A NOTE ON UNIVERSAL OPERATORS

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Abstract. An operator T is called universal for the complement of the ideal $\mathfrak{A}$ if T does not belong to $\mathfrak{A}$, and factors through every element of the complement of $\mathfrak{A}$. We show that the complements of many ideals (such as the ideal of strictly (co)singular operators, or any maximal normed ideal) have no universal operators. On the other hand, the complement of the ideal of finitely strictly singular operators has a universal operator. Moreover, we show that, for many ideals $\mathfrak{A}$, any positive operator which factors positively through any positive member of the complement of $\mathfrak{A}$ must be compact.

1. Introduction

Suppose $\mathfrak{A}$ is an operator ideal (see e.g. [2] for a concise introduction). We denote by $\mathrm{C}\mathfrak{A}$ the complement of $\mathfrak{A}$. We say that $T \in B(X,Y)\backslash\mathfrak{A}(X,Y)$ is *universal* for $\mathrm{C}\mathfrak{A}$ if it *factors* through every operator $S \in \mathrm{C}\mathfrak{A}$ – that is, for any Banach spaces X_0 and Y_0, and any $S \in B(X_0,Y_0)\backslash\mathfrak{A}$, there exist $A \in B(X,X_0)$ and $B \in B(Y_0,Y)$ so that $T = BSA$. In the cases mentioned below, the existence of universal operators is known:

(1) $\mathfrak{A}$ is the ideal of compact operators: the formal identity from ℓ_1 to ℓ_∞ is a universal operator for $\mathrm{C}\mathfrak{A}$ [8].
(2) $\mathfrak{A}$ is the ideal of weakly compact operators: the summing operator from ℓ_1 to ℓ_∞ is a universal operator for $\mathrm{C}\mathfrak{A}$ [9].
(3) $\mathfrak{A}$ is the ideal of ℓ_p-strictly singular or c_0-strictly singular operators: any isomorphic embedding from ℓ_p (or c_0) to ℓ_∞ is a universal operator for $\mathrm{C}\mathfrak{A}$, cf. [2, Section 1].
(4) $\mathfrak{A}$ is the ideal of (c_0,p,q)-summing operators: the formal identity from ℓ_{p^*} to ℓ_q is a universal operator for $\mathrm{C}\mathfrak{A}$ [7].

The situation is more complicated for Dunford-Pettis operators [5].

In this note, we work further on the existence of universal operators. In Section 2, we examine some ideals closed in the operator norm – namely, the ideals of strictly singular, strictly cosingular, and finitely strictly singular operators. We show that the complements of the first two ideals have no universal operators, while the third one does. In Section 3 we prove that quasi-normed ideals, verifying certain (very general) conditions, have no universal elements. These results apply, for instance, to all maximal Banach ideals. In Section 4, we switch to the Banach lattice setting. Suppose X, Y, X_0, Y_0 are Banach lattices. We say that $T \in B(X,Y)_+$ *positively factors* through $S \in B(X_0,Y_0)_+$ if there exist $A \in B(X,X_0)_+$ and $B \in B(Y_0,Y)_+$

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so that $T = BSA$. $T \in \mathfrak{C}\mathfrak{A}(X,Y)_+$ is *positively universal* for $\mathfrak{C}\mathfrak{A}$ if it positively factors through every member of $\mathfrak{C}\mathfrak{A}_+$. We establish the non-existence of positively universal operators, for a wide class of operator ideals.

2. Ideals closed in the operator norm

First recall some definitions: an operator $T \in B(X,Y)$ is called

(1) *strictly singular* if, for any infinite dimensional $E \subset X$, the restriction $T|_E$ is not an isomorphism.
(2) *strictly cosingular* if, for any infinite dimensional quotient $q : Y \to F$, qT is not surjective.
(3) *finitely strictly singular* if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that, for any N-dimensional subspace $E \subset X$, we can find a norm one $e \in E$ so that $\|Te\| < \varepsilon$.

We refer the reader to [1] for the first two ideals, and to [12] for the third one. Note that recently, the class of of finitely strictly singular operators has been studied intensively, see e.g. [4] and [6].

First we show that the complements of strictly (co)singular operators possess no universal objects.

