LEBESGUE CONSTANTS FOR THE WEAK GREEDY ALGORITHM

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ABSTRACT. We estimate the Lebesgue constants for the weak thresholding greedy algorithm in a Banach space relative to a biorthogonal system. The estimates involve the weakness (relaxation) parameter of the algorithm, as well as properties of the basis, such as its quasi-greedy constant and democracy function.

1. Introduction

In this short note, we calculate the Lebesgue of the t-greedy, and the Chebyshev t-greedy, algorithms in Banach spaces (thus measuring the *efficiency* of these approximation methods, in the *worst case*).

Throughout this paper, X is a separable infinite dimensional Banach space. A family $(e_i, e_i^*)_{\in \in \mathbb{N}} \subset X \times X^*$ is called a *bounded biorthogonal system* if:

(1) $X = \text{span}\,[e_i : i \in \mathbb{N}]$.
(2) $e_i^*(e_j) = 1$ if $i = j$, $e_i^*(e_j) = 0$ otherwise.
(3) $0 < \inf_i \min\{\|e_i\|, \|e_i^*\|\} \leq \sup_i \max\{\|e_i\|, \|e_i^*\|\} < \infty$.

For brevity, we refer to (e_i) as a *basis*. Note that Condition (3) is referred to as (e_i) being *seminormalized*. In this note, only seminormalized bases are considered.

It is easy to see that, for any $x \in X$, $\lim_i e_i^*(x) = 0$, and $\sup_i |e_i^*(x)| > 0$, unless $x = 0$.

Bases as above are quite common. It is known [6, Theorem 1.27] that, for any $c > 1$ any separable Banach space has a bounded biorthogonal system (a *Markushevitch basis*) with $1 \leq \|e_i\|, \|e_i^*\| \leq c$, and $X^* = \overline{\text{span}}^{w^*}[e_i^* : i \in \mathbb{N}]$.

To consider the problem of approximating $x \in X$ by finite linear combinations of e_i's, introduce some notation. For $x \in X$ set $\text{supp}\, x = \{i \in \mathbb{N} : e_i^*(x) \neq 0\}$. For finite $A \subset \mathbb{N}$, set $P_A x = \sum_{i \in A} e_i^*(x) e_i$. If $A^c = \mathbb{N} \backslash A$ is finite, write $P_A x = x - P_{A^c} x$.

The *best n-term approximation* for $x \in X$ is defined as

$$\sigma_n(x) = \inf_{|\text{supp}\, y| \leq n} \|x - y\|,$$

while the *best n-term coordinate approximation* is

$$\tilde{\sigma}_n(x) = \inf_{|B| \leq n} \|x - P_B x\|.$$

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It is easy to see that $\lim_n \sigma_n(x) = 0$, and

$$\sigma_n(x) = \inf_{|\operatorname{supp} y|=n} \|x - y\| \text{ and } \tilde{\sigma}_n(x) = \inf_{|B|=n} \|x - P_B x\|$$

(the second equality is due to the fact that $\lim_i e_i^*(x) = 0$).

We also consider the *n term residual approximation*

$$\hat{\sigma}_n(x) = \|x - P_{[1,n]} x\|.$$

We say that (e_i) is a *Schauder basis* if $\lim_n \hat{\sigma}_n(x) = 0$ for every $x \in X$ (in this case, also $\lim_n \tilde{\sigma}_n(x) = 0$). Many commonly used bases (such as the Haar basis or the trigonometric basis in L_p, for $1 < p < \infty$) are, in fact, Schauder bases.

Note that calculating $\sigma_n(x)$ and $\tilde{\sigma}_n(x)$ is next to impossible, since all coordinates of x are in play. Therefore, one can naively look for a good n-term approximant of x by considering the n largest (or "nearly largest") coefficients. This is done using the *weak greedy algorithm*. To define this algorithm, fix the *relaxation parameter* $t \in (0, 1]$. Consider a non-zero $x \in X$. A set $A \subset \mathbb{N}$ is called *t-greedy for x* if $\inf_{i \in A} |e_i^*(x)| \geq t \sup_{i \notin A} |e_i^*(x)|$ (by the above, A is finite). Suppose $\rho = \rho_x : \mathbb{N} \to \mathbb{N}$ is a *t-greedy ordering* – that is, $\{\rho(n) : n \in \mathbb{N}\}$ is t-greedy for x, for every n. In general, a t-greedy ordering is not unique. Note that $\{\rho(n) : n \in \mathbb{N}\} = \mathfrak{S}_x := \{n \in \mathbb{N} : e_n^*(x) \neq 0\}$ if the set $\mathfrak{S}_x$ is infinite. On the other hand, if $|\mathfrak{S}_x| = m < \infty$, then $\{\rho(1), \ldots, \rho(m)\} = \mathfrak{S}_x$ while $\rho(i) \notin \mathfrak{S}_x$ for $i > m$.

An *n-term t-greedy approximant of x* is defined as $\mathbf{G}_n^t(X) = P_{A_n} x$, where $A_n = \{\rho(1), \ldots, \rho(n)\}$, and ρ is a t-greedy ordering for x. We define an *n-term Chebyshev t-greedy approximant* $\mathbf{CG}_n^t(x)$ as $y \in \operatorname{span}[e_i : i \in A_n]$ so that $\|x - y\|$ is minimal. We stress that these approximants are not unique, and *a fortiori*, the operators $x \mapsto \mathbf{G}_n^t(x)$ and $x \mapsto \mathbf{CG}_n^t(x)$ are not linear.

For more information on greedy approximation algorithms, we refer the reader to the survey papers [9] and [14], as well as to the recent monograph [10].

