a different ordering of $X_1, \ldots, X_n$.

A similar result for ideals of operators holds as well. We keep the notation for projections implementing the FDD in Banach spaces $X_1$ and $X_2$. We say that a Banach operator ideal $\mathcal{A}$ is suitable (for the pair $(X_1, X_2)$) if the finite rank operators are dense in $\mathcal{A}(X_1, X_2)$ (in its ideal norm). We say that a sequence $(w_j)_{j \in \mathbb{N}} \subseteq \mathcal{A}(X_1, X_2)$ is block diagonal if there exists a sequence $0 = N_1 < N_2 < \ldots$ so that, for any $j$, $w_j = (P_{2,N_j} - P_{2,N_j-1})w_j(P_{1,N_j} - P_{1,N_j-1})$. If $\mathcal{E}$ is an unconditional sequence space, we say that $\mathcal{A}(X_1, X_2)$ satisfies the $\mathcal{E}$-estimate for some constant $C$,

$$C^{-1}||w_j||_{\mathcal{E}} \leq \sum_j w_j \leq C||w_j||_{\mathcal{E}}$$

(3.2)

holds for any finite block-diagonal sequence $(w_j)$.

**Theorem 3.2.** Suppose $X_1$ and $X_2$ are Banach spaces with unconditional FDD, so that $X_1^*$ and $X_2$ are subprojective. Suppose, furthermore, that the ideal $\mathcal{A}$ is suitable for $(X_1, X_2)$, and $\mathcal{A}(X_1, X_2)$ satisfies the $\mathcal{E}$-estimate for some unconditional sequence $\mathcal{E}$. Then $\mathcal{A}(X_1, X_2)$ is subprojective.

Before proving these theorems, we state a few consequences.

**Corollary 3.3.** The spaces $X_1 \widehat{\otimes} \ldots \widehat{\otimes} X_n$ and $X_1 \widehat{\otimes} \ldots \widehat{\otimes} X_n$ are subprojective where $X_i$ is isomorphic to either $\ell_p$ (1 ≤ $p_i$ < ∞), or to $c_0$, for every 1 ≤ $i$ ≤ $n$.

For $n = 2$, this result goes back to [38] and [32] (the injective and projective cases, respectively).

Suppose a Banach space $X$ has an unconditional FDD implemented by projections $(P'_n)$ – that is, $P'_n P'_m = 0$ unless $n = m$, sup$_{\mathcal{N}, \pm} |\sum^n_{n=1} \pm P'_n| < \infty$, and lim$_n$ sup$_{\mathcal{N}}$ sup$_{\mathcal{N}}$ $\sum^n_{n=1} P'_n = I_X$ point-norm. We say that $X$ satisfies the lower $p$-estimate if there exists a constant $C$ so that, for any finite sequence $\xi_j \in \text{ran } P'_j$, $|\sum_j \xi_j|^p \leq C \sum_j |\xi_j|^p$. The smallest $C$ for which the above inequality holds is called the lower $p$-estimate constant. The upper $p$-estimate, and the upper $p$-estimate constant, are defined in a similar manner. Note that, if $X$ is an unconditional sequence space, then the above definitions coincide with the standard one (see e.g. [28, Definition 1.f.4])

**Corollary 3.4.** Suppose the Banach spaces $X_1$ and $X_2$ have unconditional FDD, satisfy the lower and upper $p$-estimates respectively, and both $X_1^*$ and $X_2$ are subprojective. Then $\mathcal{K}(X_1, X_2)$ is subprojective.

Before proceeding, we mention several instances where the above corollary is applicable. Note that, if $X$ has type 2 (cotype 2), then $X$ satisfies the upper (resp. lower) 2-estimate. Indeed, suppose $X$ has type 2, and $w_1, \ldots, w_n$ are such that $w_j = P_j w_j$ for any $j$. Then

$$|\sum_j w_j| \leq C \text{Ave}_\pm |\sum_j \pm w_j| \leq CT_2(X)\left(\sum_j |w_j|^2\right)^{1/2}$$
(T_2(X) is the type 2 constant of X). The cotype case is handled similarly. Thus, we can state:

**Corollary 3.5.** Suppose the Banach spaces X_1 and X_2 have unconditional FDD, cotype 2 and type 2 respectively, and both X^*_1 and X_2 are subprojective. Then K(X_1, X_2) is subprojective.

This happens, for instance, if X_1 = L_p(\mu) or \mathcal{C}_p (1 < p \leq 2) and X_2 = L_q(\mu) or \mathcal{C}_q (2 \leq q < \infty). Indeed, the type and cotype of these spaces are well known (see e.g. [36]). The Haar system provides an unconditional basis for L_p. The existence of unconditional FDD of \mathcal{C}_p spaces is given by [4].

**Proof of Theorem 3.1.** We will prove the theorem by induction on k. Clearly, we can take k = 1 as the basic case. Suppose the statement of the theorem holds for a tensor product of any k – 1 subprojective Banach spaces that satisfy \mathcal{E}-estimate. We will show that the statement holds for the tensor product of k Banach spaces X = X_1 \overline{\otimes} X_2 \overline{\otimes} \ldots \overline{\otimes} X_k.

For notational convenience, let P_{ik} = \sum_{i=1}^{n} P_{ik}^m, and I_i = I_{X_i}. If A \in B(X) is a projection, we use the notation A^\perp for IX + A. Furthermore, define the projections Q_n = P_{1n} \overline{\otimes} P_{2n} \overline{\otimes} \ldots \overline{\otimes} P_{kn} and R_n = P_{1n} \overline{\otimes} P_{2n} \overline{\otimes} \ldots \overline{\otimes} P_{kn}^\perp. Renorming all X_i’s if necessary, we can assume that their unconditional FDD constants equal 1.

First show that, for any n, ran R_n^\perp is subprojective. To this end, write \sum_{i=1}^{n} P^{(i)} are defined by

\begin{align*}
P^{(1)} &= P_{1n} \overline{\otimes} I_2 \overline{\otimes} \ldots \overline{\otimes} I_k, \\
P^{(2)} &= P_{1n} \overline{\otimes} P_{2n} \overline{\otimes} I_3 \overline{\otimes} \ldots \overline{\otimes} I_k, \\
P^{(3)} &= P_{1n} \overline{\otimes} P_{2n}^\perp \overline{\otimes} P_{3n} \overline{\otimes} I_4 \overline{\otimes} \ldots \overline{\otimes} I_k, \\
&\ldots \ldots \ldots \\
P^{(k)} &= P_{1n} \overline{\otimes} P_{2n}^\perp \overline{\otimes} P_{3n}^\perp \overline{\otimes} P_{4n}^\perp \overline{\otimes} \ldots \overline{\otimes} P_{kn}^\perp
\end{align*}

(note also that P^{(i)}P^{(j)} = 0 unless i = j). Thus, there exists i so that P^{(i)} is an isomorphism on a subspace Y' \subset Y. Now observe that the range of P^{(i)} is isomorphic to a subspace of L_2^N(X^{(i)}), where N = rank P_{1n}, and

\[ X^{(i)} = X_1 \overline{\otimes} X_2 \overline{\otimes} \ldots \overline{\otimes} X_{i-1} \overline{\otimes} X_{i+1} \overline{\otimes} \ldots \overline{\otimes} X_k. \]

By the induction hypothesis, X^{(i)} is subprojective. By Proposition 2.2, ran P^{(i)} is subprojective for every i, hence so is ran R_n^\perp.

Now suppose Y is an infinite dimensional subspace of X. We have to show that Y contains a subspace Z, complemented in X. If there exists n \in \mathbb{N} so that R_n^\perp|_Y is not strictly singular, then, by Corollary 2.4, Z contains a subspace complemented in X.

Now suppose R_n^\perp|_Z is strictly singular for any n. It is easy to see that, for any sequence of positive numbers (\varepsilon_m), one can find 0 = n_0 < n_1 < n_2 < \ldots, and norm one elements x_m \in Y, so that, for any m, |R_{n_{m-1}}^\perp x_m| + |x_m - Q_{n_m} x_m| <
By a small perturbation, we can assume that \( x_m = R_{n_m}^\perp Q_{n_m} x_m \). That is,

\[
x_m \in \text{ran} \left( (P_{1,n_m} - P_{1,n_m+1}) \otimes (P_{2,n_m} - P_{2,n_m+1}) \otimes \ldots \otimes (P_{k,n_m} - P_{k,n_m+1}) \right).
\]

Let \( E_{i,m} = \text{ran} (P_{i,n_m} - P_{i,n_m+1}) \), and \( W = \text{span} \{ E_{1,m} \otimes E_{2,m} \otimes \ldots \otimes E_{k,m} : m \in \mathbb{N} \} \subseteq X \). Applying "Tong's trick" (see e.g. [27, p. 20]), and taking the 1-unconditionality of our FDDs into account, we see that

\[
U : X \to W : x \mapsto \sum_m ((P_{1,n_m} - P_{1,n_m+1}) \otimes \ldots \otimes (P_{k,n_m} - P_{k,n_m+1})) x
\]
defines a contractive projection onto \( W \). Furthermore, \( Z = \text{span} \{ x_m : m \in \mathbb{N} \} \) is complemented in \( W \). Indeed, the projection \( P_{i,n_m} - P_{i,n_m+1} \) is contractive, hence we can identify \( E_{1,m} \otimes \ldots \otimes E_{k,m} \) with \( (E_{1,m} \otimes \ldots \otimes E_{k,m}) \cap X \). By the by Hahn-Banach Theorem, for each \( m \) there exists a contractive projection \( U_m \) on \( E_{1,m} \otimes \ldots \otimes E_{k,m} \), with range \( \text{span} \{ x_m \} \). By our assumption, there exists an unconditional sequence space \( E \) so that \( X_1 \otimes \ldots \otimes X_k \) satisfies the \( E \)-estimate. Then, for any finite sequence \( w_m \in E_{1,m} \otimes \ldots \otimes E_{k,m} \), (3.1) yields

\[
| \sum_k U_k w_k | \leq C | (| U_k w_k | ) | E | \leq C | (| w_k | ) | E | \leq C \sum_k U_k w_k |.
\]

Thus, \( Z \) is complemented in \( X \).

