A NOTE ON LATTICEABILITY AND ALGEBRABILITY

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Abstract. Suppose A is a subset of a Banach lattice (Banach algebra) X. We look for "large" sublattices (resp. subalgebras) of A. If X is a Banach lattice, we prove: (1) If Y is a closed subspace of X of codimension at least n, then $(X \backslash Y) \cup \{0\}$ contains a sublattice of dimension n. (2) If Y is a closed infinite codimensional ideal in X, then $(X \backslash Y) \cup \{0\}$ contains a closed infinite dimensional sublattice. (3) If the order in X is induced by a 1-unconditional basis, and Y is a closed infinite codimensional subspace of X, then $(X \backslash Y) \cup \{0\}$ contains a closed infinite dimensional ideal. Further, we show that (4) $(\ell_p \backslash (\cup_{q<p} \ell_q)) \cup \{0\}$ contains a sublattice which is dense in ℓ_p, and that (5) the sets $L_1(\mathbb{T}) \backslash (\cup_{p>1} L_p(\mathbb{T})) \cup \{0\}$ and $\mathcal{S}_\infty \backslash (\cup_{p<\infty} \mathcal{S}_p) \cup \{0\}$ contain a dense subalgebra with a continuum of free generators (here $\mathcal{S}_p$ denotes the Schatten p-space).

1. Introduction

We are motivated by the recent survey of lineability and spaceability [10]. Recall that $A \subset X$ is called *lineable* (*spaceable*, *densely lineable*) if $A \cup \{0\}$ contains an infinite dimensional subspace (resp. an infinite dimensional closed subspace, an infinite dimensional subspace dense in X).

If X is a Banach algebra, we say that $A \subset X$ is *algebrable* if $A \cup \{0\}$ contains a subalgebra B so that any family of generators of B is infinite. We say that A is *densely algebrable* if, in addition, B is dense in A.

In this paper, we search for large sublattices in subsets of Banach lattices. Suppose X is an infinite dimensional Banach lattice. A subset $A \subset X$ is (*completely*) *latticeable* if X contains a (complete) infinite dimensional sublattice Z so that $Z \subset A \cup \{0\}$.

As far as we know, the present paper is the first systematic investigation of latticeability. Even the term "latticeability" has not appeared previously – although, in fact, [1, 21] produce atomic sublattices while proving the spaceability of certain sets in rearrangement invariant spaces. One should also also mention the related notion of "coneability", studied in [13].

In Section 2, we look for sublattices in complements of closed subspaces of a Banach lattice. We start in the finite codimensional case: if Y is a subspace of a Banach lattice X of codimension $n < \infty$, then $(X \backslash Y) \cup \{0\}$ contains an n-dimensional lattice (Theorem 2.1). Further, we show that complements of finite dimensional subspaces of a Banach lattice, and of

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infinite codimensional ideals, are completely latticeable (Propositions 2.4 and 2.8).

If the order of X is determined by a 1-unconditional basis, then Proposition 2.9 shows that $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional ideal. Some partial results on complements of subspaces in Köthe function spaces are also established (Propositions 2.13, 2.15).

In Section 3 (Theorem 3.1) we prove that $\ell_p \backslash (\cup_{q<p} \ell_q)$ is *densely latticeable* – that is, $(\ell_p \backslash (\cup_{q<p} \ell_q)) \cup \{0\}$ contains a sublattice which is dense in ℓ_p.

In Section 4, we consider algebrability of subsets of a Banach algebra X. We prove that $L_1(\mathbb{T}) \backslash (\cup_{p>1} L_p(\mathbb{T}))$ is densely maximally algebrable (Proposition 4.1) – that is, it contains a dense subalgebra W so that every set generating W must have the cardinality of continuum. Previously, similar results were obtained for $c_0 \backslash (\cup_{q<\infty} \ell_q)$ [5] and $C_0(\mathbb{R}) \backslash (\cup_{q<\infty} L_q(\mathbb{R}))$ [14]. In the non-commutative setting, we establish in Proposition 4.2 that $\mathcal{S}_\infty \backslash \big(\cup_{p<\infty} \mathcal{S}_p \big)$ is densely maximally algebrable (here $\mathcal{S}_p$ is a p-Schatten space on ℓ_2). Here, we should also note a plethora of results on spaceability and dense lineability of $L_p(\mu) \backslash (\cup_{q \in S} L_q(\mu))$ (where $S = (p, \infty)$, $(0, p)$, or $\mathbb{R} \backslash \{p\}$, and μ is not necessarily σ-finite), recently established in e.g. [6], [7], [8], [9].

Finally, in Section 5, we prove (Proposition 5.1) that, for $1 < p < \infty$, the set of non-regular compact operators on ℓ_p is densely maximally lineable, and spaceable.

For the sake of simplicity, we work with real Banach lattices, while all Banach algebras are assumed to be complex. We use standard notation and results (see e.g. [3], [19]). We denote by $\mathbf{B}(\cdot)$ the closed unit ball of a space. The unit circle is denoted by $\mathbb{T}$, and equipped with the translation-invariant Lebesgue probability measure μ_0.

2. Latticeability: complements of closed subspaces

We investigate the "largest" sublattice of X contained in the complement of a closed subspace $Z \subset X$. Note that, if Y is a closed subspace of a Banach space X of finite codimension, then clearly $(X\backslash Y) \cup \{0\}$ contains a subspace Z, with $\dim W = \dim X/Y$. Furthermore, if Y is a closed subspace of X with $\dim X/Y = \infty$, then $(X\backslash Y) \cup \{0\}$ contains a closed infinite dimensional subspace W. This result (attributed to N. Kalton) goes back to [25]. Related results concerning operator ranges were established in [18].

2.1. The finite codimensional case.

In this subsection, we are assuming that Y is a finite codimensional subspace of a Banach lattice X. We are looking for a sublattice $W \subset X$, so that $W \cap Y = \{0\}$. We recall that any finite dimensional Banach lattice is spanned by its atoms, see [22, Section II.3] or [23, Proposition I.4.19].

Theorem 2.1. *If Y is a closed subspace of a Banach lattice X with $\dim X/Y \geqslant n$, then there exists an n-dimensional sublattice $W \subset X$ so that $W \cap Y = \{0\}$.*

Let us start with the case of finite dimensional X.

Lemma 2.2. *Suppose X is a Banach lattice of dimension n, and Y is a subspace of X of dimension $m < n$. Then X contains a sublattice W so that $\dim W = n - m$, and $W \cap Y = \{0\}$.*

Proof. As noted above, X is spanned by its atoms, so we can identify X (as a vector lattice) with $\mathbb{R}^n$ (or $\mathbb{C}^n$). By standard linear algebra, we can assume (up to relabeling) that $Y^\perp \subset \mathbb{R}^n$ has a basis $z_1 = (z_{1i})_{i=1}^n$, …, $z_{n-m} = (z_{n-m,i})_{i=1}^n$ so that, for $1 \leqslant k \leqslant n - m$, $z_{kk} = 1$, and $z_{ki} = 0$ if $i \in \{1, \ldots, n - m\} \backslash \{k\}$. Then

$$W = \{(a_1, \ldots, a_{n-m}, 0, \ldots, 0) : a_j \in \mathbb{R}, 1 \leqslant j \leqslant n - m\}$$

is the lattice we need. Indeed, for $w = (a_1, \ldots, a_{n-m}, 0, \ldots, 0) \in W$, $w \in Y$ if and only if, for any $k \in \{1, \ldots, n - m\}$, $a_k = \langle w, z_k\rangle = 0$. ∎

Recall that a Banach lattice is said to be *Dedekind* (or *order*) *complete* if every non-empty subset which is bounded from above has a supremum.