When $t = 1$, we omit it, and use the terms "greedy set", ("Chebyshev") "greedy approximant", as well as notation $\mathbf{G}_n(x)$ and $\mathbf{CG}_n(x)$. A basis (e_i) is called *quasi-greedy* if its *quasi-greedy constant* is finite:

$$\mathfrak{K} = \sup_{\|x\|=1} \sup_{n \in \mathbb{N}} \|\mathbf{G}_n(x)\| < \infty.$$

In [13] it was shown that a basis is quasi-greedy if and only if $\lim_n \mathbf{G}_n(x) = x$ for any $x \in X$, and any (equivalently, some) choice of the sequence $\mathbf{G}_n(x)$. By [7], for a quasi-greedy basis we also have $\lim_n \mathbf{G}_n^t(x) = x$ for any $x \in X$, and any choice of the sequence $\mathbf{G}_n^t(x)$.

The goal of this paper is to estimate the *efficiency* of the t-greedy and t-Chebyshev greedy methods (in the worst case), by comparing $\|x - \mathbf{G}_n^t(x)\|$ and $\|x - \mathbf{CG}_n^t(x)\|$ with the best n-term approximation $\sigma_n(x)$, and similar quantities. This is done through estimating the Lebesgue constants and its relatives:

$$\textit{The Lebesgue constant} \quad \mathbf{L}(n,t) = \sup_{x \in X, \sigma_n(x) \neq 0} \frac{\|x - \mathbf{G}_n^t(x)\|}{\sigma_n(x)}.$$

$$\textit{The Chebyshevian Lebesgue constant} \quad \mathbf{L}_{\mathrm{ch}}(n,t) = \sup_{x \in X, \sigma_n(x) \neq 0} \frac{\|x - \mathbf{CG}_n^t(x)\|}{\sigma_n(x)}.$$

$$\textit{The residual Lebesgue constant} \quad \mathbf{L}_{\rm re}(n,t) = \sup_{x\in X, \hat{\sigma}_n(x)\neq 0} \frac{\|x - \mathbf{G}_n^t(x)\|}{\hat{\sigma}_n(x)}.$$

We stress that the suprema in the above inequalities are taken over all $x \in X$, and all possible realizations of the (Chebyshev) weakly greedy algorithm. A basis is called *greedy* if $\sup_n \mathbf{L}(n,1) < \infty$, and *partially greedy* if $\sup_n \mathbf{L}_{\rm re}(n,1) < \infty$

To estimate the Lebesgue constants, we quantify some properties of (e_i). We use the *left* and *right democracy functions* $\phi_l(k) = \inf_{|A|=k} \|\sum_{i\in A} a_i\|$ and $\phi_r(k) = \sup_{|A|=k} \|\sum_{i\in A} a_i\|$ (sometimes, ϕ_r is also referred to as the *fundamental function*). We define the *democracy parameter*

$$\boldsymbol{\mu}(n) = \max_{k\leq n} \frac{\phi_r(k)}{\phi_l(k)} = \sup_{|A|=|B|\leq n} \frac{\|\sum_{i\in A} e_i\|}{\|\sum_{i\in B} e_i\|}.$$

Following [8], define the *disjoint democracy parameter*

$$\boldsymbol{\mu_d}(n) = \sup_{|A|=|B|\leq n, A\cap B=\emptyset} \frac{\|\sum_{i\in A} e_i\|}{\|\sum_{i\in B} e_i\|}.$$

Clearly, $\boldsymbol{\mu_d}(n) \leq \boldsymbol{\mu}(n)$. By [4, Lemma 3.2], $\boldsymbol{\mu}(n) \leq 3\mathfrak{K}\boldsymbol{\mu_d}(n)$. Related to the democracy parameter of a basis (e_i) is its *conservative parameter*:

$$\mathbf{c}(n) = \sup\Big\{\frac{\|\sum_{i\in A} e_i\|}{\|\sum_{i\in B} e_i\|} : \max A \leq n < \min B, |A| = |B|\Big\}.$$

The norms of coordinate projections in a basis (e_i) are quantified by the *unconditionality parameter* and *complemented unconditionality parameter*: $\mathbf{k}(n) = \sup_{|A|\leq n} \|P_A\|$, resp. $\mathbf{k}_c(n) = \sup_{|A|\leq n} \|I - P_A\|$ (clearly $|\mathbf{k}(n) - \mathbf{k}_c(n)| \leq 1$).

The investigation of Lebesgue constants dates back to the earliest works on greedy algorithms, see e.g. [8] (there, for instance, the Lebesgue constant of the Haar basis in the BMO, and the dyadic BMO, were computed). More recently, in [11, 12], the Lebesgue constants for tensor product bases in L_p-spaces (in particular, for the multi-Haar basis) were calculated. The Lebesgue constants for the trigonometric basis L_p (which is not quasi-greedy) are also known, see e.g. [9, Section 1.7]. The recent paper [3] estimates the Lebesgue constants for bases in L_p spaces with specific properties (such as being uniformly bounded). Lebesgue constants for redundant dictionaries are studied in [10, Section 2.6].

The paper is structured as follows: in Section 2, we gather some preliminary facts about quasi-greedy bases. In Section 3, we estimate $\mathbf{L}(n,t)$ in terms of $\mathfrak{K}$, $\boldsymbol{\mu_d}(n)$, $\mathbf{k}(n)$, and t. For $t = 1$, related results were obtained in [4]. However, the Lebesgue constant was not explicitly calculated there. Retracing the computations, one obtains worse constants than those given by Theorem 3.1. Corollary 3.5 gives an upper estimate for the Lebesgue constant of quasi-greedy bases in Hilbert spaces, by combining Theorem 3.1 with the recent results of Garrigos and Wojtaszczyk [5]. Further, we estimate the Lebesgue constant for general (not necessarily quasi-greedy) systems in Proposition 3.6.