**Sketch of the proof of Theorem 3.2.** On \( A(X_1, X_2) \) we define the projection \( R_n : A(X_1, X_2) \to A(X_1, X_2) : w \mapsto P_n^\perp w P_1 \). Then the range of \( R_n \) is isomorphic to \( X_1^\perp \otimes \ldots \otimes X_k^\perp \otimes X_2 \). Then proceed as in the the proof of Theorem 3.1 (with \( k = 2 \)).

To prove Corollary 3.3, we need two auxiliary results.

**Lemma 3.6.** Suppose \( 1 < p_i < \infty \) (\( 1 \leq i \leq n \)) and \( X = \bigotimes_{i=1}^n \ell_{p_i} \).

1. If \( \sum 1/p_i > n - 1 \), then \( X \) satisfies the \( \ell_s \)-estimate with \( 1/s = \sum 1/p_i - (n - 1) \).

2. If \( \sum 1/p_i \leq n - 1 \), then \( X \) satisfies the \( c_0 \)-estimate.

**Proof.** Suppose \( (w_j) \) is a finite block-diagonal sequence in \( X \). We shall show that \( | \sum_j w_j | = | (| w_j | ) | s \), with \( s \) as in the statement of the lemma. To this end, let \( (U_{ij}) \) be coordinate projections on \( \ell_{p_i} \), for every \( 1 \leq i \leq n \), such that \( w_j = U_{ij} \otimes \ldots \otimes U_{nj} w_j \), and for each \( i \), \( U_{ik} U_{im} = 0 \) unless \( k = m \). Letting \( p_i' = p_i / (p_i - 1) \), we see that

\[
| \sum_j w_j | = \sup_{\xi \in \ell_{p_i'}, | \xi | _s \leq 1} \left| \sum_j w_j, \xi \right|.
\]
Choose \( \otimes, \xi \) with \( |\xi| \leq 1 \), and let \( \xi_{ij} = U_{ij} \xi_i \). Then \( \sum_j |\xi_{ij}|^{p_j'} \leq 1 \), and
\[
|\sum_j w_j, \otimes \xi_{ij} \rangle | \leq \sum_j |w_j, \otimes \xi_{ij} \rangle | = \sum_j |w_j, \otimes \xi_{ij} \rangle | \leq \sum_j w_j, \Pi_{i=1}^n |\xi_{ij}|.
\]

Now let \( 1/r = \sum 1/p_i = n - \sum 1/p_i \). By Hölder’s Inequality,
\[
(\sum_j (\prod_{i=1}^n |\xi_{ij}|^r)^{1/r} \leq \left( \sum_j (\prod_{i=1}^n |\xi_{ij}|)^{p_i'} \right)^{1/p_i'} \leq 1.
\]

If \( \sum 1/p_i \leq n - 1 \), then \( r \leq 1 \), hence \( \sum \Pi_{i=1}^n |\xi_{ij}| \leq 1 \). Therefore, \( |w_j| \leq \max_j |w_j| = (|w_j|)_{c_0} \). Otherwise, \( r > 1 \), and
\[
|\sum_j w_j| \leq (\sum_j |w_j|)^{1/s} (\sum_j (\Pi_{i=1}^n |\xi_{ij}|)^r)^{1/r} \leq (\sum_j |w_j|)^{1/s} = (|w_j|)_s,
\]
where \( 1/s = 1 - 1/r = \sum 1/p_i - n + 1 \).

In a similar fashion, we show that \( |\sum_j w_j| \geq (|w_j|)_s \). For \( s = \infty \), the inequality \( |\sum_j w_j| \geq \max_j |w_j| \) is trivial. If \( s \) is finite, assume \( \sum_j |w_j|^s = 1 \) (we are allowed to do so by scaling). Find norm one vectors \( \xi_{ij} \in \ell_{p_i} \), so that \( \xi_{ij} = U_{ij} \xi_i \), and \( |w_j| = \langle w_j, \otimes \xi_{ij} \rangle \). Let \( \gamma_j = |w_j|^{n/r} \). Then \( \sum_j \gamma_j = 1 = \sum_j \gamma_j |w_j| \). Further, set \( \alpha_{ij} = \gamma_j^{\Pi_{m=1}^n (1/p_i)} \). An elementary calculation shows that \( \gamma_j = \Pi_{i=1}^n \alpha_{ij} \), and \( \sum_j \alpha_{ij}^{p_j'} = 1 \). Let \( \xi_i = \sum_j \alpha_{ij} \xi_{ij} \). Then \( |\xi_i|_{p_i'} = 1 \), and therefore,
\[
|\sum_j w_j| \geq \langle \sum_j w_j, \otimes \xi_i \rangle = \sum_j \Pi_{i=1}^n \alpha_{ij} \langle w_j, \otimes \xi_{ij} \rangle = \sum_j \gamma_j |w_j| = 1.
\]

This establishes the desired lower estimate. \( \square \)

**Lemma 3.7.** For \( 1 \leq p_i \leq \infty \), \( X = \ell_{p_1} \otimes \ell_{p_2} \otimes \ldots \otimes \ell_{p_n} \) satisfies the \( \ell_r \)-estimate, where \( 1/r = \sum 1/p_i \) if \( \sum 1/p_i < 1 \), and \( r = 1 \) otherwise. Here, we interpret \( \ell_X \) as \( c_0 \).

**Proof.** The spaces involved all have the Contractive Projection Property (the identity can be approximated by contractive finite rank projections). Thus, the duality between injective and projective tensor products of finite dimensional spaces (see e.g. [9, Section 1.2.1]) shows that, for \( w \in X \),
\[
|w| = \sup \{|\langle x, w \rangle| : x \in \ell_{p_1} \otimes \ldots \otimes \ell_{p_n}, |x| \leq 1 \}
\]
(here, as before, \( 1/p_i' + 1/p_i = 1 \)). Abusing the notation somewhat, we denote by \( P_{m} \) the projection on the span of the first \( m \) basis vectors of both \( \ell_{p_i} \) and \( \ell_{p_i'} \).

Suppose a finite sequence \( (w_k)_{k=1}^N \in X \) is block-diagonal, or more precisely, \( w_k = ((P_{m_k} - P_{m_k-1}) \otimes \ldots \otimes (P_{n_k} - P_{n_k-1})) w_k \) for every \( k \). Define the operator \( U \) on \( X \) by setting \( Ux = \sum_{k=1}^N ((P_{m_k} - P_{m_k-1}) \otimes \ldots \otimes (P_{n_k} - P_{n_k-1})) x \).
We also use $U_0$ to denote the similarly defined operator on $X^\ast$. By “Tong’s trick” (see e.g. [27, p. 20]), since $X$ and $X^\ast$ has an unconditional basis, $U_0(U_0)$ is a contractive projection onto its range $W(U_0)$. Then

$$|\sum_k w_k| = \sup \{|\sum_k w_k, x\}|: |x|_{X^\ast} \leq 1 = \sup \{|U(\sum_k w_k), x\}|: |x|_{X^\ast} \leq 1 = \sup \{|(\sum_k w_k, U_0 x)|: |x|_{X^\ast} \leq 1\}.$$ 

Write $U_0 x = \sum_{k=1}^N x_k$. By Lemma 3.6 there is an $s$ (either $1/s = \sum 1/p_k - (n - 1) = 1 - \sum 1/p_k$ or $s = \infty$) $|(x_k)|_s = |U_0 x| \leq |x| \leq 1$. Moreover,

$$\langle \sum_k w_k, U_0 x \rangle = \langle \sum_k w_k, \sum_k x_k \rangle = \sum_k \langle w_k, x_k \rangle,$$

and therefore,

$$|\sum_k w_k| = \sup \{\sum_k |w_k, x_k|: |(x_k)|_s \leq 1\} = |(|w_k|)_r.$$ 

Proof of Corollary 3.4. Combine Theorem 3.1 with Lemma 3.6 and 3.7.

