ORDER EXTREME POINTS AND SOLID CONVEX HULLS

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To the memory of Victor Lomonosov

Abstract. We consider the "order" analogues of some classical notions of Banach space geometry: extreme points and convex hulls. A Hahn-Banach type separation result is obtained, which allows us to establish an "order" Krein-Milman Theorem. We show that the unit ball of any infinite dimensional reflexive space contains uncountably many order extreme points, and investigate the set of positive norm-attaining functionals. Finally,we introduce the "solid" version of the Krein-Milman Property, and show it is equivalent to the Radon-Nikodým Property.

1. Introduction

At the very heart of Banach space geometry lies the study of three interrelated subjects: (i) separation results (starting from the Hahn-Banach Theorem), (ii) the structure of extreme points, and (iii) convex hulls (for instance, the Krein-Milman Theorem on convex hulls of extreme points). Certain counterparts of these notions exist in the theory of Banach lattices as well. For instance, there are positive separation/extension results, see e.g. [1, Section 1.2]. One can view solid convex hulls as lattice analogues of convex hulls; these objects have been studied, and we mention some of their properties in the paper. However, no unified treatment of all three phenomena listed above has been attempted.

In the present paper, we endeavor to investigate the lattice versions of (i), (ii), and (iii) above. We introduce the order version of the classical notion of an extreme point: if A is a subset of a Banach lattice X, then $a \in A$ is called an *order extreme point* of A if the inequality $a \leq (1-t)x_0 + tx_1$ ($x_0, x_1 \in A$, $0 < t < 1$) implies $x_0 = a = x_1$. Note that, in this case, if $x \geq a$ and $x \in A$, then $x = a$ (write $a \leq (x+a)/2$).

Throughout, we work with real spaces. We will be using the standard Banach lattice results and terminology (found in, for instance, [1], [20] or [22]). Some special notation is introduced in Section 2. In the same section, we establish some basic facts about order extreme points and solid hulls.

Date: May 25, 2019.

2010 *Mathematics Subject Classification.* 46B22, 46B42.

Key words and phrases. Banach lattice, extreme point, convex hull, Radon-Nikodým Property.

In Section 3 we prove a "Hahn-Banach" type result (Proposition 3.1), involving separation by positive functionals. This result is used in Section 4 to establish a "solid" analogue of the Krein-Milman Theorem. We prove that solid compact sets are solid convex hulls of their order extreme points (see Theorem 4.1). A "solid" Milman Theorem is also proved (Theorem 4.4).

Order extreme points, and their relation to "classical" extreme points, are further investigated in Section 5.

In Section 6 we study the order extreme points in AM-spaces. For instance, we show that, for an AM-space X, the following three statements are equivalent: (i) X is a $C(K)$ space; (ii) the unit ball of X is the solid convex hull of finitely many of its elements; (iii) the unit ball of X has an order extreme point (Propositions 6.3 and 6.4).

Further in Section 6 we investigate positive norm-attaining positive functionals. Functionals attaining their maximum on certain sets have been investigated since the early days of functional analysis; here we must mention V. Lomonosov's papers on the subject (see e.g. the excellent summary [4], and there references contained there). In this paper, we show that a separable AM-space is a $C(K)$ space iff any positive functional on it attains its norm (Proposition 6.5). On the other hand, an order continuous lattice is reflexive iff every positive operator on it attains its norm (Proposition 6.6).

In Section 7 we show that the unit ball of any reflexive infinite-dimensional Banach lattice has uncountably many order extreme points (Theorem 7.1).

Finally, in Section 8 we define the "solid" version of the Krein-Milman Property, and show that it is equivalent to the Radon-Nikodym Property (Theorem 8.1).

To close this introduction, we would like to mention that related ideas have been explored before, in other branches of functional analysis. In the theory of C^* algebras, and, later, operator spaces, the notions of "matrix" or "C^*" extreme points and convex hulls have been used. The reader is referred to e.g. [11], [12], [14], [23] for more information; for a recent operator-valued separation theorem, see [19].

2. Preliminaries

In this section, we introduce the notation commonly used in the paper, and mention some basic facts.

The closed unit ball (sphere) of a normed space Z is denoted by $\mathbf{B}(Z)$ (resp. $\mathbf{S}(Z)$). We will denote the set of order extreme points of C (defined in Section 1) by $\mathrm{OEP}(C)$; the set of "classical" extreme points is denoted by $\mathrm{EP}(C)$.

If Z is a normed lattice, and $C \subset Z$, write $C_+ = C \cap Z_+$, where Z_+ stands for the positive cone of Z. Further, we say that $C \subset Z$ is *solid* if $z \in C$ whenever there exists $x \in C$ so that $|z| \leq |x|$ (hence $|x| \in C$ whenever

$x \in C$). Note that any solid set is automatically *balanced* – that is, $C = -C$. Sometimes, we need to restrict our attention to the positive cone Z_+; we say that $C \subset Z_+$ is *positive-solid* if $z \in C$ whenever there exists $x \in C$ so that $z \leq x$.

Denote by $\mathrm{S}(C)$ the *solid hull* of C – that is, the smallest solid set containing C. It is easy to see that $\mathrm{S}(C)$ is the set of all $z \in Z$ for which there exists $x \in C$ satisfying $|z| \leq |x|$. Clearly $\mathrm{S}(C) = \mathrm{S}(|C|)$, where $|C| = \{|x| : x \in C\}$. Further, we denote by $\mathrm{CH}(C)$ the *convex hull* of C. For future reference, observe:

Proposition 2.1. *If X is a Banach lattice, then $\mathrm{S}(\mathrm{CH}(|C|)) = \mathrm{CH}(\mathrm{S}(C))$ for any $C \subset X$.*

Proof. Let $x \in \mathrm{CH}(\mathrm{S}(C))$. Then $x = \sum a_i y_i$, where $\sum a_i = 1, a_i > 0$, and $|y_i| \leq |k_i|$ for some $k_i \in C$. Then

$$|x| \leq \sum a_i |y_i| \leq \sum a_i |k_i| \in \mathrm{CH}(|C|),$$

so $x \in \mathrm{S}(\mathrm{CH}(|C|))$. If $x \in \mathrm{S}(\mathrm{CH}(|C|))$, then

$$|x| \leq \sum_1^n a_i y_i, \quad y_i \in |C|, \quad 0 < a_i, \quad \sum a_i = 1.$$

We use induction on n to prove that $x \in \mathrm{CH}(\mathrm{S}(C))$. If $n = 1$, $x \in \mathrm{S}(C)$ and we are done. Now, suppose we have shown that if $|x| \leq \sum_1^{n-1} a_i y_i$ then there are $z_1, ..., z_{n-1} \in \mathrm{S}(C)_+$ such that $|x| = \sum_1^{n-1} a_i z_i$. From there, we have that

$$|x| = (\sum_1^n a_i y_i) \wedge |x| \leq (\sum_1^{n-1} a_i y_i) \wedge |x| + (a_n y_n) \wedge |x|.$$