In Section 4, we estimate $\mathbf{L}_{\rm ch}(n,t)$. The estimates involve only t, $\mathfrak{K}$, and $\boldsymbol{\mu_d}(n)$. Finally, in Section 5, we provide upper and lower bounds for $\mathbf{L}_{\rm re}(n,t)$, involving t, $\mathfrak{K}$, and $\mathbf{k}(n)$. The main results are given in Theorems 4.1 and 5.1, respectively.

Most of the work in this paper is done in the real case. In Section 6, we indicate that the complex versions of the results of this paper also hold, albeit perhaps with different numerical constants.

2. Preliminary results

In this section we prove two lemmas, which will be needed throughout the paper, and may be of interest in their own right. The first lemma sharpens some results from [7, Section 2].

Lemma 2.1. *Suppose $(e_i) \subset X$ is a basis with a quasi-greedy constant $\mathfrak{K}$, and a set A is t-greedy for $x \in X$. Then $\|P_A x\| \leq (1 + 4t^{-1}\mathfrak{K})\mathfrak{K}\|x\|$.*

Proof. For the sake of brevity, set $a_i = e_i^*(x)$. Let $M = \min_{i \in A} |a_i|$, then $|a_i| \leq t^{-1}M$ for $i \notin A$. Define $B = \{i : |a_i| \geq t^{-1}M\}$ and $C = \{i : |a_i| \geq M\}$. Then $B \subset A \subset C$, and $P_A x = P_B x + P_{A\backslash B} x$. By the definition of $\mathfrak{K}$, $\|P_B x\| \leq \mathfrak{K}\|x\|$, and $\|P_C x\| \leq \mathfrak{K}\|x\|$. Write $P_C x = \sum_{i \in C} a_i e_i$. The proof of [2, Lemma 2.2] shows that $M\|\sum_{i \in C} e_i'\| \leq 2\mathfrak{K}\|x\|$, where

$$e_i' = \begin{cases} \operatorname{sign}(a_i) e_i & i \in C \\ e_i & \text{otherwise} \end{cases} .$$

Note that the basis (e_i') has the same quasi-greedy constant as (e_i). For $i \in C$, set

$$b_i = \begin{cases} |a_i| & i \in A\backslash B \\ 0 & \text{otherwise} \end{cases} .$$

For any i, $|b_i| \leq t^{-1}M$, hence, by [2, Lemma 2.1],

$$\|\sum_{i \in A\backslash B} a_i e_i\| = \|\sum_{i \in C} b_i e_i'\| \leq 2t^{-1}M\mathfrak{K}\|\sum_{i \in C} e_i'\| \leq 4t^{-1}M\mathfrak{K}^2\|x\|.$$

By the triangle inequality, $\|P_A x\| \leq \|P_B x\| + \|P_{A\backslash B} x\|$. □

Lemma 2.2. *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in X. Consider $x \in X$, and let $a_i = e_i^*(x)$, for $i \in \mathbb{N}$. Suppose a finite set $A \subset \mathbb{N}$ satisfies $\min_{i \in A} |a_i| \geq M$. Then $M\|\sum_{i \in A} \operatorname{sign}(a_i) e_i\| \leq (1+3\mathfrak{K})\mathfrak{K}\|x\|$. Furthermore, $M\|\sum_{i \in A} e_i\| \leq 2(1+3\mathfrak{K})\mathfrak{K}\|x\|$.*

Here and throughout the paper, we shall use the function

$$f_M : \mathbb{R} \to \mathbb{R} : t \mapsto \begin{cases} -M & t < -M \\ t & -M \leq t \leq M \\ M & t > M \end{cases} .$$

Abusing the notation slightly, we shall write

$$f_M(x) = x - \sum_i \Big(e_i^*(x) - f_M\big(e_i^*(x)\big)\Big) e_i.$$

The sum above converges, since the set $\{i \in \mathbb{N} : |e_i^*(x)| > M\}$ is finite. Moreover, $f_M(x)$ is the only element $y \in X$ with the property that, for every i, $e_i^*(y) = f_M(e_i^*(x))$. By [1], $\|f_M(x)\| \leq (1 + 3\mathfrak{K})\|x\|$.

Proof. Let $y = f_M(x)$. Then A is a greedy set for y, and $P_A y = M \sum_{i \in A} \text{sign}\,(a_i) e_i$. Therefore, $M \| \sum_{i \in A} \text{sign}\,(a_i) e_i \| = \|P_A y\| \leq \mathfrak{K} \|y\|$. This proves the first statement of our lemma. To establish the "moreover" part, let $A_+ = \{i \in A : \text{sign}\,(a_i) = 1\}$, and $A_- = \{i \in A : \text{sign}\,(a_i) = -1\}$. By the above, $M \| \sum_{i \in A_+} \text{sign}\,(a_i) e_i \| \leq (1 + 3\mathfrak{K})\mathfrak{K}\|x\|$, and the same holds for A_-. Complete the proof using the triangle inequality. ☐

We close this section with a brief discussion about the values of $\boldsymbol{\mu_d}(n)$, $\mathbf{k}(n)$, and $\mathbf{c}(n)$. It was shown in [1] and [4] that, for a $\mathfrak{K}$-quasi-greedy basis, $\mathbf{k}(n) \leq C \log(en)$, where the constant C depends on the particular basis. For bases in L_p spaces, sharper estimates were obtained in [5]. It is easy to see that $\mathbf{c}(n) \leq \boldsymbol{\mu_d}(n) \leq Cn$, where C depends on a basis. These estimates are optimal: indeed, an appropriate enumeration of the canonical (normalized and 1-unconditional) basis in $c_0 \oplus_2 \ell_1$ gives $\mathbf{c}(n) \geq cn$.

3. The Lebesgue constant

In this section, we use some of the techniques of [4] to estimate the Lebesgue constants $\mathbf{L}(n, t)$.