Lemma 2.3. *Theorem 2.1 is satisfied if X is Dedekind complete.*

Proof. Find an n-dimensional subspace $E \subset X$ so that $E \cap Y = \{0\}$. By compactness, there exists $c \in (0, 1/(9n))$ so that for every $x \in E$, $\mathrm{dist}(x, Y) \geqslant 9cn\|x\|$. By [15, P. 23], there exists a finite dimensional sublattice $G \subset X$ so that, for every $x \in E$, $\mathrm{dist}(x, G) \leqslant c\|x\|$. Let $Z = Y \cap G$. We need to show that $\dim G/Z \geqslant n$. Once this is done, an application of Lemma 2.2 would provide us with a sublattice $W \subset G$ with the desired properties.

Suppose $(e_i)_{i=1}^n$ is a normalized Auerbach basis in E (that is, $\|e_i\|_E = 1 = \|e_i^*\|_{E^*}$ for $1 \leqslant i \leqslant n$, where $e_1^*, \ldots, e_n^* \in E^*$ are the biorthogonal functionals). For each i, find $e_i' \in E$ so that $\mathrm{dist}(e_i, e_i') \leqslant c$. Note that $\big|\|e_i'\| - 1\big| \leqslant c$. Let $f_i = e_i'/\|e_i'\|$. Then

$$\|e_i - f_i\| \leqslant \|e_i - e_i'\| + \big|\|e_i'\| - 1\big| \leqslant 2c.$$

Let $F = \mathrm{span}[f_i : 1 \leqslant i \leqslant n]$, and consider a linear map $T : E \to F$, taking e_i to f_i, for every i. Then, for $e = \sum_{i=1}^n \alpha_i e_i$, we have $e - Te = \sum_{i=1}^n \alpha_i(e_i - f_i)$. As $\max_i |\alpha_i| \leqslant \|e\| \leqslant \sum_i |\alpha_i|$, we obtain $\|e - Te\| \leqslant 2cn\|e\|$. By the triangle inequality, $\|Te\| \geqslant (1 - 2cn)\|e\|$, hence, for any $f \in F$, $\|f - T^{-1}f\| \leqslant 4cn\|f\|$, and $\|T^{-1}f\| \geqslant \|f\|/2$.

Now, for $f \in F$,

$$\begin{aligned}\mathrm{dist}(f, Y) &\geqslant \mathrm{dist}(T^{-1}f, Y) - \|f - T^{-1}f\| \geqslant 9cn\|T^{-1}f\| - 4cn\|f\| \\ &\geqslant 9cn\frac{\|f\|}{2} - 4cn\|f\| = \frac{cn}{2}\|f\|,\end{aligned}$$

hence $F \cap Y = \{0\}$. ∎

Proof of Theorem 2.1. Consider $Y \subset X$, with $\dim X/Y \geqslant n$. Denote by J the canonical embedding of X into X^{**}. The latter lattice is Dedekind complete (see e.g. [19, Theorem 1.3.2]), hence, by Lemma 2.3, there exist an n-dimensional sublattice $E \subset X^{**}$ and $c \in (0, 1/9)$ so that, for any $e \in E$,

$\operatorname{dist}(e, Y^{\perp\perp}) \geqslant 3c\|e\|$. As $Y^{\perp\perp}$ is weak* closed, we can find $x_1^*, \ldots, x_N^* \in \mathbf{B}(X^*) \cap Y^\perp$ so that, for every $e \in E$, $\max_{1 \leqslant i \leqslant N} |\langle x_i^*, e\rangle| \geqslant 2c\|e\|$. Let $V = \{x^{**} \in X^{**} : \max_{1 \leqslant i \leqslant N} |\langle x_i^*, x^{**}\rangle| < c\}$. By [11], there exists a lattice isomorphism $T : E \to X$ so that $\|T\|, \|T^{-1}\| < 1 + c$, and, for every $e \in E$, $e - JTe \in \|e\|V$. Clearly $W = T(E)$ is an n-dimensional sublattice of X.

It remains to show that $J(W) \cap Y^{\perp\perp} = \{0\}$. To this end, for $e \in E\backslash\{0\}$, find i so that $|\langle x_i^*, e\rangle| \geqslant 2c\|e\|$. By our choice of T, $|\langle x_i^*, e - JTe\rangle| \leqslant c\|e\|$. Then

$$\big|\langle x_i^*, JTe\rangle\big| \geqslant \big|\langle x_i^*, JTe\rangle\big| - \big|\langle x_i^*, e - JTe\rangle\big| \geqslant 2c\|e\| - c\|e\| > 0.$$

As x_i^* annihilates $Y^{\perp\perp}$, we are done. ■

2.2. **Complements of ideals.** In the previous subsection, we found a sublattice in the complement of a finite codimensional subspace. Now we do the same for ideals of infinite codimension.

Proposition 2.4. *Suppose J is a closed ideal in X, with $\dim X/J = \infty$. Then $X\backslash J$ is completely latticeable.*

Note that, in this situation, the quotient map $Q : X \to X/J$ is a lattice homomorphism (see e.g. [19, Proposition 1.3.13]).

The proof of the following lemma essentially follows from [20, Theorem 9.1].

Lemma 2.5. *Suppose J is an ideal in a Banach lattice X, $\lambda > 1$, $Q : X \to X/J$ is a quotient map, and (y_i) is a sequence of disjoint positive elements in X/J. Then X contains disjoint positive elements (x_i) so that $\|x_i\| < \lambda\|y_i\|$ and $Qx_i = y_i$ for every i.*

Sketch of the proof. We can assume that $\|y_i\| = C^{1-i}$, where $C > \lambda/(\lambda - 1)$. We follow the recursive procedure from proof of [20, Theorem 9.1], keeping their notation.

Suppose we have already constructed $x_1, \ldots, x_n, u_n$, where $Qx_i = y_i$ and $\|x_i\| < (C + 1)C^{-i}$ for $1 \leqslant i \leqslant n$, $Qu_n = z_{n+1} = \sum_{j>n} y_j$, and $\|u_n\| < cC^{-n}/(1 - 1/C)$, where $c > 1$ is such that $c/(1 - 1/C) < \lambda$. Clearly, we can make such a selection for $n = 1$. To make the next step, find (using [19, Proposition 1.3.12]) $\tilde{x}_{n+1}$ and u'_{n+1} in $[0, u_n]$ so that $Q\tilde{x}_{n+1} = y_{n+1}$ and $Qu'_{n+1} = z_{n+2}$. Furthermore, there exists $u''_{n+1} \in X_+$ of norm less than $cC^{-n}/(1 - 1/C)$, so that $Qu''_{n+1} = z_{n+2}$. Then $\tilde{u}_{n+2} = u'_{n+1} \wedge u''_{n+1}$ belongs to $[0, u_n]$, has norm less than $cC^{-n}/(1 - 1/C)$, and satisfies $Q\tilde{u}_{n+1} = z_{n+2}$ (due to Q being a lattice homomorphism). Then define x_{n+1} and u_{n+1} as in proof of [20, Theorem 9.1]. ■

The following lemma is folklore.

Lemma 2.6. *Any infinite dimensional Banach lattice contains an infinite disjoint sequence.*

Sketch of a proof. Suppose E is an infinite dimensional Banach lattice. The result is clear if E has infinitely many atoms. Otherwise, a sequence with the desired properties can be constructed using the following fact: if a non-zero $x \in E_+$ is not an atom, then there exist non-zero disjoint $y, z \in E_+$ so that $y \vee z \leqslant x$ (see [3, p. 111]). ■

Proof of Proposition 2.4. Combining Lemmas 2.5 and 2.6, we conclude that X contains a normalized disjoint positive sequence (x_i) so that (Qx_i) is seminormalized and disjoint (as before, $Q : X \to X/J$ is the quotient map). We claim that $W = \overline{\operatorname{span}}[x_1, x_2, \ldots]$ meets J only at $\{0\}$. Indeed, any $w \in W$ has a unique expression $w = \sum_i a_i x_i$. Then $\operatorname{dist}(w, Y) = \|\sum_i a_i Q x_i\| \geqslant \sup_i |a_i| \|Qx_i\| > 0$. ■

Remark 2.7. In [26], a similar result is established for the case when X is an ordered Frechet space. Namely, X is shown to possess a sublattice W (not necessarily closed), so that $W \cap J = \{0\}$, and $W + Y$ is not closed.