Now

$$0 \leq |x| - (\sum_1^{n-1} a_i y_i) \wedge |x| \leq a_n (y_n \wedge \frac{|x|}{a_n}).$$

Let $z_n := \frac{1}{a_n}(|x| - (\sum_1^{n-1} a_i y_i) \wedge |x|)$. By the above, $z_n \in \mathrm{S}(C)_+$. Furthermore,

$$\frac{1}{1-a_n}(|x| \wedge \sum_1^{n-1} a_i y_i) \leq \sum_1^{n-1} \frac{a_i}{1-a_n} y_i \in \mathrm{CH}(|C|),$$

so by induction there exist $z_1, .., z_{n-1} \in \mathrm{S}(C)_+$ such that

$$|x| \wedge (\sum_1^{n-1} a_i y_i) = \sum_1^{n-1} \frac{a_i}{1-a_n} z_i$$

Therefore $|x| = \sum_1^n a_i z_i$. Now for each n, $a_i z_i \leq |x|$, so $|x| = \sum \big((a_i z_i) \wedge |x|\big)$, and

$$a_i z_i = a_i z_i \wedge x_+ + a_i z_i \wedge x_- = a_i (z_i \wedge (\frac{x_+}{a_i}) + z_i \wedge (\frac{x_-}{a_i})).$$

Let $w_i = z_i \wedge (\frac{x_+}{a_i}) - z_i \wedge (\frac{x_-}{a_i})$. Note that $|w_i| = z_i$, so $w_i \in \mathrm{S}(C)$. It follows that $x = \sum a_i w_i \in \mathrm{CH}(\mathrm{S}(C))$. □

For $C \subset Z$ we define the *solid convex hull* of C to be the smallest convex, solid set containing C, and denote it by $\mathrm{SCH}(C)$; the norm (equivalently, weak) closure of the latter set is denoted by $\mathrm{CSCH}(C)$, and referred to as the *closed solid convex hull* of C.

Corollary 2.2. *Let $C \subseteq X$. Then*

(1) $\mathrm{SCH}(C) = \mathrm{CH}(\mathrm{S}(C)) = \mathrm{SCH}(|C|)$*, and consequently,* $\mathrm{CSCH}(C) = \mathrm{CSCH}(|C|)$.
(2) If $C \subseteq X_+$, then $\mathrm{SCH}(C) = \mathrm{S}(\mathrm{CH}(C))$.

Proof. (1) Suppose $C \subseteq D$, where D is convex and solid. Then $\mathrm{CH}(\mathrm{S}(C)) \subseteq D$. By Proposition 2.1, $\mathrm{CH}(\mathrm{S}(C))$ is also solid, so $\mathrm{SCH}(C) = \mathrm{CH}(\mathrm{S}(C)) = \mathrm{CH}(\mathrm{S}(|C|)) = \mathrm{SCH}(|C|)$.
(2) This follows from (1) and the equality in Proposition 2.1. □

Remark 2.3. In general, C being closed does not imply that $\mathrm{S}(C)$ is closed. Below we provide a couple of examples.

(1) Let X be a Banach lattice of dimension at least two, and consider disjoint norm one $e_1, e_2 \in B(X)_+$. Let $C = \{x_n : n \in \mathbb{N}\}$, where $x_n = \frac{n}{n+1} e_1 + n e_2$. Now, C is norm-closed: if $m > n$, then $\|x_m - x_n\| \geq \|e_2\| = 1$. However, $\mathrm{S}(C)$ is not closed: it contains re_1 for any $r \in (0,1)$, but not e_1.

(2) If X is infinite dimensional, then there exists a closed bounded $C \subset X_+$, for which $\mathrm{S}(C)$ is not closed. Indeed, find disjoint norm one elements $e_1, e_2, \ldots \in X_+$. For $n \in \mathbb{N}$ let $y_n = \sum_{k=1}^n 2^{-k} e_k$ and $x_n = y_n + e_n$. Then clearly $\|x_n\| \leq 2$ for any n; further, $\|x_n - x_m\| \geq 1$ for any $n \neq m$, hence $C = \{x_1, x_2, \ldots\}$ is closed. However, $y_n \in \mathrm{S}(C)$ for any n, and the sequence (y_n) converges to $\sum_{k=1}^\infty 2^{-k} e_k \notin \mathrm{S}(C)$.

However, under certain conditions we can show that the solid hull of a convex set is closed.

Proposition 2.4. *A Banach lattice X is a KB-space if and only if, for any norm closed bounded convex $C \subset X_+$, $\mathrm{S}(C)$ is norm closed.*

Proof. Suppose first X is a KB-space, and C is a norm closed bounded convex subset of X_+. Suppose (x_n) is a sequence in $\mathrm{S}(C)$, which converges to some x in norm; show that x belongs to $\mathrm{S}(C)$ as well. Clearly $|x_n| \to |x|$ in norm. For each n find $y_n \in C$ so that $|x_n| \leq y_n$. By passing to a subsequence if necessary, we can assume that the sequence (y_n) converges to some $y \in X^{**}$ in the weak* topology. By [20, Theorem 2.5.6], a Banach lattice is a KB-space iff it is weakly sequentially complete, hence $y \in X$, and $y_n \to y$ weakly. For convex sets, norm and weak closures coincide, hence y belongs to C. For each n, $\pm x_n \leq y_n$; passing to the weak limit gives $\pm x \leq y$, hence $|x| \leq y$.

Now suppose X is not a KB-space. By [20, Sections 2.4-5], there exists a sequence of disjoint elements $e_i \in \mathbf{S}(X)_+$, equivalent to the natural basis of c_0. Let C be the closed convex hull of

$$x_1 = \frac{e_1}{2}, \ x_n = \big(1 - 2^{-n}\big)e_1 + \sum_{j=2}^n e_j \ (n \geq 2).$$

We shall show that any element of C can be written as $ce_1 + \sum_{i=2}^\infty c_i e_i$, with $c < 1$. This will imply that $\mathrm{S}(C)$ is not closed: clearly $e_1 \in \overline{\mathrm{S}(C)}$.

