GREEDY ALGORITHM WITH GAPS

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Abstract. We generalize the well-known greedy approximation algorithm, by allowing gaps in the approximating sequence. We give examples of bases which are "quasi-greedy with gaps," in spite of failing to be quasi-greedy in the usual sense. However, we also show that for some classical bases (such as the normalized Haar basis in L_1, and the trigonometric basis in L_p for $p \neq 2$), the greedy algorithm may diverge, even if gaps are introduced into the approximating sequence.

1. Introduction

We introduce and investigate a generalization of the well-known Thresholding Greedy Algorithm. First we need to fix some notation. If $(e_i, e_i^*)_{i \in I} \subset X \times X^*$ is a biorthogonal system, and A is a finite subset of $\mathbb{N}$, we write $Ax = \sum_{i \in A} \langle e_i^*, x \rangle e_i$, and $A^c x = x - Ax$. By default we assume that all our biorthogonal systems are infinite (that is, X is infinite dimensional), complete ($\overline{\mathrm{span}[e_i : i \in I]} = X$) and total ($\overline{\mathrm{span}[e_i^* : i \in I]}^{\mathrm{weak}*} = X^*$), and further, $\sup_i \max\{\|e_i\|, \|e_i^*\|\} < \infty$ (the systems are *semi-normalized*). We would sometimes refer to $(e_i)_{i \in I}$ as a *basis*. The *support* of x ($\mathrm{supp}\, x$) is the set $\{i : \langle e_i^*, x \rangle \neq 0\}$. By our assumption, elements with finite support are dense in X.

We say that a set $A \subset I$ is *a t-greedy set for $x \in X$* (here $t \in (0, 1]$) if

$$\inf_{i \in A} \left|\langle e_i^*, x \rangle\right| \geq t \sup_{i \notin A} \left|\langle e_i^*, x \rangle\right|$$

(if x has infinite support, then such a set A is necessarily finite, since, for every $\varepsilon > 0$, the set $\{i : |\langle e_i^*, x \rangle| > \varepsilon\}$ is finite, see [5]). A *t-greedy projection of x* $\mathbf{G}_m^t(x)$ is Ax, where A is a t-greedy set of cardinality m (such a projection need not be unique). When $t = 1$, we simply talk about greedy sets, and greedy projections $\mathbf{G}_m$.

Suppose $\mathbf{n} = (n_1 < n_2 < \ldots)$ is an increasing sequence of positive integers, and $0 < t \leq 1$. The basis (e_i) is **n**-*quasi-greedy with parameter t* (**n**-*t-QG* for short) if $\lim_i \mathbf{G}_{n_i}^t x = x$ for any $x \in X$, and any choice of t-greedy projections $\mathbf{G}_{n_i}^t x$. When $t = 1$, we use the term **n**-*quasi-greedy* (**n**-*QG*). When $\mathbf{n} = \mathbb{N}$, we recover the usual notion of a basis being t-quasi-greedy, or quasi-greedy for $t = 1$ (we use the acronyms t-QG and QG).

This paper is structured as follows. In Section 2, we obtain a criterion for a basis being **n**-t-QG (similar to the criterion for the quasi-greediness of a basis from [18]). We also show that any basis which is **n**-QG with constant 1 must be 1-suppression unconditional. Examples of **n**-QG bases which are not QG are given in Section 3. In Section 4 we show that, if a set **n** is "large enough" (more precisely, if it "almost" forms an additive basis in $\mathbb{N}$), then any **n**-QG basis is automatically QG. In Section 5, we show that certain "classical" bases (the Haar basis in L_1, and the trigonometric basis in a weighted L_p space) are not **n**-QG, for any sequence **n**.

Throughout the paper, we use standard facts and notation from Banach spaces and approximation theory. We refer the reader to e.g. [15], [16], [17] for the necessary background.