Proposition 2.1. *If an operator T factors through every non-strictly (co)singular operator, then T itself is strictly (co)singular.*

Proof. Suppose, for the sake of contradiction, that $T \in B(X,Y)$ is not strictly singular, and factors through any non-strictly operator. By restricting the domain of X, and extending its range, we may assume that $T : X \to \ell_\infty(\Gamma)$ is an isomorphism into. Clearly T factors through I_{ℓ_p} for $1 \leqslant p \leqslant \infty$, and through I_{c_0}. Thus, X embeds isomorphically into ℓ_p for any p. Thus, X is ℓ_p-saturated for any p, which is clearly impossible.

The case of strictly cosingular operators is handled similarly. Suppose $T \in B(X,Y)$ is not strictly cosingular, and factors through any non-strictly cosingular operator. By composing X with quotient maps $\ell_1(\Gamma) \to X$ and $q : Y \to Y/Y_0$, we can assume that $T \in B(\ell_1(\Gamma), Y)$ is such that $T(\mathbf{B}(\ell_1(\Gamma))) \supset c\mathbf{B}(Y)$ (here and below, $\mathbf{B}(Z)$ stands for the closed unit ball of the normed space Z). T factors through ℓ_p for $1 < p < \infty$, hence the same is true for $T^* \in B(Y^*, \ell_\infty(\Gamma))$. As T^* is an isomorphism, be obtain a contradiction, as in the strictly singular case. ■

The situation is different for finitely strictly singular operators.

Proposition 2.2. *There exists an operator $T \in B((\sum_n \ell_2^n)_{\ell_1}, \ell_\infty)$ which is not finitely strictly singular, and factors through every non-finitely strictly singular operator.*

The folklore result below (needed to prove Proposition 2.2) can be established by emulating the proof of [10, Lemma 1.a.6].

Lemma 2.3. *Suppose E is a finite dimensional subspace of an infinite dimensional space X. For any $c > 1$, X contains a finite codimensional subspace Y so that, for any $e \in E$ and $y \in Y$, $c\|e + y\| \geqslant \|e\|$.*

Proof of Proposition 2.2. Let $id : (\sum_n \ell_2^n)_{\ell_1} \to (\sum_n \ell_2^n)_{c_0}$ be the formal identity, let $j : (\sum_n \ell_2^n)_{c_0} \to \ell_\infty$ be an isometric embedding, and set $T = j \circ id$. Clearly, T is not finitely strictly singular. We establish its universality.

Suppose a contraction $S \in B(X, Y)$ is not FSS. By Dvoretzky Theorem, X contains 2-Hilbertian subspaces E of arbitrarily large dimension so that $\|Sx\| \geqslant c\|x\|$ for any $x \in E$ (here $c > 0$ is a constant depending only on S). We now construct a sequence of 2-Hilbertian subspaces E_n, of dimension n, so that $\|Sx\| \geqslant c\|x\|$ for any $x \in E_n$, and let $F_n = S(E_n)$. The sequence (E_n) must have a special property: for any finite sequences $e_1 \in E_1, \ldots, e_n \in E_n, f_1 \in F_1, \ldots, f_n \in F_n$, we have $\|e_1 + \ldots + e_k\| \leqslant 2\|e_1 + \ldots + e_n\|$, $\|f_1 + \ldots + f_k\| \leqslant 2\|f_1 + \ldots + f_n\|$ whenever $k \leqslant n$. Once this is done, we obtain a factorization of T through S: for each n, pick a contraction $u_n \in B(\ell_2^n, E_n)$ so that $\|u_n^{-1}\| \leqslant 2$. Then $U : (\sum_n \ell_2^n)_1 \to \mathrm{span}[E_1, E_2, \ldots] \subset X : (\xi_1, \xi_2, \ldots) \mapsto \sum_k u_k \xi_k$ is a contraction. Let $v_n = u_n^{-1}(S|_{E_n})^{-1} : F_n \to \ell_2^n$, and observe that $\|v_n\| \leqslant 2c^{-1}$. Define $V : \mathrm{span}[F_1, F_2, \ldots] \to (\sum_n \ell_2^n)_{c_0}$ via $V(f_1 + \ldots + f_n) = (v_1 f_1, \ldots, v_n f_n)$. We claim that V is a bounded well-defined operator. Indeed, by construction, for $k \leqslant n$ we have

$$\|f_k\| \leqslant \|f_1 + \ldots + f_k\| + \|f_1 + \ldots + f_{k-1}\| \leqslant 4\|f_1 + \ldots + f_n\|.$$