The elements of $\mathrm{CH}(x_1, x_2, \ldots)$ are of the form $\sum_{i=1}^n t_i x_i = ce_1 + \sum_{i=2}^n c_i e_i$; here, $t_i \geq 0$ and $\sum_i t_i = 1$. Here, $c_i = \sum_{j=i}^n t_i$ for $i \geq 2$; for convenience, let $c_1 = \sum_{j=1}^n t_i = 1$, and $c_j = 0$ for $j > n$. Then $t_i = c_i - c_{i+1}$; Abel's summation technique gives

$$c = \sum_{i=1}^n \big(1 - 2^{-i}\big)t_i = 1 - \sum_{i=1}^n 2^{-i}\big(c_i - c_{i+1}\big) = 1 - \frac{1}{2} + \sum_{j=2}^n 2^{-j} c_j.$$

Now consider $x \in C$. Then x is the norm limit of the sequence

$$x^{(m)} = c^{(m)} e_1 + \sum_{i=2}^\infty c_i^{(m)} e_i \in \mathrm{CH}(x_1, x_2, \ldots);$$

for each m, the sequence $(c_i^{(m)})$ has only finitely many non-zero terms, $c^{(m)} = 1 - \frac{1}{2} + \sum_{j=2}^n 2^{-j} c_j^{(m)}$, and $|c_i^{(m)} - c_i^{(n)}| \leq \|x^{(m)} - x^{(n)}\|$. Thus, $x = ce_1 + \sum_{i=2}^\infty c_i e_i$, with $c = 1 - \frac{1}{2} + \sum_{j=2}^n 2^{-j} c_j$. As $0 \leq c_j \leq 1$, and $\lim_j c_j = 0$, we conclude that $c < 1$, as claimed. □

3. Separation by positive functionals

Throughout the section, X is a Banach lattice, equipped with a locally convex Hausdorff topology τ. This topology is called *sufficiently rich* if the following conditions are satisfied:

(i) The space X^τ of τ-continuous functionals on X is a Banach lattice (with lattice operations defined by Riesz-Kantorovich formulas).
(ii) X_+ is τ-closed.

Note that (i) and (ii) together imply that positive τ-continuous functionals separate points – that is, for every $x \in X \backslash \{0\}$ there exists $f \in X_+^\tau$ so that $f(x) \neq 0$. Indeed, without loss of generality, $x_+ \neq 0$. Then $-x_+ \notin X_+$, hence there exists $f \in X_+^\tau$ so that $f(x_+) > 0$. By [20, Proposition 1.4.13], there exists $g \in X_+^\tau$ so that $g(x_+) > f(x_+)/2$ and $g(x_-) < f(x_+)/2$. Then $g(x) > 0$.

Clearly, the norm and weak topologies are sufficiently rich; in this case, $X^\tau = X^*$. The weak* topology on X, induced by the predual Banach lattice X_*, is sufficiently rich as well; then $X^\tau = X_*$.

Proposition 3.1 (Separation)**.** *Suppose τ is a sufficiently rich topology on a Banach lattice X, and $A \subset X_+$ is a τ-closed positive-solid bounded subset of X_+. Suppose, furthermore, $x \in X_+$ does not belong to A. Then there exists $f \in X_+^\tau$ so that $f(x) > \sup_{a \in A} f(a)$.*

Lemma 3.2. *Suppose A and X are as above, and $f \in X^\tau$. Then $\sup_{a\in A} f(a) = \sup_{a\in A} f_+(a)$.*

Proof. Clearly $\sup_{a\in A} f(a) \leq \sup_{a\in A} f_+(a)$. To prove the reverse inequality, write $f = f_+ - f_-$, with $f_+ \wedge f_- = 0$. Fix $a \in A$; then

$$0 = \big[f_+ \wedge f_-\big](a) = \inf_{0 \leq x \leq a} \big(f_+(a-x) + f_-(x)\big).$$

For any $\varepsilon > 0$ we can find $x \in A$ so that $f_+(a-x), f_-(x) < \varepsilon$. Then $f_+(x) = f_+(a) - f_+(a-x) > f_+(a) - \varepsilon$, and therefore, $f(x) = f_+(x) - f_-(x) > f_+(a) - 2\varepsilon$. Now recall that $\varepsilon > 0$ and $a \in A$ are arbitrary. □

Proof of Proposition 3.1. Use Hahn-Banach Theorem to find f strictly separating x from A. By Lemma 3.2, f_+ achieves the separation as well. □

Remark 3.3. In this paper, we do not consider separation results on general ordered spaces. Our reasoning will fail without lattice structure. For instance, Lemma 3.2 is false when X is not a lattice, but merely an ordered space. Indeed, consider $X = M_2$ (the space of real 2×2 matrices), $f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $A = \{ta_0 : 0 \leq t \leq 1\}$, where $a_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; one can check that $A = \{x \in M_2 : 0 \leq x \leq a_0\}$. Then $f|_A = 0$, while $\sup\limits_{x\in A} f_+(x) = 1$. The reader interested in the separation results in the non-lattice ordered setting can be referred to the recent monograph [2], as well as to a very thorough treatment of separation in [15]. Of particular interest is Sandwich Theorem [16], recently re-proved in [3].

4. Solid convex hulls: theorems of Krein-Milman and Milman

Throughout this section, the topology τ is assumed to be sufficiently rich (defined in the beginning of Section 3).

Theorem 4.1 ("Solid" Krein-Milman)**.** *Any τ-compact positive-solid subset of X_+ coincides with the τ-closed positive-solid convex hull of its order extreme points.*

Proof. Denote the τ-closed positive convex hull of $\mathrm{OEP}(A)$ by B; then clearly $B \subset A$. The proof of the reverse inclusion is similar to that of the "usual" Krein-Milman.

Suppose C is a τ-compact subset of X. We say that a non-void closed $F \subset C$ is an *order extreme subset* of C if, whenever $x \in F$ satisfies $x \leq (a_1 + a_2)/2$ $(a_1, a_2 \in C)$, then necessarily $a_1, a_2 \in F$. The set $\mathcal{F}(C)$ of order extreme subsets of C can be ordered by reverse inclusion (this makes C itself the smallest order extreme subset of itself). By compactness, each chain has an upper bound; therefore, by Zorn's Lemma, $\mathcal{F}(C)$ has a maximal element. We claim that these maximal elements are singletons – that is, order extreme points of C.

We need to show that, if $F \in \mathcal{F}(C)$ is not a singleton, then there exists $G \subsetneq F$ which is also an order extreme set. To this end, find distinct $a_1, a_2 \in F$, and $f \in X_+^\tau$ which separates them – say $f(a_1) > f(a_2)$. Let $\alpha = \max_{x \in F} f(x)$, then $G = F \cap f^{-1}(\alpha)$ is a proper, order extreme subset of F.

Suppose, for the sake of contradiction, that there exists $x \in A \backslash B$. Use Proposition 3.1 to find $f \in X_+^\tau$ so that $f(x) > \max_{y \in B} f(y)$. Let $\alpha = \max_{x \in A} f(x)$, then $A \cap f^{-1}(\alpha)$ is an order extreme subset of A, disjoint from B. As noted above, this subset contains at least one extreme point. This yields a contradiction, as we started out assuming all order extreme points lie in B. ☐

Corollary 4.2. *Any τ-compact solid subset of X coincides with the τ-closed solid convex hull of its order extreme points.*

Of course, there exist Banach lattices whose unit ball has no order extreme points at all – $L_1(0,1)$, for instance. However, an order analogue of [17, Lemma 1] holds.