For a normed space X, we denote by $\mathbf{B}(X) = \{x \in X : \|x\| \leq 1\}$ its closed unit ball. We write "$x \precsim y$" ("$x \succsim y$", "$x \sim y$") as a shorthand for "there exists $c > 0$ so that $x/y \leq c$" (resp. $c^{-1} \leq x/y$, $c^{-1} \leq x/y \leq c$). For a collection of vectors $(e_i)_{i \in I}$, and a finite set $A \subset I$, we let $\mathbf{1}_A = \sum_{i \in A} e_i$. More generally, for $\varepsilon_i = \pm 1$, we set $\mathbf{1}_{\varepsilon A} = \sum_{i \in A} \varepsilon_i e_i$.

Remark 1.1. One can define $\mathbf{n}$-greedy, and $\mathbf{n}$-almost greedy bases, in the manner similar to the definition of $\mathbf{n}$-QG bases. More precisely, suppose $(e_i, e_i^*)_{i \in I} \subset X \times X^*$ is a biorthogonal system. For $x \in X$, we can define

$$\sigma_m(x) = \inf_{|\operatorname{supp} y| \leq m} \|x - y\| \text{ and } \tilde{\sigma}_m(x) = \inf_{|A| \leq m} \|x - Ax\|.$$

The above basis is said to be $\mathbf{n}$-*greedy* (resp. $\mathbf{n}$-*almost greedy*) if there exists an absolute constant $C > 0$ so that, for any x, and any $n \in \mathbf{n}$, we have $\|x - \mathbf{G}_n x\| \leq C\sigma_n(x)$, resp. $\|x - \mathbf{G}_n x\| \leq C\tilde{\sigma}_n(x)$ (that is, the greedy approximation is "almost optimal"). However, such bases are necessarily greedy, resp. almost greedy, when I is infinite.

We consider the greedy case, as the almost greedy one is handled similarly. Suppose (e_i) is $\mathbf{n}$-greedy, with constant C. We have to show that, for any x, and any greedy set A, we have $\|x - Ax\| \leq C\sigma_m(x)$, where $m = |A|$. For $\varepsilon > 0$ pick $y \in X$ so that $|\operatorname{supp}(y)| = m$ and $\|x - y\| < \sigma_m(x) + \varepsilon$.

Pick $n \in \mathbf{n} \cap [m, \infty)$. If $n = m$, we are done. Otherwise, by a "small perturbation argument", we can assume that x has finite support. Find a set B of cardinality $n - m$, disjoint from $\operatorname{supp} x \cup \operatorname{supp} y$. Set $x' = x + \sum_{k \in B} r e_i$, where $r > \max_i |\langle e_i^*, x \rangle|$. Then $A \cup B$ is a greedy set (of cardinality n) for x', and $x' - (A \cup B)x' = x - Ax$. However, $x' - (Bx' + y) = x - y$, hence $\|x - y\| \geq \sigma_n(x')$. Thus,

$$\|x - Ax\| = \|x' - (A \cup B)x'\| \leq C\sigma_n(x') \leq C\|x - y\| \leq C(\sigma_m(x) + \varepsilon).$$

As ε can be arbitrarily small, we are done.

2. Connection to norms of projections

In this section, we develop some technical tools, and prove a characterization of a basis being $\mathbf{n}$-QG similar to the characterization of quasi-greediness from [18]:

Theorem 2.1. *For a biorthogonal system $(e_i, e_i^*)_{i \in I} \subset X \times X^*$, and a sequence $\mathbf{n} = (n_1 < n_2 < \ldots) \subset \mathbb{N}$, the following are equivalent:*

1. *If $\emptyset = B_0 \subset B_1 \subset B_2 \subset \ldots$ are subsets of I, so that, for any k, $|B_k| = n_k$, and $B_k \backslash B_{k-1}$ is a t-greedy set for $B_{k-1}^c x$, then $\lim_k B_k x = x$.*
2. *The basis (e_i) is $\mathbf{n}$-t-QG.*
3. *There exists a constant $C > 0$ so that, for any $x \in X$, any $n \in \mathbf{n}$, and any t-greedy projection $\mathbf{G}_n^t x$, we have $\|\mathbf{G}_n^t x\| \leq C\|x\|$.*

Throughout this section we set $c = (\sup_i \|e_i^*\| + 1)(\sup_i \|e_i\| + 1)$ (by our assumptions, $c \in [1, \infty)$). We make use of the following "small perturbations" lemma.