Consequently,

$$\|V(f_1 + \ldots + f_n)\| = \max_{1 \leqslant k \leqslant n} \|v_k f_k\| \leqslant 8c^{-1}\|f_1 + \ldots + f_n\|.$$

Now extend V to $W : Y \to \ell_\infty$ using injectivity. Then $T = WSU$.

The sequences (E_n) and (F_n) are constructed recursively. E_1 can be selected arbitrarily. Set $F_1 = S(E_1)$. Use Lemma 2.3 to find finite codimensional $X_1 \subset X$ and $Y_1 \subset Y$ so that $2\|e_1 + x_1\| \geqslant \|e_1\|$ and $2\|f_1 + y_1\| \geqslant \|f_1\|$ whenever $e_1 \in E_1$, $f_1 \in F_1$, $x_1 \in X_1$, and $y_1 \in Y_1$.

Now suppose we have already constructed $E_1, \ldots, E_n$, $F_1, \ldots, F_n$, $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ in such a way that, for $1 \leqslant k \leqslant n$:

(1) $E_k \subset X$ and $F_k = S(E_k) \subset Y$ are finite dimensional, while $X_n \subset \ldots \subset X_1 \subset X$ and $Y_n \subset \ldots \subset Y_1 \subset Y$ are finite codimensional.
(2) For $k > 1$, $E_k \subset X_{k-1}$, and $F_k \subset Y_{k-1}$.
(3) For any $x \in X_k$, $e \in E_1 + \ldots + E_k$, $y \in Y_k$, and $f \in F_1 + \ldots + F_k$, we have $\|e\| \leqslant 2\|e + x\|$ and $\|f\| \leqslant 2\|f + y\|$.

Pick a 2-Hilbertian $E \subset X$, of dimension $n + 1 + \mathrm{codim}\, X_n + \mathrm{codim}\, Y_n$, so that $\|Se\| \geqslant c\|e\|$ for any $e \in E$. Then the dimension of $E' = E \cap X_n \cap S^{-1}(Y_n)$ is at least $n+1$. Clearly, any $(n+1)$-dimensional subspace of E' can serve as E_{n+1}. Complete the procedure by selecting $X_{n+1} \subset X_n$ and $Y_{n+1} \subset Y_n$ using Lemma 2.3. ■

3. Normed ideals

In this section, $(\mathfrak{A}, \boldsymbol{\alpha})$ is a quasi-Banach ideal of operators on Banach spaces. It is well-known that such ideals have an equivalent ideal p-norm, for some $p \in (0, 1]$ (see e.g. [2, Section 2]). Throughout this section, we assume that $\mathfrak{A}$ is p-normed, and satisfies two conditions:

$$\sup\left\{\boldsymbol{\alpha}(I_E) : \dim E < \infty\right\} = \infty. \tag{1}$$

and

$$\text{If } T \notin \mathfrak{A}(X,Y), \text{ then } \sup_{E \subset X, \dim E < \infty} \boldsymbol{\alpha}(T|_E) = \infty. \tag{2}$$

These conditions hold, for instance, for all maximal Banach operator ideals, different from the ideal of all bounded operators (see e.g. [3, Chapter 6]).

Theorem 3.1. *If $(\mathfrak{A}, \boldsymbol{\alpha})$ is a quasi-Banach operator ideal satisfying (1) and (2), then $C\mathfrak{A}$ has no universal operator.*

Begin by establishing a simple lemma.

Lemma 3.2. *Suppose $(\mathfrak{A}, \boldsymbol{\alpha})$ is a p-normed quasi-Banach operator ideal. Then there exists a constant C so that, for any finite dimensional normed space E, $\boldsymbol{\alpha}(I_E) \leqslant C(\dim E)^{1/p}$.*

Proof. Note first that, for any rank one operator $x^* \otimes y \in B(X,Y)$ (X and Y are Banach spaces), $\boldsymbol{\alpha}(x^* \otimes y) = C\|x^*\|\|y\|$, where the constant C depends solely on $\mathfrak{A}$. Indeed, let $C = \boldsymbol{\alpha}(I_{\mathbb{F}})$, where $\mathbb{F}$ is the underlying vector field ($\mathbb{R}$ or $\mathbb{C}$).