Proposition 4.3. *For a Banach lattice X, the following two statements are equivalent:*

(1) Every bounded closed solid subset of X has an order extreme point.
(2) Every bounded closed solid subset of X is the closed solid convex hull of its order extreme points.

Proof. (2) $\Rightarrow$ (1) is evident; we shall prove (1) $\Rightarrow$ (2). Suppose $A \subset X$ is closed, bounded, convex, and solid. Let $B = \mathrm{CSCH}(\mathrm{OEP}(A))$ (which is not empty, by (1)). Suppose, for the sake of contradiction, that B is a proper subset of A. Find $a \in A_+ \backslash B$; then there exists $f \in \mathbf{S}(X^*)_+$ which strictly separates a from B; consequently,

$$\sup_{x \in A} f(x) \geq f(a) > \sup_{x \in B} f(x).$$

Fix $\varepsilon > 0$ so that

$$2\sqrt{2\varepsilon}\alpha < \sup_{x \in A} f(x) - \sup_{x \in B} f(x), \text{ where } \alpha = \sup_{x \in A} \|x\|.$$

By Bishop-Phelps-Bollobás Theorem (see e.g. [5] or [9]), there exists $f' \in \mathbf{S}(X^*)$, attaining its maximum on A, so that $\|f - f'\| \leq \sqrt{2\varepsilon}$.

Let $g = |f'|$, then $\|f - g\| \leq \|f - f'\| \leq \sqrt{2\varepsilon}$. Further, g attains its maximum on A_+, and $\max_{g \in A} g(x) > \sup_{x \in B} g(x)$. Indeed, the first statement follows immediately from the definition of g. To establish the second one, note that the triangle inequality gives us

$$\sup_{x\in B} g(x) \leq \sqrt{2\varepsilon}\alpha + \sup_{x\in B} f(x), \quad \sup_{x\in A} g(x) \geq \sup_{x\in A} f(x) - \sqrt{2\varepsilon}\alpha.$$

Our assumption on ε gives us $\max_{g\in A} g(x) > \sup_{x\in B} g(x)$.
Let $D = \{a \in A : g(a) = \sup_{x\in A} g(x)\}$. Due to (1), D has an order extreme point, which is an order extreme point of A as well; this point lies inside of B, leading to the desired contradiction. □

Milman's theorem [21, 3.25] states that, if both K and $\overline{\mathrm{CH}(K)}^\tau$ are compact, then $\mathrm{EP}\big(\overline{\mathrm{CH}(K)}^\tau\big) \subset K$. An order analogue of Milman's theorem exists:

Theorem 4.4. *Suppose X is a Banach lattice.*

(1) If $K \subset X_+$ and $\overline{\mathrm{CH}(K)}^\tau$ are τ-compact, then $\mathrm{OEP}\big(\overline{\mathrm{SCH}(K)}^\tau\big) \subseteq K$.
(2) If $K \subset X_+$ is weakly compact, then $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq K$.
(3) If $K \subset X$ is norm compact, then $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq |K|$.

The following lemma describes the solid hull of a τ-compact set.

Lemma 4.5. *Suppose a Banach lattice X is equipped with a sufficiently rich topology τ. If $C \subset X_+$ is τ-compact, then $\mathrm{S}(C)$ is τ-closed.*

Proof. Suppose a net $(y_i) \subset \mathrm{S}(C)$ τ-converges to $y \in X$. For each i find $x_i \in C$ so that $|y_i| \leq x_i$ – or equivalently, $y_i \leq x_i$ and $-y_i \leq x_i$. Passing to a subnet if necessary, we can assume that $x_i \to x \in C$ in the topology τ. Then $\pm y \leq x$, which is equivalent to $|y| \leq x$. □

Proof of Theorem 4.4. (1) We first consider a τ-compact $K \subseteq X_+$. Milman's traditional theorem holds that $\mathrm{EP}\big(\overline{\mathrm{CH}(K)}^\tau\big) \subseteq K$. Every order extreme point of a set is extreme, hence the order extreme points of $\overline{\mathrm{CH}(K)}^\tau$ are in K. Therefore, by Lemma 4.5 and Corollary 2.2,

$$\overline{\mathrm{SCH}(K)}^\tau = \overline{\mathrm{S}(\mathrm{CH}(K))}^\tau \subseteq \mathrm{S}\big(\overline{\mathrm{CH}(K)}^\tau\big) = \{x : |x| \leq y \in \overline{\mathrm{CH}(K)}^\tau\}.$$

Thus, the points of $\overline{\mathrm{SCH}(K)}^\tau \backslash \overline{\mathrm{CH}(K)}^\tau$ cannot be order extreme due to being dominated by $\overline{\mathrm{CH}(K)}^\tau$. Therefore $\mathrm{OEP}\big(\overline{\mathrm{SCH}(K)}^\tau\big) \subseteq \mathrm{OEP}\big(\overline{\mathrm{CH}(K)}^\tau\big) \subseteq K$.
(2) Combine (1) with Krein's Theorem (see e.g. [13, Theorem 3.133]), which states that $\overline{\mathrm{CH}(K)}^w = \overline{\mathrm{CH}(K)}$ is weakly compact.
(3) Finally, suppose $K \subseteq X$ is norm compact. By Corollary 2.2, $\mathrm{CSCH}(K) = \mathrm{CSCH}(|K|)$. $|K|$ is norm compact, hence by [21, Theorem 3.20], so is $\overline{\mathrm{CH}(|K|)}$. By the proof of part (1), $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq |K|$. □

We conclude this section with some results about interchanging "solidification" and norm closure. We work with the norm topology, unless specified otherwise.

Lemma 4.6. *Let $C \subseteq X$, where X is a Banach lattice, and suppose that $\mathrm{S}(\overline{|C|})$ is closed. Then $\overline{\mathrm{S}(C)} = \mathrm{S}(\overline{|C|})$.*

Proof. One direction is easy: $\mathrm{S}(C) = \mathrm{S}(|C|) \subseteq \mathrm{S}(\overline{|C|})$, hence $\overline{\mathrm{S}(C)} \subseteq \overline{\mathrm{S}(\overline{|C|})} = \mathrm{S}(\overline{|C|})$.