2.3. **Complements of finite dimensional subspaces.**

Proposition 2.8. *If Y is a finite dimensional subspace of an infinite dimensional Banach lattice X, then $(X \backslash Y) \cup \{0\}$ contains a infinite dimensional closed ideal.*

Proof. By Lemma 2.6, X contains an infinite sequence of pairwise disjoint positive elements, which we label by (x_{ij}) $(i, j \in \mathbb{N})$. Suppose, for the sake of contradiction, that, for any i, the Banach lattice generated by $(x_{ij})_{j \in \mathbb{N}}$ meets $Y \backslash \{0\}$. For any i, find scalars (a_{ij}) so that $\sum_j a_{ij} x_{ij} = y_i \in Y \backslash \{0\}$. For n large enough, $y_1, \ldots, y_n$ are linearly dependent. Thus, there exist scalars $(c_i)_{i=1}^n$, some of them non-zero, so that $\sum_i c_i y_i = 0$. Then $\sum_{i,j} c_i a_{ij} x_{ij} = 0$, and this can only happen when $c_i a_{ij} = 0$ for any i and j. This is the desired contradiction. ■

2.4. **Complements of subspaces: separable atomic case.** Throughout this section, the order on our Banach lattice X is determined by its 1-unconditional basis $(\sigma_i)_{i \in \mathbb{N}}$ (that is, $(\sum_i a_i \sigma_i) \vee (\sum_i b_i \sigma_i) = \sum_i (a_i \vee b_i) \sigma_i$, etc.).

Proposition 2.9. *Suppose the order in a Banach lattice X is given by a 1-unconditional basisand Y is a subspace of X of infinite codimension. Then $(X \backslash Y) \cup \{0\}$ contains an infinite dimensional closed ideal.*

Proof. Denote the basis of X by $(\sigma_i)_{i=1}^\infty$. We need to find an infinite set I so that $\overline{\operatorname{span}}[\sigma_i : i \in I] \cap Y = \{0\}$. To this end, consider the quotient map $q : X \to X/Y$, and set $z_i = q\sigma_i$. One can then find $i_1 < i_2 < \ldots$ so that any finite family of the vectors $(z_{i_k})_{k \in \mathbb{N}}$ is linearly independent. By [24, Theorem III.6.1], we can find $k(1) < k(2) < \ldots$ with the following property: if $|\alpha_s| \leqslant 1$ for $s \in \mathbb{N}$, and $\sum_{s=1}^\infty \alpha_s z_{i_{k(s)}}$ converges to 0, then $\alpha_s = 0$ for any s. We claim that $I = \{i_{k(s)} : s \in \mathbb{N}\}$ has the desired property. Indeed, if $w = \sum_{s=1}^\infty \alpha_s \sigma_{i_{k(s)}} \in Y$, then $0 = qw = \sum_{s=1}^\infty \alpha_s z_{i_{k(s)}}$, which gives us $w = 0$. ■

If $X = \ell_p$ $(p > 1)$ or c_0, we prove that $(X\backslash Y) \cup \{0\}$ contains a closed sublattice "at positive angle" to Y.

Proposition 2.10. *Suppose X is either ℓ_p $(1 < p < \infty)$ or c_0, and Y is a closed subspace of X, with $\dim X/Y = \infty$. Then $X\backslash Y$ is completely latticeable. Moreover, there exists a closed sublattice W and a constant c so that $\|w + y\| \geqslant c\|w\|$ for any $w \in W$ and $y \in Y$.*

This clearly fails for $X = \ell_1$. Indeed, find Y so that X/Y is isometric to ℓ_2. Any subspace of X contains an isomorphic copy of ℓ_1, hence we cannot lift a Hilbert space into X.

For the proof we need an easy consequence of [16].

Lemma 2.11. *Suppose X is as in Proposition 2.10, and W is an infinite dimensional quotient of X. Then W contains an isomorphic copy of X. Furthermore, any weakly null sequence in W contains a subsequence equivalent to the canonical basis of X.*

Proof of Proposition 2.10. We work with $X = \ell_p$. The case of $X = c_0$ is handled in a similar fashion. By [16], X/Y contains an isomorph of ℓ_p. Find a sequence $(\tilde{x}_i)$ in X/Y, equivalent to the ℓ_p-basis. Find liftings $x_i \in X$, so that $qx_i = \tilde{x}_i$ ($q : X \to X/Y$ is the quotient map), and $\|x_i\| < 2\|\tilde{x}_i\|$ for any i. Passing to a subsequence, we may assume that (x_i) is weakly convergent. Furthermore, by considering $x_{2i-1} - x_{2i}$ instead of x_i, we can assume that (x_i) is weakly null. Note that this sequence is norm bounded away from 0 (just look at the images under the quotient maps). Passing to a further subsequence, we assume that the sequence (x_i) is "almost disjoint". Perturbing the x_i's slightly, we can assume that they are disjoint and consecutive. Write $x_i = u_i - v_i$, with u_i and v_i disjoint and positive. By passing to $-x_i$, we can assume that $c_1 = \inf_i \|u_i\| > 0$ for every i.

Now let $\tilde{u}_i = qu_i$ and $\tilde{v}_i = qv_i$. Then $\tilde{u}_i - \tilde{v}_i = \tilde{x}_i$. The sequence (u_i) is weakly null, hence so is $(\tilde{u}_i)$. By Lemma 2.11, we can assume, by passing to further subsequence, that $(\tilde{u}_i)$ is equivalent to the ℓ_p-basis. Therefore, for any finite sequence (a_i),

$$\Big(\sum_i |a_i|^p\Big)^{1/p} \succ \|\sum_i a_i u_i\| \geqslant \mathrm{dist}\big(\sum_i a_i u_i, Y\big) = \|\sum_i a_i \tilde{u}_i\| \succ \Big(\sum_i |a_i|^p\Big)^{1/p}.$$

Consequently, for $w \in W$ and $y \in Y$, $\|w + y\| \sim \|w\|$. ■

Remark 2.12. In contrast to Proposition 2.9, $C[0,1]$ contains an infinite codimensional subspace Y so that $(X\backslash Y) \cup \{0\}$ contains no ideals. To construct such Y, denote by $\mathcal{I}$ the family of all ternary intervals of the form $[a/3^k, (a+1)/3^k]$, with $a \in \{1, 4, 7, \ldots, 3^k - 2\}$. Note that the family $\mathcal{I}$ is self-similar, in the following sense. Consider $I = [\alpha, \beta] \in \mathcal{I}$, and define the affine map $\phi_I : [0,1] \to I : t \mapsto \alpha + t(\beta - \alpha)$ (hence, $\phi_I(0) = \alpha$ and $\phi_I(1) = \beta$). For any $J \in \mathcal{I}$, $\phi_I(J) \in \mathcal{I}$.

Let f denote the usual "Cantor staircase" (an increasing function with $f(0) = 0$, $f(1) = 1$, and constant on every interval $I \in \mathcal{I}$). We further introduce the "Cantor bridge" $g = f \wedge (1 - f)$. For $I \in \mathcal{I}$, define the function

$$g_I(t) = \begin{cases} g(\phi_I(t)) & t \in I \\ 0 & t \notin I \end{cases}$$

For an interval $I = [a, b]$, consider the measure $\mu_I = \delta_a - \delta_b$. Consider the family of measures $\mathcal{F} = \{\mu_I : I \in \mathcal{I}\}$, and define Y as the family of all $f \in C[0,1]$ so that $\langle \mu_I, f\rangle = 0$ for any $I \in \mathcal{I}$. By the above, $g_I \in Y$ for every I. Moreover, $\dim X/Y = \infty$. Indeed, otherwise the set $\{\mu_I : I \in \mathcal{I}\}$ would be spanned by finitely many of its elements. But this is impossible: for any $I_1, \ldots, I_n \in \mathcal{I}$, we can find $I \in \mathcal{I}$ so that the endpoints of I are different from those of $I_1, \ldots I_n$, hence μ_I cannot be represented as a linear combination of $\mu_{I_1}, \ldots, \mu_{I_n}$.