Now consider non-zero $x^* \in X^*$ and $y \in Y$. To show that $\boldsymbol{\alpha}(x^* \otimes y) \geqslant C\|x^*\|\|y\|$, consider the contractions $j : \mathbb{F} \to X : 1 \mapsto x$ (with $\|x\| = 1$) and $q : Y \to \mathbb{F} : z \mapsto \langle y^*, z\rangle$ (with norm one y^* selected in such a way that $\|y\| = \langle y^*, y\rangle$). Note that $q \circ (x^* \otimes y) \circ j = \langle x^*, x\rangle\|y\|I_{\mathbb{F}}$, hence $\boldsymbol{\alpha}(x^* \otimes y) \geqslant \boldsymbol{\alpha}\big(q \circ (x^* \otimes y) \circ j\big) \geqslant |\langle x^*, x\rangle|\|y\|\boldsymbol{\alpha}(I_{\mathbb{F}}) = C|\langle x^*, x\rangle|\|y\|$. Taking the supremum over all $x \in X$ of norm one, we obtain $\boldsymbol{\alpha}(x^* \otimes y) \geqslant C\|x^*\|\|y\|$. The inequality $\boldsymbol{\alpha}(x^* \otimes y) \leqslant C\|x^*\|\|y\|$ is proved in a similar fashion.

Now suppose E is an n-dimensional space. Using the Auerbach basis of E, write $I_E = P_1 + \ldots + P_n$, where $P_1, \ldots, P_n$ are contractive projections of rank one. Then $\boldsymbol{\alpha}(I_E) \leqslant \big(\sum_i \boldsymbol{\alpha}(P_i)^p\big)^{1/p} = Cn^{1/p}$. ■

Next we quantify the "non-belonging" to $\mathfrak{A}$. For $T \in B(X,Y)$ and $n \in \mathbb{N}$, define

$$\beta_n(T) = \sup\big\{\boldsymbol{\alpha}(T|_E) : E \subset X, \dim E = n\big\}.$$

Note that, by the ideal property, $\boldsymbol{\alpha}(T|_E) \geqslant \boldsymbol{\alpha}(T|_F)$ if $F \subset E$, hence, in the definition of β_n, we could have taken the supremum over all $E \subset X$ of dimension not exceeding n. We have:

Lemma 3.3. *If $S \in B(X_0, Y_0)$ factors through $T \in B(X,Y)$, then there exists a constant c so that $\beta_n(S) \leqslant c\beta_n(T)$ for every n.*

Proof. Write $S = VTU$. If E is a finite dimensional subspace of X_0, then, by the ideal property, $\boldsymbol{\alpha}(S|_E) \leqslant \|V\|\boldsymbol{\alpha}(T|_{U(E)})\|U\|$, and therefore,

$$\begin{aligned}\beta_n(S) = \sup_{\dim E = n} \boldsymbol{\alpha}(S|_E) &\leqslant \|V\| \sup_{\dim E = n} \boldsymbol{\alpha}(T|_{U(E)})\|U\| \\ &\leqslant \|U\|\|V\| \sup_{\dim F \leqslant n} \boldsymbol{\alpha}(T|_F) = \|U\|\|V\|\beta_n(T),\end{aligned}$$

as claimed. ■

The following technical lemma is crucial.