Theorem 3.1. *For any $\mathfrak{K}$-quasi-greedy basis,*

$$\max\left\{\mathbf{k}(n) - 1, t^{-1}\boldsymbol{\mu_d}(n)\right\} \leq \mathbf{L}(n, t) \leq 1 + 2\mathbf{k}(n) + 4t^{-1}(1 + 3\mathfrak{K})\mathfrak{K}^2 \boldsymbol{\mu_d}(n).$$

The proof of the theorem relies on several lemmas, whose proofs closely resemble those given in [4] (Lemma 3.4 yields better upper estimates).

Lemma 3.2. *For any $\mathfrak{K}$-quasi-greedy basis,* $\mathbf{L}(n, t) \geq t^{-1}\boldsymbol{\mu_d}(n)$.

Proof. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Find $A, B \subset \mathbb{N}$, so that $A \cap B = \emptyset$, $|A| = |B| = k \leq n$, and

$$\| \sum_{i \in A} e_i \| \geq (\boldsymbol{\mu_d}(n) - \varepsilon) \| \sum_{i \in B} e_i \|.$$

Pick a set C, disjoint from A and B, so that $|C| = n - k$. Consider

$$x = (t + \varepsilon) \sum_{i \in B \cup C} e_i + \sum_{i \in A} e_i.$$

Then $(t + \varepsilon) \sum_{i \in B \cup C} e_i$ is a t-greedy approximant of x, for which $\|x - \mathbf{G}_n^t(x)\| = \| \sum_{i \in A} e_i \|$. However, $|A \cup C| = n$, hence

$$\sigma_n(x) \leq \tilde{\sigma}_n(x) \leq \|x - P_{A \cup C} x\| = (t + \varepsilon) \| \sum_{i \in B} e_i \|.$$

Thus,

$$\mathbf{L}(n, t) \geq \frac{\|x - \mathbf{G}_n^t(x)\|}{\sigma_n(x)} = (t + \varepsilon)^{-1} \frac{\| \sum_{i \in A} e_i \|}{\| \sum_{i \in B} e_i \|} \geq \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{t + \varepsilon}.$$

As ε can be arbitrarily small, the desired estimate follows. ☐

Lemma 3.3. *For any $\mathfrak{K}$-quasi-greedy basis,* $\mathbf{L}(n, t) \geq \mathbf{k}_c(n)$.

Proof. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$, and find a finitely supported $x \in X$ of norm 1, so that $\|P_{A^c}x\| > \mathbf{k}_c(n) - \varepsilon$ for some set A of cardinality not exceeding n. Find $C \subset A^c$ so that $C \cap \mathrm{supp}(x) = \emptyset$, and $|C| = n - |A|$. Pick $c > \sup_i |e_i^*(x)|$, and consider $y = x - P_A x + 2c \sum_{i \in A \cup C} e_i$. Then $|A \cup C| = n$, and

$$\min_{i \in A \cup C} |e_i^*(y)| > c > \min_{i \notin A \cup C} |e_i^*(y)|.$$

Thus, for any t, we can have $\mathbf{G}_n^t(y) = P_{A \cup C} y = P_{A^c} x$, hence $\|y - \mathbf{G}_n^t y\| = \|P_{A^c} x\| > \mathbf{k}_c(n) - \varepsilon$. However, $z = 2c \sum_{i \in A \cup C} e_i - P_A x$ is supported on $A \cup C$, hence $\sigma_n(y) \leq \|y - z\| = \|x\| = 1$, yielding the desired estimate. □

Lemma 3.4. *For any $\mathfrak{K}$-quasi-greedy basis,* $\mathbf{L}(n,t) \leq \mathbf{k}(n) + \mathbf{k}_c(n) + 4t^{-1}(1 + 3\mathfrak{K})\mathfrak{K}^2 \boldsymbol{\mu_d}(n)$.

Proof. For $x \in X$, let $a_i = e_i^*(x)$, and fix $\varepsilon > 0$. Suppose $A \subset \mathbb{N}$ is a t-greedy set for x, of cardinality n. Find $z \in X$, supported on a set B of cardinality n, so that $\|x - z\| < \sigma_n(x) + \varepsilon$. Let $M = \sup_{i \notin A} |a_i|$, then $|a_i| \geq tM$ whenever $i \in A$. By the triangle inequality,

$$\|x - P_A x\| \leq \|x - P_B x\| + \|P_{A \backslash B} x\| + \|P_{B \backslash A} x\|.$$

We have

$$\|P_{A \backslash B} x\| = \|P_{A \backslash B}(x - z)\| \leq \mathbf{k}(n)\|x - z\|,$$

and

$$\|x - P_B x\| = \|x - P_B x + z - P_B z\| = \|(1 - P_B)(x - z)\| \leq \mathbf{k}_c(n)\|x - z\|.$$

It remains to estimate the third summand, in the non-trivial case of $|B \backslash A| = k > 0$. For $i \in B \backslash A$, $|a_i| \leq M$, hence by [2],

$$\|P_{B \backslash A} x\| = \|\sum_{i \in B \backslash A} a_i e_i\| \leq 2M\mathfrak{K}\|\sum_{i \in B \backslash A} e_i\|.$$

By Lemma 2.2, $M \leq t^{-1} 2(1 + 3\mathfrak{K})\mathfrak{K}\|\sum_{i \in A \backslash B} e_i\|^{-1}\|x - z\|$. Thus,

$$\|P_{B \backslash A} x\| \leq 2M\mathfrak{K}\|\sum_{i \in B \backslash A} e_i\|$$
$$\leq 4t^{-1}(1 + 3\mathfrak{K})\mathfrak{K}^2 \frac{\|\sum_{i \in B \backslash A} e_i\|}{\|\sum_{i \in A \backslash B} e_i\|}\|x - z\| \leq 4t^{-1}(1 + 3\mathfrak{K})\mathfrak{K}^2 \boldsymbol{\mu_d}(n)\|x - z\|.$$

As $\|x - z\|$ can be arbitrarily close to $\sigma_n(x)$, we are done. □

We use Theorem 3.1 to estimate the Lebesgue constant for quasi-greedy bases in a Hilbert spaces. Recall that a basis (e_i) is called *hilbertian* (*besselian*) if there exists a constant c so that, for every finite sequence of scalars (α_i). we have $\sum_i |\alpha_i|^2 \geq c\|\sum_i \alpha_i e_i\|^2$ (resp. $\sum_i |\alpha_i|^2 \leq c\|\sum_i \alpha_i e_i\|^2$).