Proof of Corollary 3.3. To apply Theorem 3.2, we have to show that $K(X_1, X_2)$ satisfies the $c_0$-estimate. By renorming, we can assume that the FDD constants of $X_1$ and $X_2$ equal 1. Suppose $(w_k)_{k=1}^N$ is a block-diagonal sequence, with $w_k = (P_{2,n_k} - P_{2,n_k-1})w_k(P_{1,n_k} - P_{1,n_k-1})$. Let $w = \sum_k w_k$. Then $|w| \geq |(P_{2,n_k} - P_{2,n_k-1})w(P_{1,n_k} - P_{1,n_k-1})| = |w_k|$, hence $|w| \geq \max_k |w_k|$. To prove the reverse inequality (with some constant), pick a norm one $\xi \in X_1$, and let $\xi_k = (P_{1,n_k} - P_{1,n_k-1})x$. Then $\eta_k = w\xi_k$ satisfies $(P_{2,n_k} - P_{2,n_k-1})\eta_k = \eta_k$. Set $\eta = w\xi = \sum_k \eta_k$. Denote by $C_1$ $(C_2)$ lower (upper) $p$-estimate constants of $X_1$ (resp. $X_2$). Then

$$|w\xi|^p = |\eta|^p \leq C_1 \sum_k |\eta_k|^p \leq C_2 \sum_k |w_k|^p |\xi_k|^p \leq \max_k |w_k|^p C_2 \sum_k |\xi_k|^p \leq \max_k |w_k|^p C_2 C_1 \sum_k |\xi_k|^p = C_2 C_1 |\xi|^p.$$ 

Taking the supremum over all $\xi \in UB(X_1)$, $|w| \leq (C_1 C_2)^{1/p} \max_k |w_k|$. 

We do not know whether every space with a symmetric basis must contain a subprojective subspace. However, we have the following example:

**Proposition 3.8.** There exists a non-subprojective Banach space with a symmetric basis.
Proof. Fix $1 < p < 2$. The Haar system forms an unconditional basis in $L_p(0, 1)$, hence, by [26, Theorem 3.b.1], $L_p(0, 1)$ embeds (complementably) into a space $E$ with a symmetric basis. The space $L_p(0, 1)$ is not subprojective (see Section 1), hence, by Proposition 2.1, neither is $E$. 

Furthermore, a tensor product of subprojective spaces (in fact, of Hilbert spaces) need not be subprojective.

**Proposition 3.9.** There exists a tensor norm $\otimes$, so that $\ell_2 \otimes \ell_2$ is not subprojective.

Proof. By Proposition 3.8, there exists a non-subprojective space $E$ with 1-symmetric basis. For Banach spaces $X$ and $Y$, and $a \in X \otimes Y$, we set $|a|_\otimes = \sup\{|(u \otimes v)(a)|_{E[H,K]}\}$, where the supremum is taken over all contractions $u : X \to H$ and $v : Y \to K$ ($H$ and $K$ are Hilbert spaces, and $E(H,K)$ is the Schatten space; we identify $H$ with its dual). Clearly $\otimes$ is a norm on $X \otimes Y$. It is easy to see that, for any $a \in X \otimes Y$, $T_X \in B(X, X_0)$, and $T_Y \in B(Y, Y_0)$, $|(T_X \otimes T_Y)(a)|_\otimes \leq |T_X| |T_Y| |a|_\otimes$. Thus, to prove that $\otimes$ determines a tensor norm, it suffices to establish that, for any $z \in X \otimes Y$, we have

$$|z|_\otimes \leq |z|_\otimes \leq |z|_\otimes. \tag{3.3}$$

To establish the left hand side of (3.3), let $H$ be a 1-dimensional Hilbert space (the field of scalars). For any $f \in UB(X^*)$ and $g \in UB(Y^*)$, consider the contractions $u_f : X \to H : x \mapsto \langle f, x \rangle$ and $v_g : Y \to H : y \mapsto \langle g, y \rangle$. Then, for $z = \sum_i x_i \otimes y_i \in X \otimes Y$, we have

$$|(u \otimes v)(z)|_{E[H]} = \sup_{f \in UB(X^*)} \sup_{g \in UB(Y^*)} |\langle f, x \rangle \langle g, y \rangle|_{E[H]} = |z|_\otimes.$$ 

To prove the right hand side of (3.3), note that, for any two contractions $u : X \to H$ and $v : Y \to K$,

$$|(u \otimes v)(x \otimes y)|_E = |ux \otimes vy|_E = |ux||vy| \leq |x||y|.$$ 

The triangle inequality establishes $|z|_\otimes \leq |z|_\otimes$ for any $z$.

If $X$ and $Y$ are Hilbert spaces, then for $a \in X \otimes Y$ we have $|a|_\otimes = |a|_{E(X^*, Y)}$. Identifying $\ell_2$ with its dual, we see that $E$ embeds into $\ell_2 \otimes \ell_2$ as the space of diagonal operators. As $E$ is not subprojective, neither is $\ell_2 \otimes \ell_2$. \hfill \blacksquare

Here is another wide class of non-subprojective spaces.

**Theorem 3.10.** Let $X$ be an infinite dimensional Banach space. Then $B(X)$ is not subprojective.
Proof. Suppose, for the sake of contradiction, that $B(X)$ is subprojective. Fix a norm one element $x^* \in X^*$. For $x \in X$ define $T_x \in B(X) : y \mapsto \langle x^*, y \rangle x$. Clearly $M = \{ T_x : x \in X \}$ is a closed subspace of $B(X)$, isomorphic to $X$. Therefore, $X$ is subprojective. By Proposition 2.10, we can find a subspace $N \subset M$ with an unconditional basis. We shall deduce that $B(X)$ contains a copy of $\ell_1$, which is not subprojective.

If $N$ is not reflexive, then $N$ contains either a copy of $c_0$ or a copy of $\ell_1$, see [27, Proposition 1.e.13]. By [27, Proposition 2.a.2], any subspace of $\ell_p$ ($c_0$) contains a further subspace isomorphic to $\ell_p$ (resp. $c_0$) and complemented in $\ell_p$ (resp. $c_0$), hence we can pass from $N$ to a further subspace $W$, isomorphic to $\ell_1$ or $c_0$, and complemented in $X$ by a projection $P$. Embed $B(W)$ isomorphically into $B(X)$ by sending $T \in B(W)$ to $PTP \in B(X)$, where $P$ is a projection from $X$ onto $W$. It is easy to see that $B(W)$ contain subspaces isomorphic to $\ell_1$, thus, $B(X)$ is not subprojective.

There is only one option left: $N$ is reflexive. Pick a subspace $W \subset N$, complemented in $X$. It has the Bounded Approximation Property [27, Theorem 1.e.13]. As in the previous paragraph, $B(W)$ embeds isomorphically into $B(X)$. Since $B(W) \not= K(W)$, [12, Theorem 4(1)] shows that $B(W)$ contains an isomorphic copy of $\ell_1$. This rules out the subprojectivity of $B(X)$. ■

Question 3.11. Suppose $X$ is a subprojective Banach space. (i) Is $\text{Rad}(X)$ subprojective? (ii) If $2 \leq p < \infty$, must $L_p(X)$ be subprojective?

Question 3.12. Is a “classical” (injective, projective, etc.) tensor product of subprojective spaces necessarily subprojective? Note that the Fremlin tensor product $\otimes_{\ell_1}$ of Banach lattices (the ordered analogue of the projective product) can destroy subprojectivity. Indeed, by [6], $L_2 \otimes_{\ell_1} L_2$ contains a copy of $L_1$. $L_2$ is clearly subprojective, while $L_1$ is not (see Section 1).

4. Spaces of continuous functions

In this section we deal with $C(K)$ spaces, mostly separable ones. It is well known that, if $K$ is a compact Hausdorff set, then $C(K)$ is separable if and only if $K$ is metrizable.

Recall that a topological space is scattered (or dispersed) if every compact subset has an isolated point. It is known that a compact set is scattered and metrizable if and only if it is countable (in this case, $C(K)$, and its dual, are separable). For more information on scattered spaces, see e.g. [11, Section 12], [26, Chapter 2], [35], or [39, Sections 8.5-6].

To examine subprojectivity, recall some relevant facts from [35]. If a compact Hausdorff space $K$ is scattered, then $C(K)$ is $c_0$-saturated. If, in addition, $K$ is metrizable, then $C(K)$ is subprojective (by Sobczyk’s Theorem, the copies of $c_0$ are complemented). If, on the other hand, a compact Hausdorff space $K$ is not scattered, then $C(K)$ contains a copy of $C[0,1]$, hence it cannot be complemented. Furthermore, if $K$ is scattered, then $C(K)^*$ is isometric to $\ell_1(|K|)$, while, if $K$ is not scattered, then $C(K)^*$ contains a copy of $L_1(0,1)$. Thus, $C(K)^*$ is subprojective if and only if $K$ is scattered.
4.1. Tensor products of $C(K)$

In this subsection we study the subprojectivity of projective and injective tensor products of $C(K)$. Our main result is:

**Theorem 4.1.** Suppose $K$ is a compact metrizable space, and $X$ is a Banach space. Then the following are equivalent:

1. $K$ is scattered, and $X$ is subprojective.
2. $C(K, X)$ is subprojective.

**Proof.** The implication $(2) \Rightarrow (1)$ is easy. The space $C(K, X)$ contains copies of $C(K)$ and of $X$, hence, by Proposition 2.1, the last two spaces are subprojective.

By the preceding paragraph, $K$ must be scattered.

To prove $(1) \Rightarrow (2)$, first fix some notation. Suppose $\lambda$ is a countable ordinal.

We consider the interval $[0,\lambda]$ with the order topology – that is, the topology generated by the open intervals $(\alpha, \beta)$, as well as $[0,\beta]$ and $(\alpha,\lambda]$. Abusing the notation slightly, we write $C(\lambda, X)$ for $C([0,\lambda], X)$.