Lemma 3.4. *Suppose the quasi-normed ideal* $(\mathfrak{A}, \boldsymbol{\alpha})$ *satisfies* (1)*. Then, for any sequence* $0 < \alpha_1 < \alpha_2 < \ldots$*, increasing without a bound, there exists an operator* T*, so that* $\lim_n \beta_n(T) = \infty$*, and* $\beta_n(T) \leqslant \alpha_n$ *for infinitely many values of* n*.*

Proof. By the discussion in the beginning of the section, and by Lemma 3.2, we can assume that $\boldsymbol{\alpha}$ is a p-norm, and furthermore (after a renorming) that $\boldsymbol{\alpha}(I_E) \leqslant (\dim E)^{1/p}$ for any finite dimensional normed space E. We shall find finite dimensional spaces E_i and an operator $T = \oplus \gamma_i I_{E_i}$, acting on $X = \big(\sum_i E_i\big)_2$, with the desired properties. We use the notation $k_i = \dim E_i$, $c_i = \boldsymbol{\alpha}(I_{E_i})$, and $\sigma_i = \gamma_i c_i$. The parameters will be selected in such a way that:

$$\begin{aligned} &\sigma_i < \frac{\sigma_{i+1}}{2^{1/p}} \text{ for any } i. \\ &\gamma_n < \frac{\sigma_i}{2^{(n-i)/p} k_i^{1/p}} \text{ whenever } n > i. \\ &\sigma_i < \frac{\alpha_{k_i}}{3} \text{ for any } i. \end{aligned} \tag{3}$$

Suppose for a moment the selection has already been made. As $E_n = T(E_n)$ is contractively complemented in X,

$$\beta_{k_n}(T) \geqslant \boldsymbol{\alpha}(T|_{E_n}) = \sigma_n,$$

and we conclude that $\lim_n \beta_{k_n}(T) = \infty$. On the other hand, suppose F is a subspace of X, with $\dim F = k_n$. Let P_i be the canonical projection onto E_i, and set $F_i = P_i(F)$. Then $\boldsymbol{\alpha}(T|_F)^p \leqslant \sum_i \boldsymbol{\alpha}(T|_{F_i})^p$. Note that, for $i \leqslant n$, $\boldsymbol{\alpha}(T|_{F_i}) \leqslant \boldsymbol{\alpha}(T|_{E_i}) = \sigma_i$, while for $i > n$, by the conditions imposed on $\mathfrak{A}$,

$$\boldsymbol{\alpha}(T|_{F_i}) \leqslant \gamma_i (\dim F_i)^{1/p} \leqslant \gamma_i k_n^{1/p} \leqslant \frac{\sigma_n}{2^{(i-n)/p}}.$$

Therefore,

$$\boldsymbol{\alpha}(T|_F)^p \leqslant \sum_{i \leqslant n} \sigma_i^p + \sum_{i > n} \frac{\sigma_n^p}{2^{i-n}} \leqslant \sum_{i \leqslant n} \frac{\sigma_n^p}{2^{n-i}} + \sum_{i > n} \frac{\sigma_n^p}{2^{i-n}} \leqslant 3\sigma_n^p < \alpha_{k_n}^p.$$

It remains to construct the sequences satisfying (3). To "prime the pump", select $k_1 = 1$. Then $c_1 = 1$. Let $\gamma_1 = \alpha_1 / c_1$.

Now suppose $E_1, \gamma_1, \ldots, E_s, \gamma_s$ have already been selected to satisfy (3). Let

$$\gamma = \min_{1 \leqslant i \leqslant s} 2^{i-1-s} \sigma_i k_i^{-1/p}, \quad \sigma = 2^{1/p} \sigma_s, \text{ and } c = \frac{\sigma}{\gamma}.$$

Find $k > 3k_s$ so that $\boldsymbol{\alpha}(I_E) > c$ for some k-dimensional E, and $\alpha_k > 8^{1/p}\sigma$. Set $E_{s+1} = E$. Then $k_{s+1} = k$, and $c_{s+1} = \boldsymbol{\alpha}(I_E)$. Finally, set

$$\gamma_{s+1} = \min\left\{\gamma, \frac{\alpha_{k_{s+1}}}{4c_{s+1}}\right\}.$$

Then

$$\sigma_{s+1} = \gamma_{s+1} c_{s+1} = \min\left\{\gamma c_{s+1}, \frac{\alpha_{k_{s+1}}}{4}\right\} \geqslant \sigma \geqslant 2^{1/p} \sigma_s.$$

Thus, the first part of (3) holds for $s+1$. It is even easier to check the second and third parts. ∎

Proof of Theorem 3.1. Suppose, for the sake of contradiction, that $\mathbf{C}\mathfrak{A}$ contains a universal operator S. By Lemma 3.3, for any $T \in \mathbf{C}\mathfrak{A}$ there exists $c > 0$ so that $\beta_n(T) \geqslant c\beta_n(S)$ for any n. Furthermore, by the conditions imposed of $\mathfrak{A}$, $\lim_n \beta_n(S) = \infty$. However, by Lemma 3.4, there exists $T \in \mathbf{C}\mathfrak{A}$ for which $\beta_n(T) \leqslant \sqrt{\beta_n(S)}$ infinitely often. This yields a contradiction. ∎

4. Positively universal operators

In this section we narrow our attention to positive operators on Banach lattices.