Corollary 3.5. *For any $\mathfrak{K}$-quasi-greedy basis in a Hilbert space, there exists $\alpha \in (0,1)$ and $C > 0$ so that, for any $n \in \mathbb{N}$ and $t \in (0,1)$, $\mathbf{L}(n,t) \leq C(t^{-1} + (\log(en))^\alpha)$. If, moreover, the basis is either besselian or hilbertian, then there exists $\alpha \in (0,1/2)$ with the above property.*

Proof. By [5], there exists a constant c_1, and α as above, so that $\mathbf{k}(n) \le c_1(\log(en))^\alpha$. By [13], $\boldsymbol{\mu}(n) \le c_2$, for some constant c_2. To finish the proof, apply Theorem 3.1. □

We conclude this section with an estimate for $\mathbf{L}(n,t)$ for bounded Markushevitch bases which are not necessarily quasi-greedy. Let $1 \le p \le q \le \infty$. We say that (e_i) satisfies weak upper p- and lower q-estimates if there exists $K > 0$ such that for all $x \in X$,

$$\frac{1}{K}\|(e_i^*(x))\|_{q,\infty} \le \|x\| \le K\|(e_i^*(x))\|_{p,1},$$

where, letting (a_n^*) denote the decreasing rearrangement of the sequence $(|a_n|)$,

$$\|(a_n)\|_{q,\infty} := \sup_{n\ge1} n^{1/q}a_n^*$$

and

$$\|(a_n)\|_{p,1} := \sum_{n\ge1} n^{1/p-1}a_n^*$$

are the usual Lorentz sequence norms. Note that $p = 1$ and $q = \infty$ are just the ℓ_1 and c_0 norms, respectively.

The following result slightly extends [12, Th. 5] by incorporating the weakness parameter t and replacing upper ℓ_p-and lower ℓ_q-estimates by weaker Lorentz sequence space estimates.

Proposition 3.6. *Suppose (e_i) satisfies weak upper $p-$ and lower q-estimates. Then there exists $D := D(p,q,K)$ such that*

$$\mathbf{L}(n,t) \le \begin{cases} Dn^{1/p-1/q}/t, & p \ne q \\ (D\log n)/t, & p = q. \end{cases}$$

Proof. First suppose $q < p$. Let $x \in X$ and set $a_i := e_i^*(x)$. Let A be a t-greedy set for x, with $|A| = n$, and let $\mathbf{G}_n^t(x) := \sum_{i\in A} a_ie_i$. Given $\varepsilon > 0$, choose $B \subset \mathbb{N}$, with $|B| = n$, such that $\|x - \sum_{i\in B} b_ie_i\| \le \sigma_n(x) + \varepsilon$. For convenience, set $b_i = 0$ if $i \notin B$. Set $C = C(p,q) := 1 + (q-p)/pq$.

$$\begin{aligned} \|x - \mathbf{G}_n^t(x)\| &\le \|x - \sum_{i\in B} b_ie_i\| + \|\sum_{i\in B} b_ie_i - \sum_{i\in A} a_ie_i\| \\ &\le \sigma_n(x) + \varepsilon + \|\sum_{i\in B} b_ie_i - \sum_{i\in A} a_ie_i\|. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} \|\sum_{i\in A}(b_i - a_i)e_i\| &\le K\|(b_i - a_i)_{i\in A}\|_{p,1} \\ &\le KCn^{1/p-1/q}\|(b_i - a_i)_{i\in A}\|_{q,\infty} \\ &\le K^2Cn^{1/p-1/q}\|x - \sum_{i\in B} b_ie_i\| \\ &\le K^2Cn^{1/p-1/q}(\sigma_n(x) + \varepsilon). \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}\|\sum_{i\in B\setminus A} b_ie_i\| &\le \|\sum_{i\in B\setminus A}(b_i-a_i)e_i\| + \|\sum_{i\in B\setminus A} a_ie_i\| \\ &\le K^2Cn^{1/p-1/q}(\sigma_n(x)+\varepsilon) + \|\sum_{i\in B\setminus A} a_ie_i\|\end{aligned} \tag{3.3}$$

Finally, since A is a t-greedy set for x and $|A\setminus B| = |B\setminus A|$,

$$\begin{aligned}\|\sum_{i\in B\setminus A} a_ie_i\| &\le K^2Cn^{1/p-1/q}\|(a_i)_{i\in B\setminus A}\|_{q,\infty} \\ &\le \frac{K^2Cn^{1/p-1/q}}{t}\|(a_i)_{i\in A\setminus B}\|_{q,\infty} \\ &\le \frac{K^3Cn^{1/p-1/q}}{t}\|x-\sum_{i\in B} b_ie_i\| \\ &\le \frac{K^3Cn^{1/p-1/q}}{t}(\sigma_n(x)+\varepsilon).\end{aligned} \tag{3.4}$$

Since $\varepsilon > 0$ is arbitrary, combining (3.1)-(3.4) gives

$$\|x - \mathbf{G}_n^t(x)\| \le \Big(1 + 2K^2C + \frac{K^3C}{t}\Big)n^{1/p-1/q}\sigma_n(x),$$

and hence $\mathbf{L}(n,t) \le \Big(1 + 2K^2C + \frac{K^3C}{t}\Big)n^{1/p-1/q}$. The case $p = q$ is similar except $Cn^{1/p-1/q}$ is replaced by $1 + \log n$ throughout. □

Since every bounded Markushevitch basis satisfies a lower ∞-estimate, we get the following corollary.