Now consider $x \in \mathrm{S}(\overline{|C|})$ – that is $|x| \le y$ for some $y \in \overline{|C|}$. Take $y_n \in |C|$ such that $y_n \to y$. Then $|x| \wedge y_n \in \mathrm{S}(|C|) = \mathrm{S}(C)$ for all n. Furthermore,

$$|x_+ \wedge y_n - x_- \wedge y_n| = |x| \wedge y_n,$$

so, $x_+ \wedge y_n - x_- \wedge y_n \in \mathrm{S}(C)$. By norm continuity of $\wedge$,

$$x_+ \wedge y_n - x_- \wedge y_n \to x_+ \wedge y - x_- \wedge y = x,$$

hence $x \in \overline{\mathrm{S}(C)}$. □

Remark 4.7. The assumption of $\mathrm{S}(\overline{|C|})$ being closed is necessary: Remark 2.3 shows that, for a closed $C \subset X_+$, $\mathrm{S}(C)$ need not be closed.

Corollary 4.8. *Suppose $C \subseteq X$ is relatively compact in the norm topology. Then $\overline{\mathrm{S}(C)} = \mathrm{S}(\overline{C})$.*

Proof. The set $\overline{C}$ is compact, hence, by the continuity of $|\cdot|$, the same is true for $|\overline{C}|$. Consequently, $|\overline{C}| \subseteq \overline{|C|} \subseteq \overline{|\overline{C}|} = |\overline{C}|$, hence $|\overline{C}| = \overline{|C|}$. By Lemmas 4.5 and 4.6, $\mathrm{S}(\overline{C}) = \mathrm{S}(|\overline{C}|) = \mathrm{S}(\overline{|C|}) = \overline{\mathrm{S}(C)}$. □

Remark 4.9. In the weak topology, the equality $|\overline{C}| = \overline{|C|}$ may fail. Indeed, equip the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ with its uniform probability measure μ. Define $x_i \in L_2(\mu)$ by setting, for $t = (t_1, t_2, \ldots) \in \Delta$, $x_i(t) = t_i - 1/4$ (that is, x_i equals to either $3/4$ or $-1/4$, depending on whether t_i is 1 or 0). Then $C = \{x_i : i \in \mathbb{N}\}$ belongs to the unit ball of $L_2(\mu)$, hence it is relatively compact. It is clear that $\overline{C}$ contains $\mathbf{1}/4$ (here and below, $\mathbf{1}$ denotes the constant 1 function). On the other hand, $\overline{C}$ does not contain $\mathbf{1}/2$, which can be witnessed by applying the integration functional. Conversely, $\overline{C}$ contains $\mathbf{1}/2$, but not $\mathbf{1}/4$.

Remark 4.10. Relative weak compactness of solid hulls have been studied before. If X is a Banach lattice, then, by [1, Theorem 4.39], it is order continuous iff the solid hull of any weakly compact subset of X_+ is relatively weakly compact. Further, by [8], the following three statements are equivalent:

(1) The solid hull of any relatively weakly compact set is relatively weakly compact.
(2) If $C \subset X$ is relatively weakly compact, then so is $|C|$.
(3) X is a direct sum of a KB-space and an atomic order continuous space.

5. Connections between order and “canonical” extreme points

Note that every order extreme point is an extreme point in the usual sense, but the converse is not true: for instance, $\mathbf{1}_{(0,1)}$ is an extreme point of $\mathbf{B}(L_\infty(0,2))_+$, but not its order extreme point. In this section, we investigate the connections between order and “classical” extreme points.

Theorem 5.1. *Suppose A is a closed bounded solid subset of a Banach lattice X. Then a is an extreme point of A if and only if $|a|$ is its order extreme point.*

Proof. Suppose $|a|$ is order extreme. Let $0 < t < 1$ such that $a = tx+(1-t)y$. Then since A is balanced, $|a| \leq t|x| + (1-t)|y|$, so $|x| = |y| = |a|$. Thus the latter inequality is in fact equality. Thus $|a| + a = 2a_+ = 2tx_+ + 2(1-t)y_+$, so $a_+ = tx_+ + (1-t)y_+$. Similarly, $a_- = tx_- + (1-t)y_-$. It follows that $x_+ \perp y_-$ and $x_- \perp y_+$. Since $x_+ + x_- = |x| = |y| = y_+ + y_-$, we have that $x_+ = x_+ \wedge (y_+ + y_-) = x_+ \wedge y_+ + x_+ \wedge y_-$ (since y_+, y_- are disjoint). Now since $x_+ \perp y_-$, the latter is just $x_+ \wedge y_+$, hence $x_+ \leq y_+$. By similar argument one can show the opposite inequality to conclude that $x_+ = y_+$, and likewise $x_- = y_-$, so $x = y = a$.

Now suppose a is extreme, and suppose $|a| \leq tx + (1-t)y$, where $0 < t < 1$. It is sufficient to show that $|a|$ is order extreme for A_+. Indeed, if so, then since $|a| \leq t|x| + (1-t)|y|$, it follows that $|x| = |y| = |a|$, so $|a| = tx + (1-t)y = t|x| + (1-t)|y|$. The latter implies that $x_- = y_- = 0$. Hence $x = |x| = |a| = |y| = y$.

Therefore, suppose $x, y \geq 0$. We show that $|a|$ is a quasi-unit of x (and by similar argument y). Then we have $a_+ - tx \wedge a_+ \leq (1-t)y \wedge a_+$. Since A is solid,

$$A \ni z_+ := \frac{1}{1-t}(a_+ - tx \wedge a_+)$$

and similarly, since $a_- - tx \wedge a_- \leq (1-t)y \wedge a_-$,

$$A \ni z_- := \frac{1}{1-t}(a_- - tx \wedge a_-)$$

These inequalities imply that $z_+ \perp z_-$, so they correspond to the positive and negative parts of some $z = z_+ - z_-$. Also, $z \in A$ since $|z| \leq |a|$. Now $a_+ = t(x \wedge \frac{a_+}{t}) + (1-t)z_+$ and $a_- = t(x \wedge \frac{a_-}{t}) + (1-t)z_+$. In addition, $|x \wedge \frac{a_+}{t} - x \wedge \frac{a_-}{t}| \leq x$, so since A is solid and balanced,

$$z' := x \wedge \frac{a_+}{t} - x \wedge \frac{a_-}{t} \in A.$$

Therefore $a = a_+ - a_- = tz' + (1-t)z$. Since a is an extreme point, $a = z$, hence

$$(1-t)z_+ = (1-t)a_+ = a_+ - tx \wedge a_+$$

so $tx \wedge a_+ = ta_+$ which implies that $(t(x - a_+)) \wedge ((1-t)a_+) = 0$. As $0 < t < 1$, we have that a_+ (and likewise a_-) is a quasi-unit of x (and similarly y). Thus $|a|$ is a quasi-unit of x and of y. Now let $s = x - |a|$. Then $a + s, a - s \in A$, since $|a \pm s| = x$. Then we have

$$a = \frac{a-s}{2} + \frac{a+s}{2},$$

but since a is extreme, s must be 0. Hence $x = |a|$, and similarly $y = |a|$. □

The situation is different if A is a positive-solid set: as shown above, A can have extreme points which are not order extreme. However, we have:

Lemma 5.2. *Suppose τ is a sufficiently rich topology, and A is a τ-compact positive-solid convex subset of X_+. Then for any extreme point $a \in A$ there exists an order extreme point $b \in A$ so that $a \leq b$.*

Remark 5.3. The compactness assumption is essential. Consider, for instance, the closed set $A \subset C[-1, 1]$, consisting of all functions f so that $0 \leq f \leq \mathbf{1}$, and $f(x) \leq x$ for $x \geq 0$. Then $g(x) = x \vee 0$ is an extreme point of A; however, A has no order extreme points.