Theorem 4.1. *Suppose $\mathfrak{A}$ is an operator ideal, not containing an isomorphic embedding of ℓ_1 into ℓ_∞. If T positively factors through every $S \in \mathbf{C}\mathfrak{A}$, then T is compact.*

Throughout this section, we work with real lattices. The complex case can be obtained with minor modifications.

Remark 4.2. (1) If an ideal $\mathfrak{A}$ contains an into isomorphism $S : \ell_1 \to \ell_\infty$, then $B(\ell_1, \ell_\infty) \subset \mathfrak{A}$. Indeed, consider $T \in B(\ell_1, \ell_\infty)$. Find $U : S(\ell_1) \to \ell_1$, so that $US = I_{\ell_1}$. By the injectivity of ℓ_∞, the operator $V = TU : S(\ell_1) \to \ell_\infty$ has an extension $W : \ell_\infty \to \ell_\infty$. It is easy to see that $T = WS$, hence T belongs to $\mathfrak{A}$.

(2) Any into isomorphism $S : \ell_1 \to \ell_\infty$ is universal for the complement of the ideal of ℓ_1-strictly singular operators, see [2, 1.18].

The following lemma (needed to establish Theorem 4.1) may be folklore, but we have not seen it in the literature. As before, we denote by $\mathbf{B}(Z)$ the unit ball of Z.

Lemma 4.3. *Suppose Z is a Banach lattice.*

1. *There exists a set I and a contractive positive map $q : \ell_1(I) \to Z$ so that $\mathbf{B}(Z) \subset 2q(\mathbf{B}(\ell_1(I)))$.*
2. *There exists a set J and a contractive positive map $j : Z \to \ell_\infty(J)$ so that $\|jz\| \geqslant \|z\|/2$ for any $z \in Z$.*

Proof. (1) Denote by I the positive part of $\mathbf{B}(Z)$. Then the map $q : \ell_1(I) \to Z : \delta_z \mapsto z$ is positive and contractive. Moreover, $\mathbf{B}(Z) \subset I - I \subset 2q(\mathbf{B}(\ell_1(I)))$.

(2) Let J be the set of all contractive positive functionals on Z, and define $j : Z \to \ell_\infty(J) : z \mapsto \big(z^*(z)\big)_{z^* \in J}$. Clearly, j is positive and contractive. Moreover, for any $z \in Z$ there exists $z^* \in \mathbf{B}(Z^*)_+$ so that $|z^*(z)| \geqslant \|z\|/2$. Indeed, consider the disjoint decomposition $z = z_+ - z_-$. Without loss of generality, assume that $\|z_+\| \geqslant 1/2$. Consider the subspace $W \subset Z$, spanned by z_+ and all elements w disjoint with z_+. By [11, Theorem 1.1.1], $\|z_+ + w\| \geqslant \|z_+\|$, for any w like this. By Hahn-Banach Theorem, there exists a norm one functional $w^* \in W^*$ so that $w^*(z_+) = \|z_+\|$, and $w^*(w) = 0$ whenever $w \perp z_+$. Then $z^* = w^*_+$ has norm not exceeding 1. Recall that (cf. [11, Section 1.3]), for any $u \in Z_+$, $z^*(u) = \sup\big\{w^*(v) : 0 \leqslant v \leqslant u\big\}$. Consequently, $z^*(z_+) = \|z_+\|$, and $z^*(z_-) = 0$. We conclude that $|z^*(z)| \geqslant \|z\|/2$. Therefore, $\|jz\| \geqslant \|z\|/2$ for any $z \in Z$. ∎