Suppose $K$ is scattered. By [39, Chapter 8], $K$ is isomorphic to $[0,\lambda]$, for some countable limit ordinal $\lambda$. Fix a subprojective space $X$. We use induction on $\lambda$ to show that, for any countable ordinal $\lambda$,

$$C(\lambda, X) \text{ is subprojective.} \quad (4.1)$$

By Proposition 2.2, $(4.1)$ holds for $\lambda \leq \omega$ (indeed, $c$ is isomorphic to $c_0$, hence $c(X) = c\otimes X$ is isomorphic to $c_0(X) = c_0\otimes X$). Let $F$ denote the set of all countable ordinals for which $(4.1)$ fails. If $F$ is non-empty, then it contains a minimal element, which we denote by $\mu$. Note that $\mu$ is a limit ordinal. Indeed, otherwise it has an immediate predecessor $\mu - 1$. It is easy to see that $C(\mu, X)$ is isomorphic to $C(\mu - 1, X) \oplus X$, hence, by Proposition 2.2, $C(\mu - 1, X)$ is not subprojective. Let $C_0(\mu, X) = \{ f \in C(\mu, X) : \lim_{\nu \to \mu} f(\nu) = 0 \}$. Clearly $C(\mu, X)$ is isomorphic to $C_0(\mu, X) \oplus X$, hence we obtain the desired contradiction by showing that $C_0(\mu, X)$ is subprojective.

To do this, suppose $Y$ is a subspace of $C_0(\mu, X)$, so that no subspace of $Y$ is complemented in $C_0(\mu, X)$. For $\nu < \mu$, define the projection $P_\nu : C(\mu, X) \to C(\nu, X) : f \mapsto f1_{[0,\nu]}$. If, for some $\nu < \mu$ and some subspace $Z \subset Y$, $P_\nu|_Z$ is an isomorphism, then $Z$ contains a subspace complemented in $X$, by the induction hypothesis and Corollary 2.4. Now suppose $P_\nu|_Y$ is strictly singular for any $\nu$.

We construct a sequence of “almost disjoint” elements of $Y$. To do this, take an arbitrary $y_1$ from the unit sphere of $Y$. Pick $\nu_1 < \mu$ so that $\|y_1 - P_{\nu_1}y_1\| < 10^{-1}$. Now find a norm one $y_2 \in Y$ so that $\|P_{\nu_2}y_2\| < 10^{-2}/2$. Proceeding further in the same manner, we find a sequence of ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \ldots$, and a sequence of norm one elements $y_1, y_2, \ldots \in Y$, so that $\|y_k - z_k\| < 10^{-k}$, where $z_k = (P_{\nu_k} - P_{\nu_{k-1}})y_k$. The sequence $(z_k)$ is equivalent to the $c_0$ basis, and the same is true for the sequence $(y_k)$.

Moreover, span$[z_k : k \in \mathbb{N}]$ is complemented in $C(\mu, X)$. Indeed, let $\nu = \sup_k \nu_k$. We claim that $\mu = \nu$. If $\nu < \mu$, then $P_\nu$ is an isomorphism on
complemented in $z_k$ of the elements. Note that $\lim\lambda$ is a limit ordinal. If $\ell$ is finite, then $\ell$ is clearly subprojective. We handle the infinite case by transfinite induction on $\lambda$. The base is easy: if $\lambda = 0$, then $C(\lambda) = 0$, and $C(\lambda)$ is isomorphic to $c_0 \otimes \ell_p$. The latter space is subprojective, by Corollary 3.3.

Suppose, for the sake of contradiction, that $\lambda$ is the smallest countable ordinal so that $C(\lambda) \otimes \ell_p$ is not subprojective. Reasoning as before, we conclude that $\lambda$ is a limit ordinal. Furthermore, $C(\lambda) \sim C_0(\lambda)$, hence $C_0(\lambda) \otimes \ell_p$ is not subprojective.

Denote by $Q_n : \ell_p \to \ell_p$ the projection on the first $n$ basis vectors in $\ell_p$, and let $Q_n = I - Q_n$. For $f \in C_0(\lambda)$ and an ordinal $\nu < \lambda$, define $P_\nu f = \chi_{[0,\nu]} f$, and $P_\nu^* = I - P_\nu$. Suppose $X$ is a subspace of $C_0(\lambda) \otimes \ell_p$, which has no subspaces complemented in $C_0(\lambda) \otimes \ell_p$. By the induction hypothesis, $(P_\nu \otimes I_p)$ is strictly singular for any $\nu < \lambda$. Furthermore, $(I_{C_0(\lambda)} \otimes Q_n)|Y$ must be strictly singular. Indeed, otherwise $Y$ has a subspace $Z$ so that $(I_{C_0(\lambda)} \otimes Q_n)|Z$ is an isomorphism, whose range is subprojective (the range of $I_{C_0(\lambda)} \otimes Q_n$ is isomorphic to the sum of $n$ copies of $C(\lambda)$, hence subprojective). Therefore, for any $\nu < \lambda$ and $n \in \mathbb{N}$, $(I - P_\nu^* \otimes Q_n)|Y$ is strictly singular. Therefore we can find a normalized basis $(x_i)$ in $Y$, and sequences $0 = \nu_0 < \nu_1 < \ldots < \lambda$, and $0 = n_0 < n_1 < \ldots$, so that $|x_i - (P_{\nu_0}^* \otimes Q_{n_0}) x_i| < 10^{-3}\cdot 2$. By passing to a further subsequence, we can assume that $|(P_\nu \otimes Q_n) x_i| < 10^{-3}\cdot 2$. Thus, by the Small Perturbation Principle, it suffices to show the following statement: If $(y_i)$ is a normalized sequence is $C_0(\lambda) \otimes \ell_p$, so that there exist non-negative integers $0 = n_0 < n_1 < n_2 < \ldots$, and ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \ldots < \lambda$, with the property that $y_i = ((P_{\nu_0} - P_{\nu_1}) \otimes (Q_{n_0} - Q_{n_1})) y_i$ for any $i$, then $Y = \text{span}\{y_i : i \in \mathbb{N}\}$ is contractively complemented in $C(\lambda) \otimes \ell_p$. 

\[ Q : C_0(\mu, X) \to C_0(\mu, X) : f \mapsto \sum_k w_k ((P_{\nu_k} - P_{\nu_{k-1}}) f) z_k. \]

Note that $\lim_k |(P_{\nu_k} - P_{\nu_{k-1}}) f| = 0$, hence the range of $Q$ is precisely the span of the elements $z_k$. By Small Perturbation Principle, $Y$ contains a subspace complemented in $C_0(\mu, X)$.  

The above theorem shows that $C(K) \otimes X$ is subprojective if and only if both $C(K)$ and $X$ are. We do no know whether a similar result holds for other tensor products. We do, however, have:

**Proposition 4.2.** Suppose $K$ is a compact metrizable space, and $W$ is either $\ell_p$ ($1 \leq p < \infty$) or $c_0$. Then $C(K) \otimes W$ is subprojective if and only if $K$ is scattered.

**Proof.** Clearly, if $K$ is not scattered, then $C(K)$ is not subprojective. So suppose $K$ is scattered. We deal with the case of $W = \ell_p$, as the $c_0$ case is handled similarly. As before, we can assume that $K = [0, \lambda]$, where $\lambda$ is a countable ordinal. If $\lambda$ is finite, then $C(\lambda) \otimes \ell_p = \ell_2^\lambda \otimes \ell_p$ is clearly subprojective. We handle the infinite case by transfinite induction on $\lambda$. The base is easy: if $\lambda = \omega$, then $C(\lambda) = c$, and $C(\lambda)$ is isomorphic to $c_0 \otimes \ell_p$. The latter space is subprojective, by Corollary 3.3.

Suppose, for the sake of contradiction, that $\lambda$ is the smallest countable ordinal so that $C(\lambda) \otimes \ell_p$ is not subprojective. Reasoning as before, we conclude that $\lambda$ is a limit ordinal. Furthermore, $C(\lambda) \sim C_0(\lambda)$, hence $C_0(\lambda) \otimes \ell_p$ is not subprojective.

Denote by $Q_n : \ell_p \to \ell_p$ the projection on the first $n$ basis vectors in $\ell_p$, and let $Q_n = I - Q_n$. For $f \in C_0(\lambda)$ and an ordinal $\nu < \lambda$, define $P_\nu f = \chi_{[0,\nu]} f$, and $P_\nu^* = I - P_\nu$. Suppose $X$ is a subspace of $C_0(\lambda) \otimes \ell_p$, which has no subspaces complemented in $C_0(\lambda) \otimes \ell_p$. By the induction hypothesis, $(P_\nu \otimes I_p)$ is strictly singular for any $\nu < \lambda$. Furthermore, $(I_{C_0(\lambda)} \otimes Q_n)|Y$ must be strictly singular. Indeed, otherwise $Y$ has a subspace $Z$ so that $(I_{C_0(\lambda)} \otimes Q_n)|Z$ is an isomorphism, whose range is subprojective (the range of $I_{C_0(\lambda)} \otimes Q_n$ is isomorphic to the sum of $n$ copies of $C(\lambda)$, hence subprojective). Therefore, for any $\nu < \lambda$ and $n \in \mathbb{N}$, $(I - P_\nu^* \otimes Q_n)|Y$ is strictly singular. Therefore we can find a normalized basis $(x_i)$ in $Y$, and sequences $0 = \nu_0 < \nu_1 < \ldots < \lambda$, and $0 = n_0 < n_1 < \ldots$, so that $|x_i - (P_{\nu_0}^* \otimes Q_{n_0}) x_i| < 10^{-3}\cdot 2$. By passing to a further subsequence, we can assume that $|(P_\nu \otimes Q_n) x_i| < 10^{-3}\cdot 2$. Thus, by the Small Perturbation Principle, it suffices to show the following statement: If $(y_i)$ is a normalized sequence is $C_0(\lambda) \otimes \ell_p$, so that there exist non-negative integers $0 = n_0 < n_1 < n_2 < \ldots$, and ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \ldots < \lambda$, with the property that $y_i = ((P_{\nu_0} - P_{\nu_1}) \otimes (Q_{n_0} - Q_{n_1})) y_i$ for any $i$, then $Y = \text{span}\{y_i : i \in \mathbb{N}\}$ is contractively complemented in $C(K) \otimes \ell_p$. 