Now suppose Z is an ideal in $C[0,1]$. Pick $z \in Z$, and find $I \in \mathcal{I}$ so that $z > 0$ on I. Then there exists a constant C so that $g_I \leqslant C|z|$, hence $g_I \in Z \cap Y$.

2.5. **Complements of subspaces: continuous case.** In this section, we assume that X is a Köthe function space on an atomless measure space (Ω, μ). Our results in this setting are not as comprehensive as in the atomic case. Nevertheless, in certain cases we can show that $X\backslash Y$ is completely latticeable.

Recall that a subset $Y \subset X(\Omega, \mu)$ is called *uniformly equiintegrable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in Y$, $\|y\chi_A\| < \varepsilon$ whenever $\mu(A) < \delta$. Abusing the notation slightly, we say that a *subspace* $Y \subset X$ is *uniformly equiintegrable* if its unit ball is – that is, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in \mathbf{B}(Y)$, $\|y\chi_A\| < \varepsilon$ whenever $\mu(A) < \delta$. We say that a subspace $Y \subset X$ is *peaked* if it is not uniformly equiintegrable: that is, there exists $c > 0$ so that, for every $\varepsilon > 0$, there exist $g \in \mathbf{B}(Y)$ and a set A so that $\mu(A) < \varepsilon$ and $\|g\chi_A\| > c$.

Recall that a Banach lattice E is called *order continuous* if $\lim_\alpha \|x_\alpha\| = 0$ whenever a net $(x_\alpha) \subset E$ decreases to 0. For the essential properties and equivalent definitions of order continuity, the reader can consult [19, Section 2.4]. It is easy to see that, if a Köthe space $X(\Omega, \mu)$ is order continuous, then for every $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ so that $\|x\chi_A\| < \varepsilon$ whenever $\mu(A) < \delta$. Indeed, otherwise we would be able to find $x \in X_+$ and a decreasing sequence of sets (C_i) so that $\mu(C_i) < 2^{-i}$, and $\|x\chi_{C_i}\| \geqslant \varepsilon$. But $\inf_i(x\chi_{C_i}) = 0$, contradicting the order continuity.

If $X(\Omega, \mu)$ is reflexive, then, by [19, Sections 2.4, 2.6], both X and X^* are order continuous. Furthermore, X^* is a Köthe function space on (Ω, μ) (*the Köthe dual* of X). Any bounded disjoint sequence in X (or X^*) is weakly null.

Proposition 2.13. *Suppose $X(\Omega, \mu)$ is a reflexive Köthe function space on the atomless measure space (Ω, μ), and G is a peaked subspace of X^*. Then $X\backslash G^\perp$ is completely latticeable.*

Proof. We have a sequence of disjoint sets $A_i \subset \Omega$, and a normalized sequence $h_i \in G$, so that $\|h_i \chi_{A_i}\| > 5c$ for any i (c is a positive constant). By passing to a subsequence (and invoking the order continuity), we can assume that the sequence (h_i) is weakly convergent, the sets (A_i) are disjoint, and $\|h_j \chi_{A_i}\| < c4^{-(i+j)}$ whenever $i > j$ (consequently, $\|h_i - h_j\| > 4c$ if $i \neq j$).

Consider the weakly null sequence $g_i = (h_{2i} - h_{2i-1})/\|h_{2i} - h_{2i-1}\|$, then $\|g_i \chi_{A_{2i}}\| \geqslant 2c$. Moreover, changing the sign of g_i if necessary, we obtain disjoint sets B_i so that g_i is positive on B_i, and $\|g_i \chi_{B_i}\| > c$. Define a sequence of disjoint positive norm one functions $f_i \in X$, supported on B_i, so that $\langle f_i, g_i \rangle = \|g_i \chi_{B_i}\|$. As the sequences (f_i) and (g_i) are weakly null, we can assume (by passing to a subsequence if necessary) that $|\langle f_i, g_j \rangle| \leqslant \int_{B_i} f_i |g_j| < c4^{-(i+j)}$ for $i \neq j$.

We claim that the closed lattice W, generated by $f_1, f_2, \ldots$, meets $\mathrm{span}[g_i : i \in \mathbb{N}]^\perp$ only at $\{0\}$. Indeed, suppose $\langle \sum_j a_j f_j, g_i \rangle = 0$ for every i, and not all a_j's vanish. Pick j so that $|a_j| = \max_i |a_i|$. By multiplying by $\overline{a_j}/|a_j|$, we can assume that $a_j > 0$. Then

$$0 = a_j \langle f_j, g_j \rangle + \sum_{i \neq j} a_i \langle f_i, g_j \rangle \geqslant a_j c \Big(1 - \sum_{i \neq j} 4^{-(i+j)} \Big) > 0,$$

a contradiction. ■

The classical Kadec-Pełczynski dichotomy immediately yields:

Corollary 2.14. *Suppose (Ω, μ) is an atomless measure space, $1 < p < 2$, $1/p + 1/q = 1$, and the subspace $Z \subset L_q(\Omega, \mu)$ contains an isomorphic copy of ℓ_q. Then $L_p(\Omega, \mu) \backslash Z^\perp$ is completely latticeable.*

Proposition 2.15. *If (Ω, μ) is an atomless measure space, and a closed subspace Y is uniformly equiintegrable in an order continuous Köthe space $X(\Omega, \mu)$, then $X \backslash Y$ is completely latticeable.*

Proof. Pick $\delta > 0$ so that $\|\chi_A y\| < 1/3$ for any $y \in \mathbf{B}(Y)$, whenever $\mu(A) < \delta$. Consider disjoint sets A_i of measure $\delta/2^i$ each, and let $A = \cup_i A_i$. We claim that the closed span W of the norm one functions $f_i = \alpha_i \chi_{A_i}$ meets Y only at 0. In fact, W and Y form a direct sum. Indeed, consider $w = \sum_i a_i f_i$ of norm one. We claim that $\|w - y\| \geqslant 1/2$ for any $y \in Y$. Indeed, this inequality clearly holds if $\|y\| > 3/2$. Otherwise,

$$\|w - y\| \geqslant \|\chi_A (w - y)\| = \|w - \chi_A y\| \geqslant 1 - \|\chi_A y\| \geqslant 1 - \frac{1}{2} = \frac{1}{2},$$

which is what we need. ■

It is well known (see e.g. [12, Theorem 13.21]) that a subset of L_1 is relatively weakly compact if and only if it is uniformly equiintegrable. Thus, Proposition 2.15 yields:

Corollary 2.16. *If Y is a reflexive subspace of $L_1(\mu)$, then $L_1(\mu) \backslash Y$ is completely latticeable.*

Here is one application of Proposition 2.13. For $\Lambda \subset \mathbb{Z}$, we denote by $L_p^\Lambda(\mathbb{T})$ the space of such functions $f \in L_p(\mathbb{T})$ so that the n-the Fourier coefficient $\widehat{f}(n) = 0$ whenever $n \notin \Lambda$ (recall that $\mathbb{T}$ is the unit circle). The space $C^\Lambda(\mathbb{T})$ is defined similarly. We say that Λ *contains arbitrarily long arithmetic sequences* if for any $k \in \mathbb{N}$ there exist $a, n \in \mathbb{Z}$ so that $a, a+n, \ldots, a+kn \in \Lambda$.