Proof of Theorem 4.1. Suppose, for the sake of contradiction, that a non-compact $T \in B(X, Y)_+$ positively factors through every $S \in B(\ell_1, \ell_\infty)_+ \backslash \mathfrak{A}$. By Lemma 4.3, there exists a set Γ, and positive contractive maps $q : \ell_1(\Gamma) \to X$ and $j : Y \to \ell_\infty(\Gamma)$, so that $\mathbf{B}(X) \subset 2q(\mathbf{B}(\ell_1(\Gamma)))$, and $\|jy\| \geqslant \|y\|/2$ for any $y \in Y$. If T positively factors through S, then so does jTq. Furthermore, jTq is not compact. Thus, we can assume that $X = \ell_1(\Gamma)$, and $Y = \ell_\infty(\Gamma)$.

Next we show that we can take Γ to be countable. To this end, denote the canonical transfinite basis of $\ell_1(\Gamma)$ by $(\delta_i)_{i\in\Gamma}$. The convex hull of a relatively compact set is relatively compact, hence there exists a countably infinite $I \subset \Gamma$ so that the family $(T\delta_i)_{i\in I}$ is c-separated, for some $c > 0$ (that is, $\|T\delta_i - T\delta_j\| > c$ for any distinct $i, j \in I$). Moreover, there exists a countable $G \subset \Gamma$ so that, for any distinct $i, j \in I$, there exists $g \in G$ with $|(T\delta_i - T\delta_j)_g| > c$.

For $\Lambda \subset \Gamma$, let $P_\Lambda \in B(\ell_1(\Gamma))$ be the corresponding coordinate projection. By the above, the family $(P_G^* T P_I \delta_i)_{i\in I}$ is c-separated, and in particular, $P_G^* T P_I$ is not compact. We establish our claim by identifying $G \cup I$ with $\mathbb{N}$.

Find a family of infinite subsets $A_i \subset \mathbb{N}$ ($i \in \mathbb{N}$) so that, for any $N \in \mathbb{N}$, and any sequence $(\varepsilon_i)_{i=1}^N \subset \{-1, 1\}^N$, $\cap_{i=1}^N A_i^{(\varepsilon_i)} \neq \emptyset$ (here, $A^{(1)} = A$, and $A^{(-1)} = \mathbb{N}\backslash A$). One can, or instance, let (q_i) be an enumeration of prime numbers, and define A_i as the set of multiples of q_i.

Then $U : \ell_1 \to \ell_\infty : \delta_i \mapsto u_i = \chi_{A_i} - \chi_{\mathbb{N}\backslash A_i}$ is an isometric embedding. Now for each $n \in \mathbb{N}$ define the operator $U_n : \ell_1 \to \ell_\infty : \delta_i \mapsto (1 - 2^{-n})\mathbf{1} + 2^{-n} u_i$ ($(\delta_i)_{i\in\mathbb{N}}$ is the canonical basis of ℓ_1). In other words, $U_n = (1 - 2^{-n})U_0 + 2^{-n}U$, where, for $x = (x_1, x_2, \ldots) \in \ell_1$, $U_0 x = (\sum_i x_i)\mathbf{1}$. Note that U_n is a contraction. Moreover, $U_n \notin \mathfrak{A}$, for any natural n. Indeed, U_0 has rank 1, and operator ideals are stable under finite rank perturbations.

Suppose, for the sake of contradiction, that there exists a non-compact $T \in B(\ell_1, \ell_\infty)_+$ so that, for every $n \in \mathbb{N}$, $T = B_n U_n A_n$, for some positive A_n and B_n. Let $v_i = T\delta_i$. By using TP_I instead of T if necessary, we can assume $\inf \|v_i\| > 0$. By considering TD, where D is a diagonal operator, we can further assume that $\|v_i\| = 1$ for any i. Finally, in light of the previous reasoning, we can assume that the sequence (v_i) is c-separated, for some $c \in (0, 1/2)$. Then, for $i \neq j$, either $v_i - (1 - c/2)v_j$ or $v_j - (1 - c/2)v_i$ is not positive. To establish this, write $v_i = (v_{is})_{s\in\mathbb{N}}$. For $i \neq j$, there exists $s \in \mathbb{N}$ so that $|v_{is} - v_{js}| > c$. If $v_{is} - v_{js} > c$, then

$$v_{js} - \Big(1 - \frac{c}{2}\Big)v_{is} < v_{js} - \Big(1 - \frac{c}{2}\Big)(v_{js} + c) = c\Big[\frac{v_{js}}{2} - \Big(1 - \frac{c}{2}\Big)\Big] \leqslant c\Big[\frac{1}{2} - \Big(1 - \frac{c}{2}\Big)\Big] < 0.$$

The case of $v_{js} - v_{is} > c$ is handled similarly.