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Suppose first there exists a finite rank operator \( K \) and \( \Pi \). After this paper has been completed, we were informed by \( \Pi \) that, in the paper [15], currently in preparation, proves that, for compact Hausdorff spaces \( K_1 \) and \( K_2 \), the following are equivalent: (i) \( K_1 \) and \( K_2 \) are scattered; (ii) \( C(K_1) \hat{\otimes} C(K_2) \) is subprojective.

### 4.2. Operators on \( C(K) \)

**Proposition 4.4.** Suppose \( K \) is a scattered compact metrizable space, and \( 1 \leq p \leq q < \infty \). Then the space \( \Pi_{pq}(C(K), \ell_q) \) is subprojective.

Recall that \( \Pi_{pq}(X, Y) \) stands for the space of \((q, p)\)-summing operators – that is, the operators for which there exists a constant \( C \) so that, for any \( x_1, \ldots, x_n \in X \),

\[
\left( \sum_i |Tx_i|^q \right)^{1/q} \leq C \sup_{x^* \in UB(X^*)} \left( \sum_i |x^*(x_i)|^p \right)^{1/p}.
\]

The smallest value of \( C \) is denoted by \( \pi_{pq}(T) \).

Note that, if a compact Hausdorff space \( K \) is not scattered, then \( C(K)^* \) contains \( L_1 \) [35], hence \( \Pi_{pq}(C(K), \ell_q) \) is not subprojective.

The following lemma may be interesting in its own right.

**Lemma 4.5.** Suppose \( X \) is a Banach space, \( K \) is a compact metrizable scattered space, and \( 1 \leq p \leq q < \infty \). Then, for any \( T \in \Pi_{pq}(C(K), X) \), and any \( \varepsilon > 0 \), there exists a finite rank operator \( S \in \Pi_{pq}(C(K), X) \) with \( \pi_{pq}(T - S) < \varepsilon \).

In proving Proposition 4.4 and Lemma 4.5, we consider the cases of \( p = q \) and \( p < q \) separately. If \( p = q \), we are dealing with \( q \)-summing operators. By
Pietsch Factorization Theorem, $T \in B(C(K), X)$ is $q$-summing if and only if there exists a probability measure $\mu$ on $K$ so that $T$ factors as $T \circ j$, where $j : C(K) \to L_q(\mu)$ is the formal identity, and $|T| \leq \pi_q(T)$. Moreover, $\mu$ and $T$ can be selected in such a way that $|\tilde{T}| = \pi_q(T)$. As $K$ is scattered, there exist distinct points $k_1, k_2, \ldots \in K$, and non-negative scalars $\alpha_1, \alpha_2, \ldots$, so that $\sum_i \alpha_i = 1$, and $\mu = \sum_i \alpha_i \delta_{k_i}$ [35].

Now suppose $T \in B(C(K), X)$ satisfies $\pi_q(T) = 1$. Keeping the above notation, find $N \in \mathbb{N}$ so that $(\sum_{i=N+1}^\infty \alpha_i)^{1/q} < \varepsilon$. Denote by $u$ and $v$ the operators of multiplication by $\chi_{(k_1, \ldots, k_N)}$ and $\chi_{(k_{N+1}, k_{N+2}, \ldots)}$, respectively, acting on $L_q(\mu)$. It is easy to see that $\text{rank } u \leq N$, and $|vj| < \varepsilon$. Then $S = \tilde{T}uvj$ works in Lemma 4.5.

If $1 \leq p < q$, then (see e.g. [8, Chapter 10] or [40, Chapter 21]), $\Pi_{qp}(C(K), X) = \Pi_q(C(K), X)$, with equivalent norms. Henceforth, we set $p = 1$. We have a probability measure $\mu$ on $K$, and a factorization $T = \tilde{T}j$, where $j : C(K) \to L_1(\mu)$ is the formal identity, and $\tilde{T} : L_1(\mu) \to X$ satisfies $|\tilde{T}| \leq \pi_1(T)$ ($c$ is a constant depending on $q$).

In this case, the proof of Lemma 4.5 proceeds as for $q$-summing operators, except that now, we need to select $N$ so that $c\left(\sum_{i=N+1}^\infty \alpha_i\right)^{1/q} < \varepsilon$.

Proof of Proposition 4.4. Define the tensor norm $\otimes$ by setting, for $z = \sum_i x_i \otimes y_i \in X \otimes Y$, $|z|_\otimes = \pi_{qp}(\pi)$, where $\pi \in B(X^*, Y)$ is defined by $\pi f = \sum_i \langle f, x_i \rangle y_i$. Recall that, for any $T$, $\pi_{qp}(T) = \pi_{qp}(T^{**})$. Furthermore, if $T$ is weakly compact, then $T^{**}(X^{**}) \subset Y$, hence, by the injectivity of the ideal $\pi_{qp}$, $\pi_{qp}(T) = \pi_{qp}(T[1])$, where $T[1]$ is the astriction of $T^{**}$ to an operator from $X^{**}$ to $Y$. If $T \in B(X, Y)$ has finite rank, write $T = \sum_i x_i \otimes y_i$ ($x_i \in X^*$, and $y_i \in Y$), and note that, by the above, $\pi_{qp}(T) = |T|_\otimes^q$.

By Lemma 4.5, we can identify $\Pi_{qp}(C(K), X)$ with $C(K)^* \otimes X$. By [35], $C(K)^* = \ell_1$ (the canonical basis in $\ell_1$ corresponds to the point evaluation functionals), hence we can describe $\otimes$ in more detail: for $u = \sum_i a_i \otimes x_i \in \ell_1 \otimes X$, $|u|_\otimes = \pi_{qp}(\pi)$, where $\pi : \ell_1 \to X$ is defined by $\pi b = \sum_i \langle b, a_i \rangle x_i$. Note that $\kappa_X \circ \pi = \pi^{**}$ ($\kappa_X : X \to X^{**}$ is the canonical embedding), where $\tilde{u} : c_0 \to X$ defined via $\tilde{u}b = \sum_i \langle b, a_i \rangle x_i$. Thus, $|u|_\otimes = \pi_{qp}(\tilde{u})$.

To finish the proof, we need to show (in light of Theorem 3.1) that $\ell_1 \otimes \ell_q$ satisfies the $\ell_q$ estimate. To this end, suppose we have a block-diagonal sequence $(u_i)_{i=1}^N$, and show that $\|\sum_i u_i\|_\otimes^q \sim \sum_i |u_i|_\otimes^q$. Abusing the notation slightly, we identify $u_i$ with an operator from $\ell_q^N$ to $\ell_q^N$ (where $N$ is large enough), and identify $| \cdot |_\otimes$ with $| \cdot |_{\Pi_{qp}}$.

First show that $\|\sum_i u_i\|_\otimes^q \leq c\sum_i |u_i|_\otimes^q$, where $c$ is a constant (depending on $q$). We have disjoint sets $(S_i)_{i=1}^N$ in $\{1, \ldots, N\}$ so that $u_i c_j = 0$ for $j \notin S_i$. Therefore there exists a probability measure $\mu_i$, supported on $S_i$, so that

$$|u_i f|_q \leq c |\pi_{qp}(u_i)|_q f_{\ell_q^q} f_{\Pi_{qp}(\mu_i)}$$

for any $f \in \ell_q^N$ ($c_1$ is a constant). Now define the probability measure $\mu$ on...
\{1, \ldots, N\}:
\[ \mu = (\sum_i \pi_{qp}(u_i)^q)^{-1} \sum_i \pi_{qp}(u_i)^q \mu_i. \]

For \( f \in L^N_\infty \), set \( f_i = f \chi_{S_i} \). Then the vectors \( u_i f_i \) are disjointly supported in \( L^q \), and therefore,
\[
\| (\sum_i u_i f)^q \| = \sum_i \| u_i f_i \|^q \leq c_1 q \sum_i \pi_{qp}(u_i)^q \| f_i \|_p^q \| f_i \|_{L_p(\mu_i)}^p \leq c_1 q \| f \|_p^q \sum_i \pi_{qp}(u_i)^q \| f_i \|_{L_p(\mu_i)}^p.
\]

An easy calculation shows that
\[
|f_i|_{L_p(\mu_i)}^p = (\sum_i \pi_{qp}(u_i)^q)^{-1} \sum_i \pi_{qp}(u_i)^q |f_i|_{L_p(\mu_i)}^p,
\]

hence
\[
| (\sum_i u_i f)^q | \leq c_1 q \left( \sum_i \pi_{qp}(u_i)^q \right)^{1/q} \sum_i |f_i|_{L_p(\mu_i)}^p
= c_1 q \left( \sum_i \pi_{qp}(u_i)^q \right)^{1/q} |f|_{L_p(\mu_i)}^p.
\]

Therefore, \( \pi_{qp}(\sum_i u_i) \leq c \left( \sum_i \pi_{qp}(u_i)^q \right)^{1/q} \), for some universal constant \( c \).