Corollary 3.7. *Let $1 \le p < \infty$ and let (e_i) be a bounded Markushevitch basis such that $\phi_r(k) \le Ck^{1/p}$ for some $C > 0$. Then $\mathbf{L}(n,t) \le Cn^{1/p}/t$.*

Proof. By the triangle inequality $\|\sum_{i\in A} \pm e_i\| \le 2Cn^{1/p}$ for all $A \subset \mathbb{N}$ with $|A| = n$. By a standard Abel summation calculation, we get $\|\sum_{i\in A} a_ie_i\| \le C\|(a_i)_{i\in A}\|_{p,1}$ for all scalars (a_i) and finite $A \subset \mathbb{N}$. It follows easily that (e_i) satisfies weak upper p- and lower ∞-estimates, so we can apply Proposition 3.6 to get the result. □

4. The Chebyshevian Lebesgue constant

Theorem 4.1. *For any $\mathfrak{K}$-quasi-greedy basis,*

$$\frac{\boldsymbol{\mu_d}(n)}{2t\mathfrak{K}} \le \mathbf{L}_{\mathrm{ch}}(n,t) \le \frac{20\mathfrak{K}^3\boldsymbol{\mu_d}(n)}{t}.$$

As a corollary, we recover a result from [1].

Corollary 4.2. *Any almost greedy basis is semi-greedy.*

Recall that (e_i) is *almost greedy* if there exists a constant C so that $\|x - \mathbf{G}_n(x)\| \le C\tilde{\sigma}_n(x)$ for any $n \in \mathbb{N}$ and $x \in X$, and *semi-greedy* if there exists a constant C so that $\|x - \mathbf{CG}_n(x)\| \le C\sigma_n(x)$, for any n and x.

Proof. By [1], a basis is almost greedy if and only if it is quasi-greedy and democratic (that is, $\sup_n \boldsymbol{\mu}(n) < \infty$). In this case $\sup_n \mathbf{L}_{\mathrm{ch}}(n,1) < \infty$, hence the basis is semi-greedy. □

Proof of the upper estimate in Theorem 4.1. For $x \in X$ let $a_i = e_i^*(x)$, and fix $\varepsilon > 0$. Suppose a set $A \subset \mathbb{N}$ of cardinality n is t-greedy for x. Let $M = \max_{i \notin A} |a_i|$, then $\min_{i \in A} |a_i| \geq tM$. We have to show that there exists $w \in X$ so that $\mathrm{supp}(x-w) \subset A$, and $\|w\| \prec 20t^{-1}\mathfrak{K}^2 \boldsymbol{\mu_d}(n)(\sigma_n(x)+\varepsilon)$.

Pick $z = \sum_{i \in B} b_i e_i$, where $|B| \leq n$, and $\|x - z\| < \sigma_n(x) + \varepsilon$. Set $y = x - z$ and

$$y_i = e_i^*(y) = \begin{cases} a_i - b_i & i \in B \\ a_i & i \notin B \end{cases} .$$

We claim that $w = P_A f_M(y) + P_{A^c} x$ has the desired properties. Indeed, $x - w$ is supported on A. To estimate $\|w\|$, note that, for $i \notin B$, $y_i = a_i$. For $i \notin A$, $f_M(a_i) = a_i$, hence, for $i \notin A \cup B$, $a_i = f_M(y_i)$. Thus,

$$w = f_M(y) + \sum_{i \in B \backslash A} (a_i - f_M(y_i)) e_i. \tag{4.1}$$

We use [1, Proposition 3.1] to estimate on the first summand:

$$\|f_M(y)\| \leq (1 + 3\mathfrak{K})\|y\| = (1 + 3\mathfrak{K})\|x - z\|. \tag{4.2}$$

To handle the second summand, set $k = |B \backslash A|$. For $i \in B \backslash A$, $|a_i| \leq M$, hence $|a_i - f_M(y_i)| \leq 2M$. By [1, (2.5)],

$$\|\sum_{i \in B \backslash A} (a_i - f_M(y_i)) e_i\| \leq 2M\mathfrak{K} \|\sum_{i \in B \backslash A} e_i\|. \tag{4.3}$$

On the other hand, for $i \in A \backslash B$, $a_i = y_i$, and $|a_i| \geq tM$, hence by Lemma 2.2,

$$tM \|\sum_{i \in A \backslash B} e_i\| \leq 2\mathfrak{K}(1 + 3\mathfrak{K})\|y\|,$$

hence

$$M \leq t^{-1} \frac{2\mathfrak{K}(1 + 3\mathfrak{K})\|x - z\|}{\|\sum_{i \in A \backslash B} e_i\|}.$$

Plugging this into (4.3), we get:

$$\begin{aligned} \|\sum_{i \in B \backslash A} (a_i - f_M(y_i)) e_i\| &\leq 4 \frac{\|\sum_{i \in B \backslash A} e_i\|}{\|\sum_{i \in A \backslash B} e_i\|} t^{-1} \mathfrak{K}^2 (1 + 3\mathfrak{K}) \|x - z\| \\ &\leq 4 \boldsymbol{\mu_d}(n) t^{-1} \mathfrak{K}^2 (1 + 3\mathfrak{K}) \|x - z\|. \end{aligned}$$

Together with (4.2), we obtain:

$$\|w\| \leq \big(1 + \frac{4\boldsymbol{\mu_d}(n)\mathfrak{K}^2}{t}\big)(1 + 3\mathfrak{K})\|x - z\| \leq \frac{20\mathfrak{K}^3 \boldsymbol{\mu_d}(n)}{t}(\sigma_n(x) + \varepsilon).$$