Proof. If a is not an order extreme point, then we can find distinct $x_1, x_2 \in A$ so that $2a \leq x_1 + x_2$. Then $2a \leq (x_1 + x_2) \wedge (2a) \leq x_1 \wedge (2a) + x_2 \wedge (2a) \leq x_1 + x_2$. Write $2a = x_1 \wedge (2a) + (2a - x_1 \wedge (2a))$. Both summands are positive, and both belong to A (for the second summand, note that $2a - x_1 \wedge (2a) \leq x_2$). Therefore, $x_1 \wedge (2a) = a = 2a - x_1 \wedge (2a)$, hence in particular $x_1 \wedge (2a) = a$. Similarly, $x_2 \wedge (2a) = a$. Therefore, we can write x_1 as a disjoint sum $x_1 = x_1' + a$ (a, x_1' are quasi-units, or components, of x_1). In the same way, $x_2 = x_2' + a$ (disjoint sum).

Now consider the τ-closed set $B = \{x \in A : x \geq a\}$. As in the proof of Theorem 4.1, we show that the family of τ-closed extreme subsets of B has a maximal element; moreover, such an element is a singleton $\{b\}$. It remains to prove that b is an order extreme point of A. Indeed, suppose $x_1, x_2 \in A$ satisfy $2b \leq x_1 + x_2$. A fortiori, $2a \leq x_1 + x_2$, hence, by the preceding paragraph, $x_1, x_2 \in B$. Thus, $x_1 = b = x_2$. □

Remark 5.4. It is well-known that the set of all extreme points of a compact metrizable set is G_δ. The same can be said for the set of order extreme points of A, whenever A is a closed solid bounded subset of a separable reflexive Banach lattice. Indeed, then the weak topology is induced by a metric d. For each n let F_n be the set of all $x \in A$ for which there exist $x_1, x_2, \in A$ with $x \leq (x_1 + x_2)/2$, and $d(x_1, x_2) \geq 1/n$. By compactness, F_n is closed. Now observe that $\cup_n F_n$ is the complement of the set of all order extreme points.

6. Examples: AM-spaces and their relatives

The following example shows that, in some cases, $\mathbf{B}(X)$ is much larger than the closed convex hull of its extreme points, yet is equal to the closed solid convex hull of its order extreme points.

Proposition 6.1. *For a Banach lattice X, $\mathbf{B}(X)$ is the (closed) solid convex hull of n disjoint elements if and only if X is lattice isometric to $C(K_1) \oplus_1 \ldots \oplus_1 C(K_n)$.*

Proof. Clearly, the only order extreme points of $\mathbf{B}(C(K_1) \oplus_1 \ldots \oplus_1 C(K_n))$ are $\mathbf{1}_{K_i}$, with $1 \leq i \leq n$.

Conversely, suppose $\mathbf{B}(X) = \mathrm{CSCH}(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathbf{B}(X)_+$ are disjoint. It is easy to see that, in this case, $\mathbf{B}(X) = \mathrm{SCH}(x_1, \ldots, x_n)$. Let E_i be the ideal generated in X by x_i – that is, the set of all $x \in X$ for which there exists $c > 0$ so that $|x| \leq c|x_i|$. Note that, for such x, $\|x\|$ is the infimum of all c's with the above property. Indeed, if $|x| \leq |x_i|$, then clearly $x \in \mathbf{B}(X)$. Conversely, suppose $x \in \mathbf{B}(X) \cap E_i$ – that is, $|x| \leq cx_i$ for some c, and also $|x| \leq \sum_j t_j x_j$, with $t_j \geq 0$, and $\sum_j t_j = 1$. Then $|x| \leq (cx_i) \wedge (\sum_j t_j x_j) = (c \wedge t_i)x_i$. Consequently, E_i (with the norm inherited from X) is an AM-space, whose strong unit is x_i. We therefore identify E_i with $C(K_i)$, for some Hausdorff compact K_i.

Further, Proposition 2.1 shows that X is the direct sum of the ideals E_i: any $y \in X$ has a unique disjoint decomposition $y = \sum_{i=1}^n y_i$, with $y_i \in E_i$. We have to show that $\|y\| = \sum_i \|y_i\|$. Indeed, suppose $\|y\| \leq 1$ – that is, $|y| = \sum_i |y_i| \leq \sum_j t_j x_j$, with $t_j \geq 0$, and $\sum_j t_j = 1$. Note that $\|y_i\| \leq 1$ for every i, or equivalently, $|y_i| \leq x_i$. Therefore,

$$|y_i| = |y| \wedge x_i = \big(\sum_j t_j x_j\big) \wedge x_i = t_i,$$

which leads to $\|y_i\| \leq t_i$; consequently, $\|y\| \leq \sum_i t_i \leq 1$. □

Example 6.2. For $X = (C(K_1) \oplus_1 C(K_2)) \oplus_\infty C(K_3)$, order extreme points of $\mathbf{B}(X)$ are $\mathbf{1}_{K_1} \oplus_\infty \mathbf{1}_{K_3}$ and $\mathbf{1}_{K_2} \oplus_\infty \mathbf{1}_{K_3}$; $\mathbf{B}(X)$ is the solid convex hull of these points. Thus, the word "disjoint" in the statement of Proposition 6.1 cannot be omitted.

Note that $\mathbf{B}(C(K))$ is the closed solid convex hull of its only order extreme point – namely, $\mathbf{1}_K$. This is the only type of AM-spaces with this property.