Next show that \( |\sum_i u_i|_{\infty}^q \geq c' \sum_i |u_i|_{\infty}^q \), where \( c' \) is a constant. There exists a probability measure \( \mu \) on \{1, \ldots, N\} so that, for any \( f \in L^N_\infty \),
\[
| (\sum_i u_i f)^q | \geq c_2 q \pi_{qp}(\sum_i u_i)^q |f|_{L_p(\mu_i)}^p
\]

For each \( i \) let \( \alpha_i = |\mu|_{S_i} |\infty| \), and \( \mu_i = \mu_i / \alpha_i \) (if \( \alpha_i = 0 \), then clearly \( u_i = 0 \)). Then for any \( i \), and any \( f \in L^N_\infty \),
\[
|u_i f|^q = |(\sum_i u_i)(\chi_{S_i} f)^q| \leq c_2 q \pi_{qp}(\sum_i u_i)^q \alpha_i |f|_{L_p(\mu_i)}^p,
\]

hence \( \pi_{qp}(u_i) \leq c' \alpha_i^{1/q} \pi_{qp}(\sum_i u_i) \) (\( c' \) is a constant). As \( \sum_i \alpha_i = 1 \), we conclude that \( \sum_i \pi_{qp}(u_i)^q \leq c'' q \pi_{qp}(\sum_i u_i) \).

\[ \square \]

4.3. Continuous fields

We refer the reader to [10, Chapter 10] for an introduction into continuous fields of Banach spaces. To set the stage, suppose \( K \) is a locally compact Hausdorff space (the base space), and \( (X_t)_{t \in K} \) is a family of Banach spaces (the spaces \( X_t \) are called (fibers). A vector field is an element of \( \prod_{k \in K} X_t \). A linear subspace \( X \) of \( \prod_{k \in K} X_t \) is called a continuous field if the following conditions hold:

1. For any \( t \in K \), the set \( \{x(t) : x \in X\} \) is dense in \( X_t \).
2. For any \( x \in X \), the map \( t \mapsto |x(t)| \) is continuous, and vanishes at infinity.

3. Suppose \( x \) is a vector field so that, for any \( \varepsilon > 0 \) and any \( t \in K \), there exist an open neighborhood \( U \ni t \) and \( y \in X \) for which \( |x(s) - y(s)| < \varepsilon \) for any \( s \in U \). Then \( x \in X \).

Equipping \( X \) with the norm \( |x| = \max_t |x(t)| \), we turn it into a Banach space.

In a fashion similar to Theorem 4.1, we prove:

**Proposition 4.6.** Suppose \( K \) is a scattered metrizable space, \( X \) is a separable continuous vector field on \( K \), so that, for every \( t \in K \), the fiber \( X_t \) is subprojective. Then \( X \) is subprojective.

**Proof.** Using one-point compactification if necessary (as in [10, 10.2.6]), we can assume that \( K \) is compact. As before, we assume that \( K = [0, \lambda] \) (\( \lambda \) is a countable ordinal). We denote by \( X_{(0)} \) the set of all \( x \in X \) which vanish at \( \lambda \).

If \( \nu \leq \lambda \), we denote by \( X_{[\nu]} \) the set of all \( x \in X_\lambda \) which vanish outside of \([0, \nu]\).

By [10, Proposition 10.1.9], \( x \in X_{[\nu]} \) for any \( x \in X \), hence \( X_{[\nu]} \) is a Banach space. We then define the restriction operator \( P_\nu : X \to X_{[\nu]} \). We denote by \( Q_\nu : X \to X_\nu \) the operator of evaluation at \( \nu \).

We say that a countable ordinal \( \lambda \) has Property \( \mathcal{P} \) if, whenever \( X \) is a continuous separable vector field whose fibers are subprojective, then \( X \) is subprojective. Using transfinite induction, we prove that any countable ordinal has this property.

The base of induction is easy to handle. Indeed, when \( \lambda \) is finite, then \( X \) embeds into a direct sum of (finitely many) subprojective spaces \( X_\nu \). Now suppose, for the sake of contradiction, that \( \lambda \) is the smallest ideal failing Property \( \mathcal{P} \). Note that \( \lambda \) is a limit ordinal. Indeed, otherwise it has an immediate predecessor \( \lambda_- \), and \( X \) embeds into a direct sum of two subprojective spaces — namely, \( X_{\lambda_-} \) and \( X_\lambda \).

Suppose \( Y \) is a subspace of \( X \), so that no subspace of \( Y \) is complemented in \( X \). We shall achieve a contradiction once we show that \( Y \) contains a copy of \( c_0 \).

By Proposition 2.3, \( Q_\lambda \) is strictly singular on \( Y \). Passing to a smaller subsequence if necessary, we can assume that \( Y \) has a basis \( \{y_i\}_{i \in \mathbb{N}} \), so that (i) for any finite sequence \( \{\alpha_i\} \), \( \sum_i \alpha_i y_i \geq \max_i |\alpha_i|/2 \), and (ii) for any \( i \), \( |Q_\lambda y_i| < 10^{-4i} \).

Consequently, for any \( y \in \text{span}\{y_j : j > i\} \), \( |Q_\lambda y| < 10^{-4i} \). Indeed, we can assume that \( y \) is a norm one vector with finite support, and write \( y = \sum_j \alpha_j y_j \). By the above, \( |\alpha_i| \leq 2 \) for every \( i \). Consequently, \( |Q_\lambda y| \leq \sum_j |\alpha_j||Q_\lambda y_j| \leq 2 \sum_{j \geq i} 10^{-4j} < 10^{-4i} \).

Now construct a sequence \( \nu_1 < \nu_2 < \ldots < \lambda \) of ordinals, a sequence \( 1 = n_1 < n_2 < \ldots \) of positive integers, and a sequence \( x_1, x_2, \ldots \) of norm one vectors, so that (i) \( x_i \in \text{span}\{y_i : n_j \leq i < n_{j+1}\} \), (ii) \( |P_\nu x_i| < 10^{-4i} \), and (iii) \( |P_{\nu+1} x_i| < 10^{-4i} \). To this end, recall that, by Proposition 2.3 again, \( P_\nu \) is strictly singular for any \( \nu < \lambda \). Pick an arbitrary \( \nu_1 < \lambda \), and find a norm 1 vector \( x_1 \in \text{span}\{y_1, \ldots, y_{n_{\nu_1+1}}\} \) so that \( |P_{\nu_1} x_1| < 10^{-4} \). We have \( |Q_\lambda x_1| < 10^{-4} \). By continuity, we can find \( \nu_2 > \nu_1 \) so that \( |P_{\nu_2} x_1| < 10^{-4} \).
Next find a norm one \( x_2 \in \text{span}[y_{n_2}, \ldots, y_{n_3-1}] \) so that \( |P_{\ell_2}x_1| < 10^{-8} \). Proceed further in the same manner.

We claim that the sequence \( (x_i) \) is equivalent to the canonical basis in \( c_0 \). Indeed, for each \( i \) let \( x'_i = P_{\nu_i}x_i + P_{\nu_{i+1}}x_i \), and \( x'_i = x_i - x''_i \). Since we are working with the sup norm, \( |x'_i| = |x_i| = 1 \) for any \( i \). Furthermore, the elements \( x'_i \) are disjointly supported, hence, for any \( (\alpha_i) \) finite sequence of scalars \( (\alpha_i) \), \( \sum_i \alpha_i x'_i = \max_i |\alpha_i| \). By the triangle inequality,

\[
\left| \sum_i \alpha_i x_i \right| - \left| \sum_i \alpha_i x'_i \right| \leq \sum_i |\alpha_i| |x'_i| < \max_i |\alpha_i| \sum_i 2 \cdot 20^{-4i} < 10^{-3} \max_i |\alpha_i|,
\]

which yields the desired result.

To state a corollary of Proposition 4.6, recall that a \( C^* \)-algebra \( A \) is CCR (or liminal) if, for any irreducible representation \( \pi \) of \( A \) on a Hilbert space \( H \), \( \pi(A) = K(H) \). A \( C^* \)-algebra \( A \) is scattered if every positive linear functional on \( A \) is a sum of pure linear functionals (\( f \in A^* \) is called pure if it belongs to an extreme ray of the positive cone of \( A^* \)). For equivalent descriptions of scattered \( C^* \)-algebras, see e.g. [19, 20, 25].

**Corollary 4.7.** Any separable scattered CCR \( C^* \)-algebra is subprojective.

**Proof.** Suppose \( A \) is a separable scattered CCR \( C^* \)-algebra. As shown in [34, Sections 6.1-3], the spectrum of a separable CCR algebra is a locally compact Hausdorff space. If, in addition, the algebra is scattered, then its spectrum \( \hat{A} \) is scattered as well [19, 20]. In fact, by the proof of [19, Theorem 3.1], \( \hat{A} \) is separable. It is easy to see that any separable locally compact Hausdorff space is metrizable.