Now fix $n > \log_2(4/c)$. Let $w_i = U_n A_n \delta_i$. We can write $A_n \delta_i = (a_{ik})_{k=1}^\infty$. Then $a_{ik} \geqslant 0$ for any i and k, and

$$\|A_n\| \geqslant \|A_n \delta_i\| = \alpha_i := \sum_k a_{ik} \geqslant \frac{\|B_n U_n A_n \delta_i\|}{\|B_n U_n\|} \geqslant \frac{1}{\|B_n\|}.$$

Pick $\lambda > 1$ so that $(1 - 2^{1-n})/\lambda > 1 - c/2$. By compactness, we can find $i \neq j$ so that $\lambda^{-1} < \alpha_i/\alpha_j < \lambda$. The desired contradiction will be achieved once we prove that both $w_i - (1 - c/2)w_j$ and $w_j - (1 - c/2)w_i$ are positive (indeed, then the positive operator B_n cannot take w_i and w_j to v_i and v_j, respectively).

Recall that $-\mathbf{1} \leqslant u_i \leqslant \mathbf{1}$, hence

$$w_i = \sum_k U_n a_{ik} \delta_k = (1 - 2^{-n})\alpha_i + 2^{-n} \sum_k a_{ik} u_k \in [(1 - 2^{1-n})\alpha_i \mathbf{1}, \alpha_i \mathbf{1}],$$

and similarly, $(1-2^{1-n})\alpha_j\mathbf{1} \leqslant w_j \leqslant \alpha_j\mathbf{1}$. Then

$$w_i - \Big(1-\frac{c}{2}\Big)w_j \geqslant (1-2^{1-n})\alpha_i\mathbf{1} - \Big(1-\frac{c}{2}\Big)\alpha_j\mathbf{1} \geqslant \alpha_i\Big[(1-2^{1-n})\lambda^{-1} - \Big(1-\frac{c}{2}\Big)\Big]\mathbf{1} \geqslant 0,$$

and $w_j - (1-c/2)w_i$ is tackled similarly. ∎

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References

[1] P. Aiena. *Fredholm and local spectral theory, with applications to multipliers*, Kluwer Academic Publishers, Dordrecht (2004).

[2] J. Diestel, H. Jarchow, and A. Pietsch. Operator ideals, *Handbook of the geometry of Banach spaces, Vol. I*, 437–496, North-Holland, Amsterdam, 2001.

[3] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, Cambridge University Press, Cambridge, 1995.

[4] J. Flores, F. Hernández, and Y. Raynaud. Super strictly singular and cosingular operators and related classes. (English summary) *J. Operator Theory* 67 (2012), 121–152.

[5] M. Girardi and W. Johnson. Universal non-completely-continuous operators, *Israel J. Math.* 99 (1997), 207–219.

[6] F. Hernández, Y. Raynaud, and E. Semenov. Bernstein widths and super strictly singular inclusions, *A panorama of modern operator theory and related topics*, 359–376, Oper. Theory Adv. Appl., 218, Birkhäuser/Springer, Basel, 2012.

[7] A. Hinrichs and A. Pietsch. The closed ideal of (c_0, p, q)-summing operators. *Integral Equations Operator Theory* 38 (2000), 302–316.

[8] W. Johnson. A universal non-compact operator, *Colloq. Math.* 23 (1971), 267–268.

[9] J. Lindenstrauss and A. Pełczynski. Absolutely summing operators in L_p-spaces and their applications, *Studia Math.* 29 (1968), 275–326.

[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, Berlin (1977).

[11] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin (1991).

[12] A. Plichko. Superstrictly singular and superstrictly cosingular operators, *Functional analysis and its applications*, 239–255, Elsevier, Amsterdam (2004).

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