Proposition 2.17. *Suppose Λ is a subset of $\mathbb{Z}$, containing arbitrarily long arithmetic sequences. Then the spaces $L_p^\Lambda(\mathbb{T})$ ($1 \leqslant p < \infty$) and $C^\Lambda(\mathbb{T})$ are peaked.*

Proof. We examine the L_p case. Continuous functions can be handled similarly. It suffices to check that, for every $\varepsilon \in (0,1)$, there exists a norm one $f \in L_p^\Lambda$ and a set A of measure less than ε, so that $\|f|_A\| > 1-\varepsilon$. Find a set B with $\mu_0(B) = c \in (0, \varepsilon/2)$. Approximating $c^{-1/p}\chi_B$ by a polynomial in the L_p norm, obtain a norm one polynomial g with $\|g|_B\| > 1-\varepsilon$. Write $g(z) = \sum_{j=-N}^N a_j z^j$. Now find $a \in \mathbb{Z}$ and $k \in \mathbb{N}$ so that $a - Nk, a - NK + k, \ldots, a + Nk \in \Lambda$. Let $h(z) = z^a g(z^k)$. This norm one polynomial belongs to L_p^Λ. Moreover, $\|h|_A\| > 1-\varepsilon$, where $A = \cup_{j=0}^{k-1}[2\pi j/k, (2\pi j + c)/k]$. ∎

Corollary 2.18. *If $1 < p < \infty$, and the set $\Lambda \subset \mathbb{Z}$ is such that $\mathbb{Z}\backslash\Lambda$ contains arbitrarily long arithmetic sequences, then $L_p(\mathbb{T})\backslash L_p^\Lambda(\mathbb{T})$ is completely latticeable.*

Remark 2.19. Sometimes $Y \subset L_p$ ($1 < p < \infty$) is peaked, and $Y^\perp \subset L_q$ ($1/p + 1/q = 1$) is not peaked. In this situation, Propositions 2.13 and 2.15 cannot be used to establish the complete latticeability of $L_p \backslash Y$. This happens, for instance, if Y is the span of all Walsh functions of order $\neq 1$ (that is, all Walsh functions except for Rademachers). Then $Y^\perp$ (the span of Rademachers) is uniformly equiintegrable (see e.g. [2, Section 6.4]). On the other hand, Y contains $r_1 r_2 L_p$ ($r_1, r_2, \ldots$ are the Rademachers), hence not uniformly equiintegrable.

Proposition 2.20. *In the notation of Remark 2.19, $L_p \backslash Y$ is completely latticeable.*

Proof. View L_p as the space of functions on $\{-1,1\}^\mathbb{N}$ (with the product measure), and the Rademacher functions r_i as the coordinates. Consider the disjoint positive functions

$$f_2 = 2^{2/p}\chi_{\{-1\}\times\{1\}\times\{-1,1\}\times\{-1,1\}\times\ldots},$$

$$f_3 = 2^{3/p}\chi_{\{-1\}\times\{-1\}\times\{1\}\times\{-1,1\}\times\{-1,1\}\times\ldots},$$

$$f_4 = 2^{4/p}\chi_{\{-1\}\times\{-1\}\times\{-1\}\times\{1\}\times\{-1,1\}\times\{-1,1\}\times\ldots}, \text{ and so on.}$$

We claim that $\overline{\mathrm{span}}[f_2, f_3, \ldots] \cap Y = \{0\}$. With $1/p + 1/q = 1$, we have:

$$\langle f_i, r_j \rangle = \begin{cases} 0 & j > i \\ 2^{-i/q} & j = i \\ -2^{-i/q} & j < i \end{cases} .$$

Now suppose $\sum_{i=2}^{\infty} a_i f_i \in Y$, and show that $a_2 = a_3 = \ldots = 0$. Note that $\sum_i a_i \langle f_i, r_j \rangle = 0$ for any $j \geqslant 1$, which gives us a system of equations:

$$\begin{cases} -2^{-2/q}a_2 - 2^{-3/q}a_3 - 2^{-4/q}a_4 - \ldots & = & 0 & (j=1) \\ 2^{-2/q}a_2 - 2^{-3/q}a_3 - 2^{-4/q}a_4 - \ldots & = & 0 & (j=2) \\ 2^{-3/q}a_3 - 2^{-4/q}a_4 - 2^{-5/q}a_5 - \ldots & = & 0 & (j=3) \\ \ldots & = & 0 & \ldots \end{cases} .$$

Adding the first two equations from (2.5) (corresponding to $j = 1$ and $j = 2$), we conclude that $a_2 = 0$, and $2^{-3/q}a_3 + 2^{-4/q}a_4 + \ldots = 0$. Adding the last equation to the one equation of (2.5) corresponding to $j = 3$, we obtain $a_3 = 0$, and $2^{-4/q}a_4 + 2^{-5/q}a_5 + \ldots = 0$. Proceeding further in the same manner, we arrive at $0 = a_2 = a_3 = \ldots$, which is the desired conclusion. ∎

3. Latticeability: complements of dense subsets

The main result of this section is:

Theorem 3.1. *For $0 < p < \infty$, $\ell_p \backslash (\cup_{q<p} \ell_q)$ contains a sublattice which is dense in ℓ_p. Likewise, $c_0 \backslash (\cup_{q<\infty} \ell_q)$ contains a sublattice which is dense in c_0.*

For the proof we need:

Lemma 3.2. *There exists a sequence of functions $f_j : \mathbb{N} \to \mathbb{N}$ ($j \in \mathbb{N}$) so that for every distinct $i_1, \ldots, i_n \in \mathbb{N}$ and every $k_1, \ldots, k_n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ so that $f_{i_j}(m) = k_j$, for $1 \leqslant j \leqslant n$.*

Proof. Let S to be the set of all sequences $(x_p)_{p\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ which are eventually 1 (that is, $x_p \in \mathbb{N}$ for every p, and $x_p = 1$ for p large enough). Set $g_j((x_p)_{p\in\mathbb{N}}) = x_j$. Find a bijection $h : \mathbb{N} \to S$, then the functions $f_j = g_j \circ h$ have the desired properties. ∎

Proof of Theorem 3.1. Show first that, for $0 < p < \infty$, $\ell_p(\mathbb{N}\otimes\mathbb{N}) \backslash (\cup_{q<p} \ell_q(\mathbb{N}\otimes\mathbb{N}))$ is densely latticeable.

For $N \in \mathbb{N}$ denote by A_N a countable dense subset of the unit sphere of $\ell_p(\{1, \ldots, N\} \times \{1, \ldots, N\}) \subset \ell_p(\mathbb{N}\times\mathbb{N})$. Find a bijection $\phi : \cup_{N>10} A_N \to \mathbb{N}$. Lemma 3.2 shows that there exist mappings $f_i : \mathbb{N} \to \mathbb{K} = \mathbb{Q} \cap (-1, 1)$ ($i \in \mathbb{N}$) so that for every distinct $i_1, \ldots, i_n \in \mathbb{N}$ and every $k_1, \ldots, k_n \in \mathbb{K}$ there exists $m \in \mathbb{N}$ so that $f_{i_j}(m) = k_j$, for $1 \leqslant j \leqslant n$. For $u = (u_{ij})_{i,j=1}^N \in A_N$, define $\Phi(u) \in \ell_p(\mathbb{N} \otimes \mathbb{N})$ by setting

$$\big[\Phi(u)\big]_{ij} = \begin{cases} u_{ij} & 1 \leqslant i, j \leqslant N \\ 2^{-i/p} f_{\phi(u)}(i) j^{-1/p} (\ln j)^{-2/p} & j > N \\ 0 & i < N, j \leqslant N \end{cases} .$$

Denote by W the lattice generated by $\Phi(u)$ as above.

To show that W is dense in $\ell_p(\mathbb{N}\times\mathbb{N})$, note that, for $u \in A_N$,

$$\left\|\Phi(u) - u\right\|^p = \sum_{j=N+1}^{\infty}\sum_{i=1}^{\infty}\left(2^{-i/p} f_{\phi(u)}(i) j^{-1/p}(\ln j)^{-2/p}\right)^p < \int_N^\infty \frac{dt}{t(\ln t)^2} = \frac{1}{\ln(\ln N)}.$$

Now pick a norm one $v \in \ell_p(\mathbb{N}\times\mathbb{N})$. For $\varepsilon > 0$ find $N \in \mathbb{N}$ so that $\ln(\ln N) > (2/\varepsilon)^p$. Find $u \in A_N$ with $\|u - v\| < \varepsilon/2$. By the triangle inequality, $\|v - \Phi(u)\| < \varepsilon$.