By [10, Section 10.5], \( A \) can be represented as a vector field over \( \hat{A} \), with fibers of the form \( \pi(A) \), for irreducible representations \( \pi \). As \( A \) is CCR, the spaces \( \pi(A) = K(H_\pi) \) (\( H_\pi \) being a separable Hilbert space) are subprojective. To finish the proof, apply Proposition 4.6.

The last corollary leads us to

**Conjecture 4.8.** A separable \( C^* \)-algebra is scattered if and only if it is subprojective.

It is known ([20], see also [25]) that a scattered \( C^* \)-algebra is GCR. However, it need not be CCR (consider the unitization of \( K(\ell_2) \)).

5. **Subprojectivity of Schatten spaces**

In this section, we establish:

**Proposition 5.1.** Suppose \( E \) is a symmetric sequence space, not containing \( c_0 \). Then \( \mathcal{E}_E \) is subprojective if and only if \( E \) is subprojective.
The assumptions of this proposition are satisfied, for instance, if $\mathcal{E} = \ell_p$ ($1 \leq p < \infty$), or if $\mathcal{E}$ is the Lorentz space $l(w, p)$ (see [27, Proposition 4.e.3]). However, by Proposition 3.8, not every symmetric sequence space is subprojective.

For the proof, we need a technical result.

**Proposition 5.2.** Suppose $\mathcal{E}_E$ is a symmetric sequence space, not containing $c_0$. Suppose, furthermore, that $(z_n) \subset \mathcal{E}_E$ is a normalized sequence, so that, for every $k$, $\lim_n |Q_kz_n| = 0$. Then, for any $\varepsilon > 0$, $\mathcal{E}_E$ contains sequences $(\hat{z}_n)$ and $(\hat{z}_n')$, so that:

1. $(\hat{z}_n)$ is a subsequence of $(z_n)$.
2. $\sum_n |\hat{z}_n - \hat{z}_n'| < \varepsilon$.
3. $(\hat{z}_n')$ lies in the subspace $Z$ of $\mathcal{E}_E$, with the property that (i) $Z$ is 3-isomorphic to either $\ell_2$, $\mathcal{E}$, or $\ell_2 \oplus \mathcal{E}$, and (ii) $Z$ is the range of a projection of norm not exceeding 3.

**Proof.** [3, Corollary 2.8] implies the existence of $(\hat{z}_n)$ and $(\hat{z}_n')$, so that (1) and (2) are satisfied, and $\hat{z}_n' = a \otimes E_{1k} + b \otimes E_{k1} + c_k \otimes E_{kk}$ ($k \geq 2$). Thus, $Z' \subset Z = Z_r + Z_c + Z_d$, where $Z_r = \text{span}[a \otimes E_{1k} : k \geq 2]$ (the row component), $Z_c = \text{span}[b \otimes E_{k1} : k \geq 2]$ (the column component), and $Z_d$ (the diagonal component) contains $c_k \otimes E_{kk}$, for any $k$. More precisely, we can write $c_k = u_k d_k v_k$, where $u_k$ and $v_k$ are unitaries, and $d_k$ is diagonal. Then we set $Z_d = \text{span}[u_k E_{ii} v_k \otimes E_{kk} : i \in \mathbb{N}, k \geq 2]$.

It remains to build contractive projections $P_r$, $P_c$, and $P_d$ onto $Z_r$, $Z_c$, and $Z_d$, respectively, so that $Z_r \cup Z_d \subset \ker P_r$, $Z_r \cup Z_c \subset \ker P_c$, and $Z_r \cup Z_c \subset \ker P_d$. Indeed, then $P = P_r + P_c + P_d$ is a projection onto $Z_r + Z_c + Z_d$, and the latter space is completely isomorphic to $Z_0 = Z_r \oplus Z_c \oplus Z_d$. The spaces $Z_r$, $Z_c$, and $Z_d$ are either trivial (zero-dimensional), or isomorphic to $\ell_2$, $\ell_2$, and $\mathcal{E}$, respectively.

$P_d$ is nothing but a coordinate projection, in the appropriate basis:

$$P_d(u_k E_{ij} v_{\ell} \otimes E_{kk}) = \begin{cases} u_k E_{ii} v_k \otimes E_{kk} & k = \ell \geq 2, i = j \\ 0 & \text{otherwise} \end{cases}$$

(for the sake of convenience, we set $u_1 = v_1 = I_{\mathbb{N}}$). Next construct $P_r$ ($P_c$ is dealt with similarly). If $a = 0$, just take $P_r = 0$. Otherwise, let $a' = a/|a|$, and find $f \in \mathcal{E}^*$ so that $|f| = 1 = \langle f, a' \rangle$. For $x = \sum_{k,\ell} x_{k\ell} \otimes E_{kk}$, define

$$P_rx = a' \otimes \sum_{\ell \geq 2} \langle f, x_{1\ell} \rangle E_{1\ell},$$

hence $|P_r x|^2 = \sum_{\ell \geq 2} |\langle f, x_{1\ell} \rangle|^2$. It remains to show $|P_r x| \leq |x|$. This inequality is obvious when $P_r x = 0$. Otherwise, set, for $\ell \geq 2$,

$$\alpha_\ell = \frac{\langle f, x_{1\ell} \rangle}{\left(\sum_{\ell \geq 2} |\langle f, x_{1\ell} \rangle|^2\right)^{1/2}}.$$
\[ y = I_{\ell_2} \otimes \sum_{\ell \geq 2} \alpha_{\ell} E_{1\ell}, \quad \text{and} \quad z = I_{\ell_2} \otimes E_{11}. \] Then \[ |y|_E = (\sum_{\ell \geq 2} |\alpha_{\ell}|^2)^{1/2} = 1 = |z|_E, \] and \[ zxy = \sum_{\ell \geq 2} \alpha_{\ell} x_{1\ell} \otimes E_{11}. \] Therefore,

\[ \{ zxy \} = \{ z \} = \{ x \}, \]

which is what we need.

**Proof of Proposition 5.1.** The space \( \mathcal{C}_E \) contains an isometric copy of \( E_\ell \), hence the subprojectivity of \( \mathcal{C}_E \) implies that of \( E_\ell \). To prove the converse, suppose \( E_\ell \) is subprojective, and \( Z_0 \) is a subspace of \( \mathcal{C}_E \), and show that it contains a further subspace \( Z \), complemented in \( \mathcal{C}_E \). To this end, find a normalized sequence \( p_z \), so that \( \lim_{n} \| p_z \| = 0 \) for every \( k \). By Proposition 5.2, \( p_z \) has a subsequence \( p_z^1 \), contained in a subspace \( Z_1 \), which is complemented in \( \mathcal{C}_E \), and isomorphic either to \( E_\ell \), or \( \ell_2 \). By Proposition 2.2, \( Z_1 \) is subprojective, hence span\( \{ z_n : n \in \mathbb{N} \} \) contains a subspace complemented in \( Z_1 \), hence also in \( \mathcal{C}_E \).

As a consequence we obtain:

**Proposition 5.3.** The predual of a von Neumann algebra \( A \) is subprojective if and only if \( A \) is atomic.

We say that \( A \) is atomic if any projection in it has an atomic subprojection. By [31, Section 1], this happens if and only if \( A = (\sum_i B(H_i))_\ell \).

**Proof.** If a von Neumann algebra \( A \) is not purely atomic, then, as explained in [31, Section 1], \( A_a \) contains a (complemented) copy of \( L_1(0,1) \). This establishes the “only if” implication of Proposition 5.3. Conversely, if \( A \) is purely atomic, then \( A_a \) is isometric to a (contractively complemented) subspace of \( \mathcal{C}_1(H) \), and the latter is subprojective.

### 6. \( p \)-convex and \( p \)-disjointly homogeneous Banach lattices

We say that \( X \) is \( p \)-disjointly homogeneous (\( p \)-DH for short) if every disjoint normalized sequence contains a subsequence equivalent to the standard basis of \( \ell_p \).

For the sake of completeness we present a proof of the following statement (see [14, 4.11, 4.12]).

**Proposition 6.1.** Let \( X \) be a \( p \)-convex \( p \)-DH Banach lattice. Then every subspace, spanned by a disjoint sequence equivalent to the canonical basis of \( \ell_p \), is complemented.
Proof. Let \((x_k) \subset X\) be a disjoint normalized sequence. Since \(X\) is \(p\)-DH, by passing to a subsequence, we can assume that \((x_k)\) is an \(\ell_p\) basic sequence. Then, in the \(p\)-concavification \(X[p]\) the disjoint sequence \((x_k^p)\) is an \(\ell_1\) basic sequence. Therefore, there exists a functional \(x^* \in (x_k^p)\) such that \(x^*(x_k^p) = 1\) for all \(k\). By the Hahn-Banach Theorem \(x^*\) can be extended to a positive functional in \(X[p]\). Define a seminorm \(\|x\|_{p,p} = |x^*(|x^p|)|^{\frac{1}{p}}\) on \(X\). Denote by \(\mathcal{N}\) the subset of \(X\) on which this seminorm is equal to zero. Clearly, \(\mathcal{N}\) is an ideal, therefore, the quotient space \(\tilde{X} = X/\mathcal{N}\) is a Banach lattice, and the quotient map \(Q : X \to \tilde{X}\) is an orthomorphism. With the defined seminorm \(\tilde{X}\) is an abstract \(L_p\)-space, and the disjoint sequence \(Q(x_k)\) is normalized. Therefore it is an \(\ell_p\) basic sequence that spans a complemented subspace (in particular, \(Q\) is an isomorphism when restricted to \([x_k]\)). Let \(\tilde{P}\) be a projection from \(\tilde{X}\) onto \([Q(x_k)]\). Then \(P = Q^{-1}\tilde{P}Q\) is a projection from \(X\) onto \([x_k]\].