Proposition 6.3. *Suppose X is an AM-space, and $\mathbf{B}(X)$ is the closed solid convex hull of finitely many of its elements. Then $X = C(K)$ for some Hausdorff compact K.*

Proof. Suppose $\mathbf{B}(X)$ is the closed solid convex hull of $x_1, \ldots, x_n \in \mathbf{B}(X)_+$. Then $x_0 := x_1 \vee \ldots \vee x_n \in \mathbf{B}(X)_+$ (due to X being an AM-space), hence $x \in \mathbf{B}(X)$ iff $|x| \leq x_0$. Thus, x_0 is the strong unit of X. □

Proposition 6.4. *If X is an AM-space, and $\mathbf{B}(X)$ has an order extreme point, then X is lattice isometric to $C(K)$, for some Hausdorff compact K.*

Proof. Suppose a is order extreme point of $\mathbf{B}(X)$. We claim that a is a strong unit – that is, $a \geq x$ for any $x \in \mathbf{B}(X)_+$. Suppose, for the sake of contradiction, that the inequality $a \geq x$ fails for some $x \in \mathbf{B}(X)_+$. Then $b = a \vee x \in \mathbf{B}(X)_+$ (due to the definition of an AM-space), and $a \leq (a+b)/2$, contradicting the definition of an order extreme point. □

We next consider norm-attaining functionals. It is known that, for a Banach space X, any element of X^* attains its norm iff X is reflexive. If we restrict ourself to positive functionals on a Banach lattice, the situation is different: clearly every positive functional on $C(K)$ attains it norm at $\mathbf{1}$. Below we show that, among separable AM-spaces, only $C(K)$ has this property.

Proposition 6.5. *Suppose X is a separable AM-space, so that every positive linear functional attains its norm. Then X is lattice isometric to $C(K)$.*

Proof. Let $(x_i)_{i=1}^\infty$ be a dense sequence in $\mathbf{S}(X)_+$. For each i find $x_i^* \in \mathbf{B}(X_+^*)$ so that $x_i^*(x_i) = 1$. Let $x^* = \sum_{i=1}^\infty 2^{-i} x_i^*$. We shall show that $\|x^*\| = 1$. Indeed, $\|x^*\| \leq \sum_i 2^{-i} = 1$ by the triangle inequality. For the opposite inequality, fix $N \in \mathbb{N}$, and let $x = x_1 \vee \ldots \vee x_N$. Then $x \in \mathbf{S}(X)_+$, and

$$\|x^*\| \geq x^*(x) \geq \sum_{i=1}^N 2^{-i} x_i^*(x) \geq \sum_{i=1}^N 2^{-i} x_i^*(x_i) = \sum_{i=1}^N 2^{-i} = 1 - 2^{-N}.$$

As N can be arbitrarily large, we obtain the desired estimate on $\|x^*\|$.

Now suppose x^* attains its norm on $a \in \mathbf{S}(X)_+$. We claim that a is the strong unit for X. Suppose otherwise; then there exists $y \in \mathbf{B}(X)_+$ so that $a \geq y$ fails. Let $b = a \vee y$, then $z = b - y$ belongs to $X_+ \backslash \{0\}$. Then $1 \geq x^*(b) \geq x^*(a) = 1$, hence $x^*(z) = 0$. However, x^* cannot vanish at z. Indeed, find i so that $\|z/\|z\| - x_i\| < 1/2$. Then $x_i^*(z) \geq \|z\|/2$, hence $x(z) > 2^{-i-1} > 0$. This gives the desired contradiction. □

In connection to this, we also mention a result about norm-attaining functionals on order continuous Banach lattices.

Proposition 6.6. *An order continuous Banach lattice X is reflexive if and only if every positive linear functional on it attains its norm.*

Proof. If an order continuous Banach lattice X is reflexive, then clearly every linear functional is norm-attaining. If X is not reflexive, then, by the classical result of James, there exists $x^* \in X^*$ which does not attain its norm. We show that $|x^*|$ does not either.

Let $B_+ = \{x \in X : x_+^*(|x|) = 0\}$, and define B_- similarly. As all linear functionals on X are order continuous [20, Section 2.4], B_+ and B_- are bands [20, Section 1.4]. Due to the order continuity of X [20, Section 2.4], $B_\pm$ are ranges of band projections $P_\pm$. Let B be the range of $P = P_+ P_-$; let B_+^o be the range of $P_+^o = P_+ P_-^\perp = P_+ - P$ (where we set $Q^\perp = I_X - Q$), and similarly for B_-^o and P_-^o. Note that $P_+^o + P_-^o = P^\perp$.

Suppose for the sake of contradiction that $x \in \mathbf{S}(X)_+$ satisfies $|x^*|(x) = \|x^*\|$. Replacing x by $P^\perp x$ if necessary, we can assume that $Px = 0$, so

$x = P_+^o x + P_-^o x$. Then $\|P_+^o x - P_-^o x\| = 1$, and

$$\begin{aligned} x^*\big(P_-^o x - P_+^o x\big) &= x_+^*\big(P_-^o x\big) - x_+^*\big(P_+^o x\big) - x_-^*\big(P_-^o x\big) + x_-^*\big(P_+^o x\big) \\ &= x_+^*\big(P_-^o x\big) + x_-^*\big(P_+^o x\big) = |x^*|(x) = \|x^*\|, \end{aligned}$$

which contradicts our assumption that x does not attain its norm. □

7. On the number of order extreme points

It is shown in [18] that, if a Banach space X is reflexive and infinite-dimensional Banach lattice, then $\mathbf{B}(X)$ has uncountably many extreme points. Here, we establish a similar lattice result.

Theorem 7.1. *If X is a reflexive infinite-dimensional Banach lattice, then $\mathbf{B}(X)$ has uncountably many order extreme points.*

Note that, if X is a reflexive infinite-dimensional Banach lattice, then Theorems 5.1 and 7.1 imply that $\mathbf{B}(X)$ has uncountably many extreme points, re-proving the result of [18] in this case.

Proof. Suppose, for the sake of contradiction, that there were only countably many such points $\{x_n\}$. For each such x_n, we define $F_n = \{f \in \mathbf{B}(X^*)_+ : f(x_n) = \|f\|\}$. Clearly F_n is weak* (= weakly) compact.

By the reflexivity of X, any $f \in \mathbf{B}(X^*)$ attains its norm at some $x \in \mathrm{EP}(\mathbf{B}(X))$. Since $f(x) \leq |f|(|x|)$ we can assume that any positive functional achieves its norm on a positive extreme point in $\mathbf{B}(X)$. By Theorem 5.1, these are precisely the order extreme points. Therefore $\bigcup F_n = \mathbf{B}(X^*)_+$. By the Baire Category Theorem, one of these sets F_n must have non-empty interior in $\mathbf{B}(X^*)_+$.

Assume it is F_1. Pick $f_0 \in F_1$, and $y_1, ..., y_k \in X$, such that if $f \in \mathbf{B}(X^*)_+$ and for each y_i, $|f(y_i) - f_0(y_i)| < 1$, then $f \in F_1$. Without loss of generality, we can assume that $\|f_0\| < 1$, and also that each $y_i \geq 0$. For the latter, replace y_i with $2y_{i+}, 2y_{i-}$ and require that the difference be less than 1. This will give a non-empty open subset of the original chosen set.