It remains to prove that $W \cap \ell_q(\mathbb{N}\times\mathbb{N}) = \{0\}$ whenever $q < p$. Any $w \in W$ can be written as $w = \Psi(\Phi(u_1), \ldots, \Phi(u_L))$, where $\Psi : \mathbb{R}^L \to \mathbb{R}$ is a finite composition of linear operations (addition, multiplication by a scalar), and lattice operations (taking maximum, minimum, or absolute value), and $u_k \in A_{N_k}$ ($N_k > 10$). For $j > N = \max_k N_k$,

$$w_{ij} = 2^{-i/p} j^{-1/p}(\ln j)^{-2/p}\Psi\big(f_{\phi(u_1)}(i), \ldots, f_{\phi(u_L)}(i)\big).$$

As Ψ is positively homogeneous and continuous, either Ψ is identically 0 on $\mathbb{R}^L$, or there exists i so that $\Psi\big(f_{\phi(u_1)}(i), \ldots, f_{\phi(u_L)}(i)\big) = c \neq 0$. If this is the case, then

$$\sum_{i,j} |w_{ij}|^q \geqslant |c|^q 2^{-iq/p} \sum_{j=N+1}^{\infty} \left(j^{-1/p}(\ln j)^{-2/p}\right)^q = \infty,$$

which shows that $w \notin \ell_q(\mathbb{N}\times\mathbb{N})$.

The dense latticeability of $c_0(\mathbb{N}\times\mathbb{N})\backslash(\cup_{q<\infty}\ell_q(\mathbb{N}\times\mathbb{N}))$ can be established similarly. The only difference lies in the definition of Φ: for $j > N$ we set $(\Phi(u))_{ij} = f_{\phi(u)}(i)/\ln j$. ■

4. Algebrability

Here we study the algebrability of sets. Throughout this section we shall use the following simple observation: if A is an infinite minimal set (in particular, a free set) of generators of an algebra X, and B is another set of generators of X, then $|A| \leqslant |B|$ (as pointed out in [4], this fails for finite sets).

4.1. **The commutative case.** As before, we equip the unit circle $\mathbb{T}$ with the rearrangement invariant probability Lebesgue measure μ_0. This turns $L_1(\mathbb{T})$ into a convolution algebra.

Proposition 4.1. *The set* $\big(L_1(\mathbb{T})\backslash(\cup_{p>1}L_p(\mathbb{T}))\big)\cup\{0\}$ *is densely maximally algebrable.*

For future use we need a Hamel basis $\mathcal{G}$ for $\mathbb{R}$, considered as a vector space over $\mathbb{Q}$. Without loss of generality, we can assume that $1 \in \mathcal{G}$, and let $\mathcal{F} = \mathcal{G}\backslash\{1\}$.

We also need to recall some well known harmonic analysis facts (to be found in all standard textbooks, such as [17]). For $n \in \mathbb{N}$ consider the *Fejer kernel*

$$K_n(t) = \sum_{j=-n}^{n} \frac{n-|j|}{n} e^{\iota j t} = \frac{1}{n} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^2 .$$

Then $K_n \geqslant 0$, and $\|K_n\|_1 = \int K_n \, d\mu_0 = 1$.

Now suppose $(a_n)_{n \geqslant 0}$ is a decreasing convex sequence, converging to 0. Consider the sequences $\Delta a_k = a_k - a_{k-1}$, and $\Delta^2 a_k = \Delta a_{k+1} - \Delta a_k = a_{k+1} + a_{k-1} - 2a_k$ ($k \geqslant 1$). It is is easy to see (cf. [27, Section III.4]) that $\lim_k k \Delta a_k = 0$, and $\sum_{k=1}^n k \Delta^2 a_k = a_0 - a_n + n \Delta a_{n+1}$. Let $g = g[a] = \sum_{j=1}^\infty j \Delta^2 a_j K_j$ and $g_n = \sum_{j=1}^n j \Delta^2 a_j K_j$. By the above,

$$\|g - g_n\|_1 \leqslant \sum_{j=n+1}^{\infty} j \Delta^2 a_j = a_n - n \Delta a_{n+1},$$

hence $\lim_n \|g - g_n\|_1 = 0$. Moreover, for $0 \leqslant |m| < n$,

$$\widehat{g_n}(m) = \sum_{j=1}^{n} j \Delta^2 a_j \widehat{K_j}(m) = \sum_{j=m+1}^{n} (j - |m|) \Delta^2 a_j = a_{|m|} - a_n - (n - |m|) \Delta a_{n+1}.$$

Thus, for any m, $\widehat{g}(m) = \lim_n \widehat{g_n}(m) = a_{|m|}$. Now observe that $g \geqslant 0$, hence $\|g\|_1 = \int g \, d\mu_0 = a_0$.

Proof of Proposition 4.1. For $\alpha \in (1/2, 1)$ and $N > 10$, define the sequence $a^{[N,\alpha]} \in c_0(\mathbb{Z})$ by setting, for $k \in \mathbb{Z}$,

$$a_k^{[N,\alpha]} = \begin{cases} (\ln |k|)^{-\alpha} & |k| \geqslant N \\ (\ln N)^{-\alpha} + (N - |k|)((\ln N)^{-\alpha} - (\ln(N+1))^{-\alpha}) & |k| < N \end{cases} .$$

The sequence $(a_k^{[N,\alpha]})_{k=0}^\infty$ is convex, and decreasing to 0. Consider also the functions $g^{[N,\alpha]}$, built on the sequence $a^{[N,\alpha]}$ in the way described above. Then $\|g^{[N,\alpha]}\| < 2(\ln N)^{-1/2}$. Indeed, by the above, it suffices to prove $a_0^{[N,\alpha]} < 2(\ln N)^{-1/2}$. Let $f(t) = (\ln t)^{-\alpha}$. By convexity,

$$a_0^{[N,\alpha]} \leqslant a_N^{[N,\alpha]} + N|f'(N)| = \big(\ln N\big)^{-\alpha} \Big(1 + \frac{\alpha}{\ln N}\Big).$$

Define by $\mathcal{P}_N$ the set of all trigonometric polynomials of degree exactly N – that is, the set of all polynomials $u(j) = \sum_{|j| \leqslant N} b_j e^{\iota t j}$, with $|b_{-N}| + |b_N| > 0$. Find a bijection

$$\phi : \bigcup_{N \geqslant 10} \mathcal{P}_N \to \mathcal{F}' = \frac{1}{2} \mathcal{F} + \frac{1}{2},$$

with $\mathcal{F}$ as above. For $u \in \mathcal{P}_N$ ($N \geqslant 10$), set $\Phi(u) = u + g^{[N,\phi(u)]}$, where

$$g^{[N,\alpha]} = g\big[a^{[N,\alpha]}\big] = \sum_{j=1}^{\infty} j \Delta^2 a_j^{[N,\alpha]} K_j .$$

Let W be the algebra generated by functions $\Phi(u)$. That is, W consists of finite linear combinations of convolutions $\Phi(u_1) * \ldots * \Phi(u_n)$.