As ε can be arbitrarily close to 0, we are done. □

Proof of the lower estimate in Theorem 4.1. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Find $A, B \subset \mathbb{N}$, so that $A \cap B = \emptyset$, $|A| = |B| = k \leq n$, and

$$\|\sum_{i\in A} e_i\| \geq (\boldsymbol{\mu_d}(n) - \varepsilon)\|\sum_{i\in B} e_i\|.$$

Pick a set C, disjoint from A and B, so that $|C| = n - k$. Consider

$$x = (t+\varepsilon)\sum_{i\in B\cup C} e_i + \sum_{i\in A} e_i.$$

We can find a t-greedy approximant $\mathbf{CG}_n^t(x)$ supported on $B \cup C$, and then $y = x - \mathbf{CG}_n^t(x) = \sum_{i\in A} e_i + \sum_{i\in B\cup C} y_i e_i$. Let $D = \{i \in B \cup C : |y_i| \geq 1\}$. Both $\sum_{i\in A} e_i + \sum_{i\in D} y_i e_i$ and $\sum_{i\in D} y_i e_i$ are greedy approximants of y, hence

$$\max\{\|\sum_{i\in A} e_i + \sum_{i\in D} y_i e_i\|, \|\sum_{i\in D} y_i e_i\|\} \leq \mathfrak{K}\|y\|.$$

By the triangle inequality, $\|\sum_{i\in A} e_i\| \leq 2\mathfrak{K}\|y\|$. Thus,

$$\begin{aligned}\|x - \mathbf{CG}_n^t(x)\| &\geq \frac{1}{2\mathfrak{K}}\|\sum_{i\in A} e_i\| \geq \frac{\boldsymbol{\mu_d}(n)-\varepsilon}{2(t+\varepsilon)\mathfrak{K}}\|(t+\varepsilon)\sum_{i\in B} e_i\| = \frac{\boldsymbol{\mu_d}(n)-\varepsilon}{2(t+\varepsilon)\mathfrak{K}}\|x - P_{A\cup C}x\| \\ &\geq \frac{\boldsymbol{\mu_d}(n)-\varepsilon}{2(t+\varepsilon)\mathfrak{K}}\tilde{\sigma}_n(x) \geq \frac{\boldsymbol{\mu_d}(n)-\varepsilon}{2(t+\varepsilon)\mathfrak{K}}\sigma_n(x)\end{aligned}$$

(since $|A \cup C| = n$). As ε can be arbitrarily small, we are done. □

5. The residual Lebesgue constant

Theorem 5.1. *For any $\mathfrak{K}$-quasi-greedy basis,*

$$t^{-1}\mathbf{c}(n) \leq \mathbf{L}_{\mathrm{re}}(n,t) \leq 1 + 5t^{-1}\mathfrak{K}^2 + 40t^{-1}\mathfrak{K}^5\mathbf{c}(n).$$

Proof of the upper estimate in Theorem 5.1. For $x \in X$ set $a_i = e_i^*(x)$. Suppose A is a t-greedy subset of $\mathbb{N}$, of cardinality n, and set $B = [1, n]$. Let $M = \min_{i\in A}|a_i|$, then $|a_i| \leq t^{-1}M$ for $i \notin A$. By the triangle inequality,

$$\|x - \mathbf{G}_n^t(x)\| = \|P_{A^c}x\| \leq \|x - P_Bx\| + \|P_{A\backslash B}x\| + \|P_{B\backslash A}x\|. \tag{5.1}$$

Let $y = P_{B^c}x$, then $\|y\| = \hat{\sigma}_n(x)$. The set $A\backslash B$ is t-greedy for y, hence, by Lemma 2.1,

$$\|P_{A\backslash B}x\| = \|P_{A\backslash B}y\| \leq 5t^{-1}\mathfrak{K}^2\|y\|.$$

Furthermore,

$$M\|\sum_{i\in A\backslash B} e_i\| \leq 4\mathfrak{K}^2\|P_{A\backslash B}x\| \leq 20t^{-1}\mathfrak{K}^4\|y\|.$$

Now

$$\|P_{B\backslash A}x\| \leq 2t^{-1}M\mathfrak{K}\|\sum_{i\in A\backslash B} e_i\| \leq 2t^{-1}M\mathfrak{K}\mathbf{c}(n)\|\sum_{i\in B\backslash A} e_i\| \leq 40t^{-1}\mathfrak{K}^5\mathbf{c}(n)\|y\|.$$

Plug the above results into (5.1) to obtain the upper estimate for $\mathbf{L}_{\mathrm{re}}(n,t)$. □

Proof of the lower estimate in Theorem 5.1. Fix $\varepsilon > 0$, and find sets $A \subset [1, n]$ and $B \subset [n+1, \infty)$ so that $|A| = k = |B|$, and

$$\mathbf{c}(n) - \varepsilon < \frac{\| \sum_{i \in A} e_i \|}{\| \sum_{i \in B} e_i \|}.$$

Consider $x = \sum_{i=1}^n e_i + (t + \varepsilon) \sum_{i \in B} e_i$. Then $B \cup ([1, n] \backslash A)$ is a t-greedy set for x, hence one can run the t-greedy algorithm in such a way that $\|x - \mathbf{G}_n^t(x)\| = \| \sum_{i \in A} e_i \|$. On the other hand, $\hat{\sigma}_n(x) = \|P_{[n+1,\infty)} x\| = (t + \varepsilon) \| \sum_{i \in B} e_i \|$. The lower estimate follows from comparing these two quantities. □

6. Appendix: the complex case

The results above are stated for the real case. The complex case is similar, but the constants are different. As customary, we set

$$\operatorname{sign} z = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

The following result is present (implicitly or explicitly) in [4, Appendix]:

Lemma 6.1. *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in a Banach space X.*

(1) If A is a finite set, then $\| \sum_{i \in A} a_i e_i \| \leq 2\mathfrak{K} \max_i |a_i| \| \sum_{i \in A} e_i \|$.
(2) Suppose A is a greedy set for $x \in X$. Let $M = \min_{i \in A} |e_i^*(x)|$. Then

$$\frac{M}{8\sqrt{2}\mathfrak{K}^2} \| \sum_{i \in A} e_i \| \leq \frac{M}{2\mathfrak{K}} \| \sum_{i \in A} \operatorname{sign}\big(e_i^*(x)\big) e_i \| \leq \|x\|.$$

For $M > 0$, define

$$f_M : \mathbb{Z} \to \mathbb{Z} : z \mapsto \begin{cases} \operatorname{sign}(z) M & |z| > M \\ z & |z| \leq M \end{cases}.$$

For $x \in X$, we set $f_M(x) = x - \sum_i \big(e_i^*(x) - f_M(e_i^*(x))\big) e_i$ (the sum converges, and $e_i^*(f_M(x)) = f_M(e_i^*(x))$ for every i). As in [1, Proposition 3.1], one can prove:

Lemma 6.2. *In the above notation, $\|f_M(x)\| \leq (1 + 3\mathfrak{K}) \|x\|$.*

As in Section 2, we obtain:

Lemma 6.3. *Suppose $(e_i) \subset X$ is a basis with a quasi-greedy constant $\mathfrak{K}$, and a set A is t-greedy for $x \in X$. Then $\|P_A x\| \leq (1 + 8\sqrt{2} t^{-1} \mathfrak{K}) \mathfrak{K} \|x\|$.*

Lemma 6.4. *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in X. Consider $x \in X$, and let $a_i = e_i^*(x)$, for $i \in \mathbb{N}$. Suppose a finite set $A \subset \mathbb{N}$ satisfies $\min_{i \in A} |a_i| \geq M$. Then $M \| \sum_{i \in A} \operatorname{sign}(a_i) e_i \| \leq (1 + 3\mathfrak{K}) \mathfrak{K} \|x\|$. Furthermore, $M \| \sum_{i \in A} e_i \| \leq 4(1 + 3\mathfrak{K}) \mathfrak{K} \|x\|$.*

Proof. Use the notation $y = f_M(x)$, $\omega_i = \overline{\operatorname{sign}(a_i)}$, and $e_i' = \operatorname{sign}(a_i) e_i$. Then A is a greedy set for y, and $P_A y = M \sum_{i \in A} e_i'$. Therefore, $M \| \sum_{i \in A} e_i' \| = \|P_A y\| \leq \mathfrak{K} \|y\|$. This proves the first statement of our lemma. Moreover any $B \subset A$ is a greedy set for y, hence for any such B, $M \| \sum_{i \in B} e_i' \| = \|P_B y\| \leq \mathfrak{K} \|y\|$. Let S be the absolute convex hull of the elements $\sum_{i \in B} e_i'$ – that is,

$$S = \big\{ \sum_{B \subset A} t_B \sum_{i \in B} e_i' : \sum_{B \subset A} |t_B| \leq 1 \big\}.$$

We claim that $\sum_{i\in A} e_i = \sum_{i\in A} \omega_i e_i' \in 4S$. Otherwise, by Hahn-Banach Separation Theorem, there exists a sequence $(b_i)_{i\in A} \in \mathbb{C}^{|A|}$ so that $|\sum_{i\in B} b_i| < 1$ whenever $B \subset A$, yet $|\sum_{i\in A} \omega_i b_i| > 4$. Let $B_+ = \{i \in A : \Re b_i \geq 0\}$ and $B_- = \{i \in A : \Re b_i < 0\}$.

$$\sum_{i\in B_+} \Re b_i \leq \Big|\sum_{i\in B_+} b_i\Big| \leq 1,$$

and similarly, $\sum_{i\in B_-}(-\Re b_i) \leq 1$. Therefore,

$$\sum_{i\in A} |\Re b_i| = \sum_{i\in B_+} |\Re b_i| + \sum_{i\in B_-} |\Re b_i| \leq 2.$$

The same way, we show that $\sum_{i\in A} |\Im b_i| \leq 2$. Consequently,

$$\Big|\sum_{i\in A} \omega_i b_i\Big| \leq \sum_{i\in A} |b_i| \leq \sum_{i\in A} \big(|\Re|b_i| + |\Im b_i|\big) \leq 4,$$

yielding a contradiction. □

These results allow us to emulate the proofs of previous sections, and to estimate the Lebesgue constants:

Theorem 6.5. *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in a complex Banach space X. Then:*

(1)

$$\max\left\{\mathbf{k}(n) - 1, t^{-1}\boldsymbol{\mu}_{\boldsymbol{d}}(n)\right\} \leq \mathbf{L}(n,t) \leq 1 + 2\mathbf{k}(n) + 16\sqrt{2}t^{-1}(1+3\mathfrak{K})\mathfrak{K}^2\boldsymbol{\mu}_{\boldsymbol{d}}(n).$$

(2)

$$\frac{\boldsymbol{\mu}_{\boldsymbol{d}}(n)}{2t\mathfrak{K}} \leq \mathbf{L}_{\mathrm{ch}}(n,t) \leq \frac{100\mathfrak{K}^3\boldsymbol{\mu}_{\boldsymbol{d}}(n)}{t}.$$

(3)

$$t^{-1}\mathbf{c}(n) \leq \mathbf{L}_{\mathrm{re}}(n,t) \leq 1 + 9\sqrt{2}t^{-1}\mathfrak{K}^2 + 900t^{-1}\mathfrak{K}^5\mathbf{c}(n).$$

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