Further, we can and do assume that there exist mutually disjoint $u_1, u_2, \ldots \in \mathbf{S}(X)_+$ which are disjoint from $y = \vee_i y_i$. Indeed, find mutually disjoint $z_1, z_2, \ldots \in \mathbf{S}(X)_+$. Denote the corresponding band projections by $P_1, P_2, \ldots$ (such projections exist, due to the σ-Dedekind completeness of X). Then the vectors $P_n y$ are mutually disjoint, and dominated by y. As X is reflexive, it must be order continuous, and therefore, $\lim_n \|P_n y\| = 0$. Find $n_1 < n_2 < \ldots$ so that $\sum_j \|P_{n_j} y\| < 1/2$. Let $w_i = \sum_j P_{n_j} y_i$ and $y'_i = 2(y_i - w_i)$. Then if $|(f_0 - g)(y'_i)| < 1$, with $g \geq 0, \|g\| \leq 1$, it follows that

$$\begin{aligned} |(f_0 - g)(y_i)| &\leq \frac{1}{2}(|(f_0 - g)(y'_i)| + |(f_0 - g)(w_i)|) \\ &\leq \frac{1}{2}(1 + \|f_0 - g\|\|w_i\|) < \frac{1}{2}(1 + 2 \cdot \frac{1}{2}) = 1 \end{aligned}$$

We can therefore replace y_i with y'_i to ensure sufficient conditions for being in F_1. Then the vectors $u_j = z_{n_j}$ have the desired properties. Let P be the band projection complementary to $\sum_j P_{n_j}$ (in other words, complementary to the the band projection of $\sum_j 2^{-j} u_j$); then $Py_i = y_i$ for any i.

By [20, Lemma 1.4.3], there exist linear functionals $g_j \in \mathbf{S}(X^*)_+$ so that $g_j(u_j) = 1$, $g_j(u_k) = 0$ if $j \neq k$, and $g_j|_{\mathrm{ran}\, P} = 0$. For $j \in \mathbb{N}$ find $\alpha_j \in [1 - \|P^* f_0\|, 1]$ so that $\|f_j\| = 1$, where $f_j = P^* f_0 + \alpha_j f_j$. Then, for $1 \leq i \leq k$, $f_j(y_i) = (P^* f_0)(y_i) + \alpha_j g_j(y_i) = f_0(y_i)$, which implies that, for every j, f_j belongs to F_1, hence attains its norm at x_1. This, however, is impossible. Note that $\lim_j g_j(x_1) = 0$ (if $\inf_k g_{j_k}(x_1) > 0$, then $\mathrm{span}[g_{j_k} : k \in \mathbb{N}]$ is isomorphic to ℓ_1, contradicting the reflexivity of X). Thus, $\lim_j f_j(x_1) = f_0(Px_1) \leq \|f_0\| < 1$. □

Corollary 7.2. *Suppose C is a closed, bounded, solid, convex subset of a reflexive Banach lattice, having non-empty interior. Then C contains uncountably many order extreme points.*

Proof. We can assume without loss of generality that $\sup_{x \in C} \|x\| = 1$. Note that 0 is an interior point of C. Indeed, suppose x is an interior point – that is, $x + \varepsilon \mathbf{B}(X) \subset C$ for some $\varepsilon > 0$. For any k such that $\|k\| < \varepsilon$, we have $\frac{k}{2} = \frac{-x}{2} + \frac{x+k}{2} \in C$, since C is solid and convex. Hence $\frac{\varepsilon}{2}\mathbf{B}(X) \subseteq C$. Since C is bounded, we can then define an equivalent norm, with $\|y\|_C = \inf\{\lambda > 0 : y \in \lambda C\}$. Since C is solid, $\|y\|_C = \| \, |y| \, \|_C$, and the norm is consistent with the order. Finally, $\| \cdot \|_C$ is equivalent to $\| \cdot \|$, since for all $y \in X$, we have that $\frac{\varepsilon}{2}\|y\|_C \leq \|y\| \leq \|y\|_C$. The conclusion follows by Theorem 7.1. □

8. The solid Krein-Milman Property and the RNP

We can say that a Banach lattice (or, more generally, an ordered Banach space) X has the *Solid Krein-Milman Property* (*SKMP*) if every solid closed bounded subset of X is the closed solid convex hull of its order extreme points. This is analogous to the canonical Krein-Milman Property (KMP) in Banach spaces, which is defined in the similar manner, but without any references to order. It follows from Theorem 5.1 that the KMP implies the SKMP.

These geometric properties turn out to be related to the Radon-Nikodým Property (RNP). It is known that the RNP implies the KMP, and, for Banach lattices, the converse is also true (see [7] for a simple proof). For more information about the RNP in Banach lattices, see [20, Section 5.4]; a good source of information about the RNP in general is [6] or [10].

One of the equivalent definitions of the RNP of a Banach space X involves integral representations of operators $T : L_1 \to X$. If X is a Banach lattice, then, by [22, Theorem IV.1.5], any such operator is regular (can be expressed as a difference of two positive ones); so positivity comes naturally into the picture.

Theorem 8.1. *For a Banach lattice X, the SKMP, KMP, and RNP are equivalent.*

Proof. The implications RNP $\Leftrightarrow$ KMP $\Rightarrow$ SKMP are noted above. Now suppose X fails the RNP (equivalently, the KMP). We shall establish the failure of the SKMP in two different ways, depending on whether X is a KB-space, or not.

(1) If X is not a KB-space, then [20, Sections 2.4-5] there exist disjoint $e_1, e_2, \ldots \in \mathbf{S}(X)_+$, equivalent to the canonical basis of c_0. Then the set

$$C = \overline{\mathrm{S}\Big(\big\{\sum_i \alpha_i e_i : \max_i |\alpha_i| = 1,\, \lim_i \alpha_i = 0\big\}\Big)}$$

is solid, balanced, bounded, and closed. To give a more intuitive description of C, for $x \in X$ we let $x_i = |x| \wedge e_i$. It is easy to see that $x \in C$ if and only if $\lim_i \|x_i\| = 0$, and $|x| = \sum_i x_i$. Finally, show that $x \in C_+$ cannot be an order extreme point. Find i so that $\|x_i\| < 1/2$, and consider $x' = \sum_{j \neq i} x_j + e_i$. Then clearly $x' \in C$, and $x' - x \in X_+ \backslash \{0\}$.

(2) If X is a KB-space, then, by the reasoning in the proof of the main theorem of [7], there exists a closed convex set $D \subset \mathbf{B}(X)_+$ with no extreme points. By Propositions 2.1 and 2.4, the set $C = \mathrm{S}(D)$ is convex and closed; it is clearly bounded and solid. However, C has no order extreme points, since all such points would have to be extreme points of D. □

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