Let us show first that, for any $h \in L_1$ and $\varepsilon > 0$, there exists $u \in \mathcal{P}_n$ so that $\|h - \Phi(u)\| < \varepsilon$ (and consequently, W is dense in $L_1(\mathbb{T})$). Pick $M > 10$ such that $2(\ln M)^{-1/2} < \varepsilon/2$. Find a polynomial u of degree $N > M$ so that $\|h - u\| < \varepsilon/2$. Then

$$\Big\|h - \Phi(u)\Big\| \leqslant \Big\|h - u\Big\| + \Big\|g^{[N,\phi(u)]}\Big\| < \frac{\varepsilon}{2} + \frac{2}{\sqrt{\ln N}} < \varepsilon.$$

To show the other properties of W, we introduce some notation. We say that $f \in L_1$ has *exact coefficient decay type* $a > 0$ if, for sufficiently large k, $\widehat{f}(k) = b\big(\ln(k+1)\big)^{-a}$, with some $b \neq 0$. Furthermore, f is said to have *approximate coefficient decay type* $a > 0$ if $\lim_{k\to\infty} \big(\ln(k+1)\big)^{-a} \widehat{f}(k)$ exists, and is different from 0. It is easy to observe that, if $f_1, \ldots f_n$ have approximate coefficient decay types $a_1, \ldots, a_n$, with $a_i \neq a_j$ for $i \neq j$, and $c_1, \ldots, c_n$ are non-zero scalars, then $\sum_i c_i f_i$ has approximate coefficient decay type $\min_{1\leqslant k\leqslant n} a_k$.

Clearly,

$$\widehat{\Phi(u_1) * \ldots * \Phi(u_n)}(k) = \Big(\ln k\Big)^{-\sum_{j=1}^n \phi(u_j)} \text{ for } k \text{ large enough,}$$

hence $\Phi(u_1)*\ldots*\Phi(u_n)$ has exact coefficient decay type $\sum_{j=1}^n \phi(u_j)$. As $\mathcal{F}$ is linearly independent over $\mathbb{Q}$, $\Phi(u_1) * \ldots * \Phi(u_n)$ and $\Phi(v_1) * \ldots * \Phi(v_m)$ have different exact coefficient decay types, unless $n = m$, and $\{u_1, \ldots, u_n\} = \{v_1, \ldots, v_m\}$ (counting repetitions). Thus, the functions $\Phi(u)$ are free in $L_1(\mathbb{T})$. Furthermore, any non-zero $w \in W$ has a certain approximate coefficient decay type. In particular, $\widehat{w} \notin \ell_q(\mathbb{Z})$ for any finite q. Hence, by Hausdorff-Young Inequality, $w \notin L_p(\mathbb{T})$, for any $p > 1$. ■

4.2. **The non-commutative case.** Throughout, $\mathcal{S}_p$ denotes the Schatten p-space on ℓ_2 ($\mathcal{S}_\infty$ stands for the space of compact operators, same as $K(\ell_2)$).

Proposition 4.2. *The set* $\big(\mathcal{S}_\infty \backslash \big(\cup_{p<\infty} \mathcal{S}_p\big)\big) \cup \{0\}$ *is densely maximally algebrable.*

Proof. We denote by $\mathbf{c}$ the continuum. Find a family of infinite sets $A_i \subset \mathbb{N}$ ($i \in \mathbf{c}$) so that, for any $i \neq j$, $A_i \cap A_j$ is finite (note that then, for any finite set $S \subset \mathbf{c}$, $|\mathbb{N}\backslash(\cup_{i\in S} A_i)| = \infty$). Let $\pi_i : \mathbb{N} \to A_i$ be the increasing bijection. For an m-tuple $t = (i_1, \ldots, i_m) \in \mathbf{c}^m$, set

$$A_t = \pi_t(\mathbb{N}), \text{ where } \pi_t = \pi_{i_1} \circ \ldots \circ \pi_{i_m}$$

(in this notation, $A_i = A_{(i)}$). We show the sets A_t are "sufficiently different".

To this end, consider a partial order on the family of finite subsets of $\mathbf{c}$. For $t = (i_1, \ldots, i_m)$ and $s = (j_1, \ldots, j_n)$, write $t \precsim s$ if t is the initial segment of s – that is, $n \geqslant m$, and $i_k = j_k$ for $1 \leqslant k \leqslant m$. Write $t \prec s$ if $t \precsim s$, $t \neq s$. If neither $t \precsim s$ nor $s \precsim t$, we say that t and s are non-comparable, and write $t \nsim s$.

Note that, for $t \nsim s$, $A_t \cap A_s$ is finite. Indeed, find the smallest k so that $i_k \neq j_k$. Then

$$A_{(i_k \dots i_m)} \subset A_{i_k} \text{ and } A_{(j_k \dots j_n)} \subset A_{j_k},$$

and $|A_{i_k} \cap A_{j_k}| < \infty$. Consequently,

$$A_t \cap A_s = \pi_{i_1} \dots \pi_{i_{k-1}} (A_{i_k} \cap A_{j_k})$$

is a finite set.

Next observe that, if $t \prec s$, then $A_s \subset A_t$. Moreover, if $s_1, \dots, s_K$ are such that $t \prec s_k$ for $1 \leqslant k \leqslant K$, then $A_t \backslash (\cup_{k=1}^K A_{s_k})$ is infinite. Indeed, by truncating it suffices to consider the case of $s_k = t \smile r_k$, where $\smile$ indicates concatenation. Then $\mathbb{N} \backslash (\cup_{k=1}^K A_{r_k})$ is infinite (indeed, consider the intersection of that set and A_r with some $r \notin \{r_1, \dots, r_K\}$), hence the same is true for

$$A_t \backslash (\cup_{k=1}^K A_{s_k}) = \pi_t \big(\mathbb{N} \backslash (\cup_{k=1}^K A_{r_k}) \big).$$

The discussion above show that, if $t_k = \big((t_{ik})_{i=1}^{m_k} \big)_{k=1}^K$ are distinct tuples with $m_1 \leqslant \dots \leqslant m_K$ ($t_k \in \mathbf{c}$), then

$$A_{t_1} \backslash \big(\cup_{k=1}^K A_{t_k} \big) = \cap_{k=1}^K \big(A_{t_1} \backslash A_{t_k} \big)$$

is infinite.

Define an isometry $U_i : \ell_2 \to \ell_2(A_i)$ via $U_i(\delta_k) = \delta_{\pi_i(k)}$, where (δ_i) denote the canonical basis. For $t = (i_1, \dots, i_m) \in \mathbf{c}^m$ as above, $U_t = U_{i_1} \dots U_{i_m} : \ell_2 \to \ell_2(A_t)$ is an isometry with $0-1$ entries.

Now let $j_1 < j_2 < \dots$ be the enumeration of $A_{t_1} \backslash \big(\cup_{k>1} A_{t_k} \big)$, and let $i_s = \pi_t^{-1}(j_s)$. Then for any s,

$$\langle U_{t_k} \delta_{i_s}, \delta_{j_s} \rangle = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases} .$$

For an operator $T \in B(\ell_2)$, denote by $T^{[n]}$ the truncation of T to the upper left $n \times n$ corner. It is easy to verify that, for any tuple $t = (t_1, \dots, t_m)$, and for any $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ so that, for any $n \geqslant N$,

$$\langle U_{t_1}^{[n]} \dots U_{t_m}^{[n]} \delta_i, \delta_j \rangle = \langle U_t \delta_i, \delta_j \rangle \text{ whenever } i, j \leqslant k.$$

Let $I_n = \{1, \dots, n\}$ and $J_n = \{n(n-1)/2 + 1, \dots, n(n+1)/2\}$, and let V_n be the "increasing coordinatewise" isometry from $\ell_2(I_n)$ to $\ell_2(J_n)$. Let P_n be the orthogonal projection from ℓ_2 onto $\ell_2(J_n)$. Now consider a bijection $\phi : \cup_{n>10} \big(M_n \backslash \{0\} \big) \mapsto \mathbf{c}$ ($M_n = B(\ell_2(I_n))$ is the space of $n \times n$ matrices). Define the map $\Phi : \cup_{n>10} \big(M_n \backslash \{0\} \big) \to K(\ell_2)$ by setting, for $x \in M_n$,

$$\Phi(x) = x + \sum_{m>n} \frac{1}{\ln m} V_m U_{\phi(x)}^{[m]} V_m^*$$

(the sum above is block-diagonal). Clearly $\Phi(x)^{[n]} = x$, and $\|x - \Phi(x)\| \leqslant 1/\ln(n+1)$. Thus, the image of Φ is dense in $\mathcal{S}_\infty$. Let W be the algebra (non-closed) generated by the image of Φ. We shall show that the elements

$\Phi(x)$ are free (as generators of W), and moreover, the algebra W only meets $\mathcal{S}_p$ ($p < \infty$) at 0.