**Proposition 6.2.** Let \(X\) be a \(p\)-convex, \(p\)-disjointly homogeneous Banach lattice \((p \geq 2)\). Then any subspace of \(X\) contains a complemented copy of either \(\ell_p\) or \(\ell_2\). Consequently, \(X\) is subprojective.

**Proof.** First, note that \(X\) is order continuous. Let \(M \subseteq X\) be an infinite dimensional separable subspace. Then there exists a complemented order ideal in \(X\) with a weak unit that contains \(M\). Therefore, without loss of generality, we may assume that \(X\) has a weak unit. Then there exists a probability measure \(\mu\) such that we have continuous embeddings

\[
L_2(\mu) \subseteq X \subseteq L_p(\mu) \subseteq L_2(\mu) \subseteq L_1(\mu).
\]

Consequently, there exists a constant \(c_1 > 0\) so that \(c_1 |x|_p \leq |x|\) for any \(x \in X\).

By the proof of [28, Proposition 1.c.8], one of the following holds:

**Case 1.** \(M\) contains an almost disjoint bounded sequence. By Proposition 6.1 \(M\) contains a copy of \(\ell_p\) complemented in \(X\).

**Case 2.** The norms \(|\cdot|_p\) and \(|\cdot|_1\) are equivalent on \(M\). Thus, there exists \(c_2 > 0\) so that, for any \(y \in M\),

\[
c_2 |y|_2 \geq c_2 |y|_1 \geq |y| \geq c_1 |y|_p \geq c_1 |y|_2.
\]

In particular, \(M\) is embedded into \(L_2(\mu)\) as a closed subspace. The orthogonal projection from \(L_2(\mu)\) onto \(M\) then defines a bounded projection from \(X\) onto \(M\).

The preceding result implies that Lorentz space \(\Lambda_{p,W}(0,1)\) is subprojective for \(p \geq 2\). Indeed, \(\Lambda_{p,W}(0,1)\) is \(p\)-DH and \(p\)-convex \((p \geq 1)\), see [13, Theorem 3] and [24]. Note that, originally, the subprojectivity of \(\Lambda_{p,W}(0,1)\) (for \(p \geq 2\)) was observed in [13, Remark 5.7].
7. Lattice-valued $\ell_p$ spaces

If $X$ is a Banach lattice, and $1 \leq p < \infty$, denote by $\hat{X}(\ell_p)$ the completion of the space of all finite sequences $(x_1, \ldots, x_n)$ (with $x_i \in X$), equipped with the norm $\| (x_1, \ldots, x_n) \| = \left\| \sum |x_i|^p \right\|^{1/p}$, where

$$\left( \sum |x_i|^p \right)^{1/p} = \sup \left\{ \left\| \sum \alpha_i x_i \right\| : \sum |\alpha_i|^p \leq 1 \right\}, \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

See [28, pp. 46-48] for more information. We have:

**Proposition 7.1.** Suppose $X$ is a subprojective separable space, with the lattice structure given by an unconditional basis, and $1 \leq p < \infty$. Then $\hat{X}(\ell_p)$ is subprojective.

**Proof.** To show that any subspace $Y \subset \hat{X}(\ell_p)$ has a further subspace $Z$, complemented in $\hat{X}(\ell_p)$, let $x_1, x_2, \ldots$ and $e_1, e_2, \ldots$ be the canonical bases in $X$ and $\ell_p$, respectively. Then the elements $u_{ij} = x_i \otimes e_j$ form an unconditional basis in $\hat{X}(\ell_p)$, with

$$\left| \sum a_{ij} u_{ij} \right| = \left| \sum \left( \sum |a_{ij}|^p \right)^{1/p} x_i \right|_X = \left| \sum \left( \sup \left\{ \left| \sum_j a_{ij} a_{ij}^* \right| x_i \right\} \right. \right|_X. \quad (7.1)$$

Let $P_n$ be the canonical projection onto $\text{span}[u_{ij} : 0 \leq i \leq n, j \in \mathbb{N}]$, and set $P_n^\perp = I - P_n$. The range of $P_n$ is isomorphic to $\ell_p$, hence, if $P_n|_Y$ is not strictly singular for some $n$, we are done, by Corollary 2.4. If $P_n|_Y$ is strictly singular for every $n$, find a normalized sequence $(y_i)$ in $Y$, and $1 = n_1 < n_2 < \ldots$, so that $\|P_{n_i} y_i\|_Y, \|P_{n_i}^\perp y_i\| < 100^{-i/2}$. By small perturbation, it remains to prove the following: if $y_i = P_{n_i}^\perp P_{n_{i+1}} y_i$, then span$[y_i : i \in \mathbb{N}]$ contains a subspace, complemented in $\hat{X}(\ell_p)$. Further, we may assume that for each $i$ there exists $M_i$ so that we can write

$$y_i = \sum_{n_i < k \leq n_{i+1}, 1 \leq j \leq M_i} a_{kj} u_{kj},$$

For each $k \in [n_i+1, n_{i+1}]$ (and arbitrary $i \in \mathbb{N}$) find a finite sequence $(a_{kj})_{j=1}^{M_i}$ so that $\sum_j |a_{kj}|^p = 1$, and $|\sum_j a_{kj} a_{kj}^*| = (\sum_j |a_{kj}|^p)^{1/p}$. Define $U : \hat{X}(\ell_p) \to X : u_{kj} \mapsto a_{kj} a_{kj}^* x_k$. By (7.1), $U$ is a contraction, and $U|_{\text{span}[y_i : i \in \mathbb{N}]}$ is an isometry. To finish the proof, recall that $X$ is subprojective, and apply Corollary 2.4. \hfill \blacksquare

**Remark 7.2.** Using similar methods, one can prove: if $K$ is a compact metrizable scattered space, and $1 \leq p < \infty$, then $C(K)(\ell_p)$ is subprojective.

Recall that, for a Banach space $X$, we denote by $Rad(X)$ the completion of the finite sums $\sum_n r_n x_n (r_1, r_2, \ldots \text{ are Rademacher functions, and } x_1, x_2, \ldots \in X)$ in the norm of $L_1(X)$ (equivalently, by Khintchine-Kahane Inequality, in the norm of $L_p(X)$). If $X$ has an unconditional basis $(x_i)$ and finite cotype, then
Rad(X) is isomorphic to \( \widehat{X(\ell_2)} \) (here we can view \( X \) as a Banach lattice, with the order induced by the basis \( (x_i) \)). Indeed, by [28, Section 1.f], \( X \) is \( q \)-concave, for some \( q \). An array \( (a_{mn}) \) can be identified both with an element of \( \text{Rad}(X) \) (with the norm \( \left| \sum_m \sum_n a_{mn} r_n x_m \right| \)), and with an element of \( \hat{X(\ell_2)} \) (with the norm \( \left| \sum_m \sum_n |a_{mn}|^2 \right|^{1/2} x_m \)). Then

\[
D \left| \sum_m \sum_n |a_{mn}|^2 \right|^{1/2} x_m \leq \sum_m \int_0^1 \left| \sum_n a_{mn} r_n x_m \right| \, \mathrm{d} t \leq \sum_m \int_0^1 \left| \sum_n a_{mn} r_n x_m \right| \, \mathrm{d} t \\
\leq (\int_0^1 \left| \sum_m \sum_n a_{mn} r_n x_m \right|^q \, \mathrm{d} t)^{1/q} \leq M_q (\int_0^1 \left| \sum_m \sum_n a_{mn} r_n x_m \right|^q \, \mathrm{d} t)^{1/q} \\
\leq M_q \left( \int_0^1 \left| \sum_n a_{mn} r_n x_m \right|^q \, \mathrm{d} t \right)^{1/q} \leq C M_q \left( \sum_m \left| a_{mn} \right|^2 \right)^{1/2} x_m,
\]

where \( M_q \) is a \( q \)-concavity constant, while \( D \) and \( C \) come from Khintchine’s inequality. Thus, we have proved:

**Proposition 7.3.** If \( X \) is a subprojective space with an unconditional basis and non-trivial cotype, then \( \text{Rad}(X) \) is subprojective.

**Remark 7.4.** By [23, Theorem 2.3], if \( X \) is a non-atomic order continuous Banach lattice with an unconditional basis, then \( \hat{X(\ell_2)} \) is isomorphic to \( X \). Furthermore, if \( X \) is a non-atomic Banach lattice with an unconditional basis and non-trivial cotype, then \( \text{Rad}(X) \) is isomorphic to \( X \). Indeed, non-trivial cotype implies non-trivial lower estimate [28, p. 100], which, by [29, Theorem 2.4.2], implies order continuity. Therefore, \( X \) is isomorphic to \( \hat{X(\ell_2)} \), which, in turn, is isomorphic to \( \text{Rad}(X) \).

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**References**