Indeed, any $w \in W$ can be written as

$$w = \sum_{k=1}^{K} a_k \Phi(x_{k1}) \ldots \Phi(x_{kJ_k}),$$

with $x_{kj} \in M_{n(k,j)}$, $J_1 \leqslant \ldots \leqslant J_K$, and non-zero a_k's. By the discussion above, there exist $u_1 < u_2 < \ldots$ and $v_1 < v_2 < \ldots$ so that

$$\langle U_{\phi(x_{k1})} \ldots U_{\phi(x_{kJ_k})} \delta_{u_s}, \delta_{v_s} \rangle = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases}$$

(we can take, for instance, (v_k) to be an enumeration of $A_{t_1} \backslash (\cup_{k=2}^{K} A_{t_k})$, where $t_k = (\phi(x_{k1}), \ldots, \phi(x_{kJ_k}))$. Therefore, there exists $N > \max_{j,k} n(j,k)$ so that, for $m > N$,

$$\langle U^{[m]}_{\phi(x_{k1})} \ldots U^{[m]}_{\phi(x_{kJ_k})} \delta_{u_1}, \delta_{v_1} \rangle = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases} .$$

For such m,

$$\begin{aligned} \|P_m w P_m\| &= \left\| \sum_k a_k P_m \Phi(x_{k1}) \ldots \Phi(x_{kJ_k}) P_m \right\| \\ &\geqslant \left| \sum_k \frac{a_k}{(\ln m)^{J_k}} \left\langle U^{[m]}_{\phi(x_{k1})} \ldots U^{[m]}_{\phi(x_{kJ_k})} \delta_{u_1}, \delta_{v_1} \right\rangle \right| = \frac{|a_1|}{(\ln m)^{J_1}}. \end{aligned}$$

Consequently, for $p < \infty$,

$$\|w\|_p \geqslant \Big(\sum_{m > N} \|P_m w P_m\|^p \Big)^{1/p} = \infty,$$

which shows that $w \notin \mathcal{S}_p$. ■

Question 4.3. What can we say about $*$-subalgebras of $\mathcal{S}_\infty \backslash \big(\cup_{p<\infty} \mathcal{S}_p \big)$?

5. Non-regular operators

Recall that an operator $T = (t_{ij}) \in B(\ell_p)$ is regular if and only if $|T| = (|t_{ij}|)$ is bounded, and $\|T\|_r = \||T|\|$. It is known that, for $1 < p < \infty$, there exist bounded operators on ℓ_p which are not regular (equivalently, not order bounded). We denote the space of regular operators by B_r. K stands for the space of compact operators, and set $K_r = K \cap B_r$.

Proposition 5.1. *For $1 < p < \infty$, the set $K(\ell_p) \backslash K_r(\ell_p)$ is (i) densely maximally lineable, and (ii) spaceable.*

Recall that $A \subset X$ is called *densely maximally lineable* if $A \cup \{0\}$ contains a dense subspace of linear dimension equal to that of X.

Proof. For $k \in \mathbb{N}$, let W_k be the Walsh unitary. These square matrices of size 2^k can be defined recursively:

$$W_0 = 1, \ W_{k+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} W_k & W_k \\ -W_k & W_k \end{pmatrix}.$$

It is easy to check that $\|W_k\|_{B(\ell_2^{2^k})} = 1$ for every k. Moreover, the entries of W_k are equal $\pm 2^{-k/2}$, hence $|||W_k|||_{B(\ell_q^{2^k})} = 2^{k/2}$ for $1 \leqslant q \leqslant \infty$.

Set $U_k = 2^{-ak} W_k$, where $a = |1/2 - 1/p|$. By interpolation, U_k is a contraction (when acting on $\ell_p^{2^k}$), and $|||U_k||| = 2^{bk}$, where $b = 1/2 - a = \min\{1/p, 1/p'\}$ $(1/p + 1/p' = 1)$.

To prove (i), for $n \in \mathbb{N}$ we let

$$T_n = 0_1 \oplus \ldots \oplus 0_{n-1} \oplus 2^{-bn/2} U_n \oplus 2^{-b(n+1)/2} U_{n+1} \oplus \ldots,$$

where 0_k is a $2^k \times 2^k$ zero matrix (above, U_k is "supported on" the interval $I_k = [2^k - 1, 2^{k+1} - 2]$). Then T_n is compact, satisfies $\|T_n\| \leqslant 2^{-bn/2}$, but is not regular.

We denote by P_S the coordinate projection associated with $S \subset \mathbb{N}$. We write $Q_N = P_{I_N}$, $R_n = P_{\{1,\ldots,n\}}$, and $R_n^\perp = I - R_n$. Find a family of infinite sets $A_\alpha \subset \mathbb{N}$ ($\alpha \in \mathbf{c}$, where $\mathbf{c}$ denotes the continuum), so that $A_\alpha \cap A_\beta$ is finite whenever $\alpha \neq \beta$. Let $\psi_\alpha : \mathbb{N} \to A_\alpha$ be a bijection, and let

$$V_\alpha : \ell_p \to \ell_p(A_\alpha) : \delta_i \mapsto \delta_{\psi_\alpha(i)}$$

be the corresponding isometry. For $S \in B(\ell_p)$ define $S^\alpha = V_\alpha S V_\alpha^* \in B(\ell_p(A_\alpha))$.

Now find a bijection $\phi : \cup_{n\in\mathbb{N}} \big(M_n \backslash \{0\}\big) \to \mathbf{c}$. For a non-zero $x \in M_n$ set $\Phi(x) = x + T_n^{\phi(x)}$ (here we identify the space of $n \times n$ matrices M_n with the set of operators on ℓ_p "living" in the $n \times n$ corner). By the above, $\Phi(x) \in K(\ell_p)$. Let Y be the linear (non-closed) span of the operators $\Phi(x)$.

Note first that Y is dense in $K(\ell_p)$. Indeed, for non-zero $x \in M_n$, $T_n^\alpha = R_n^\perp \Phi(x) R_n^\perp$, and $x = R_n \Phi(x) R_n$, hence $\|x - \Phi(x)\| \leqslant 2^{-bn/2}$. To obtain the density of Y, recall that operators with finitely many non-zero entries are dense in $K(\ell_p)$.

Next observe that the operators $\Phi(x)$ are linearly independent (hence any Hamel basis for Y has cardinality of the continuum), and moreover, $Y \cap K_r(\ell_p) = \{0\}$. Indeed, suppose $x_1 \in M_{s_1}, \ldots, x_m \in M_{s_m}$ are distinct, and $a_1, \ldots, a_m$ are all different from 0. Let us show that $w = \sum_{j=1}^m a_j \Phi(x_j) \notin K_r(\ell_2)$.

Let $\alpha_j = \phi(x_j)$. The set $A_{\alpha_1} \cap (\cup_{j=2}^m A_{\alpha_j})$ is finite, hence, for all but finitely many values of $N > \max_j s_j$, the set $J_N = \psi_{\alpha_1}(I_N)$ does not meet $\cup_{j=2}^m A_{\alpha_j}$. For such an N, we have

$$\|w\|_r \geqslant \|P_{J_N} w P_{J_N}\|_r = |a_1| 2^{-bN/2} \|U_N\|_r = |a_1| 2^{bN/2}.$$

Taking the supremum over all suitable N, we see that w is not regular. This establishes (i).

The proof of (ii) is simpler, and we merely sketch it: inside of $K(\ell_p)$ find a copy of c_0, spanned by $\oplus_{k \in B_i} 2^{-bk/2} U_k$, where the infinite sets $B_1, B_2, \ldots$ are disjoint. ■

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