Lemma 2.2. *Suppose A is a t-greedy set for $x \in X$. Then for any $\varepsilon > 0$ there exists $y \in X$ with finite support, so that $\|x - y\| < \varepsilon$, and A is a t-greedy set for y.*

Proof. By scaling, we can assume $\|x\| < 1/c$. Find a finitely supported $z \in X$ so that $\|x - z\| < \delta = \varepsilon/(4c|A|)$. We claim that

$$y = z + 2c\delta \sum_{i \in A} \operatorname{sign}\langle e_i^*, x \rangle e_i$$

has the desired properties. Indeed,

$$\|x-y\| \leq \|x-z\| + 2c\delta\|x\| \sum_{i\in A} \|e_i\|\|e_i^*\| < \varepsilon.$$

To show that A is a t-greedy set for y, observe that

$$\langle e_i^*, y\rangle = \begin{cases} (|\langle e_i^*, x\rangle| + 2c\delta)\mathrm{sign}\langle e_i^*, x\rangle + \langle e_i^*, z-x\rangle & i\in A \\ \langle e_i^*, x\rangle + \langle e_i^*, z-x\rangle & i\notin A \end{cases}.$$

Let $\alpha = \min_{i\in A} |\langle e_i^*, x\rangle|$. For $i \in A$,

$$\big|\langle e_i^*, y\rangle\big| \geq |\langle e_i^*, x\rangle| + 2c\delta - |\langle e_i^*, z-x\rangle| \geq \alpha + 2c\delta - c\delta = \alpha + c\delta,$$

while for $i \notin A$,

$$\big|\langle e_i^*, y\rangle\big| \leq |\langle e_i^*, x\rangle| + |\langle e_i^*, z-x\rangle| \leq \frac{\alpha}{t} + c\delta \leq \frac{\alpha + c\delta}{t}.$$

Thus, A is t-greedy for y. □

The preceding lemma immediately implies:

Corollary 2.3. *Suppose $m \in \mathbb{N}$, $t \in (0,1]$, and $C > 0$ are such that, for any finitely supported $x \in X$, we have $\|\mathbf{G}_m^t x\| \leq C\|x\|$. Then the same inequality holds for any $x \in X$.*

Lemma 2.4. *If A and B are finite sets, then, for any $x \in X$, $\|Ax - Bx\| \leq c|A \triangle B|\|x\|$. Consequently, for any t-greedy projection $\mathbf{G}_\ell^t x$ there exists a t-greedy projection $\mathbf{G}_k^t x$ so that $\|\mathbf{G}_\ell^t x\| \leq c|k-\ell|\|x\| + \|\mathbf{G}_k^t x\|$.*

Proof. For the first statement, use the fact that

$$\|Ax - Bx\| \leq \sum_{i\in A\triangle B} |\langle e_i^*, x\rangle|\|e_i\| \leq |A\triangle B| \sup_i \|e_i^*\|\|e_i\| \cdot \|x\|.$$

For the second statement, suppose A is the set of cardinality k with $\mathbf{G}_k^t x = Ax$. If $\ell > k$, then there exists a set B so that $Bx = \mathbf{G}_\ell^t x$, and $A \subset B$. Then $|A\triangle B| = |B\backslash A| = \ell - k$. Now apply the first statement of the lemma. The case of $\ell < k$ is handled similarly (with $B \subset A$). □

Lemma 2.5. *Fix $t \in (0,1]$. Suppose a basis $(e_i)_{i\in I}$ has the property that, for any $K > 0$, there exists $x \in \mathbf{B}(X)$ and $m \in \mathbf{n}$ with $\|\mathbf{G}_m^t x\| > K$, for some realization of the t-greedy algorithm. Then, for any $C > 0$, and for any finite set $F \subset I$, there exists $y \in \mathbf{B}(X)$, whose support is finite and disjoint from F, and $\|\mathbf{G}_n^t y\| > C$, for some realization of the t-greedy algorithm, and for some $n \in \mathbf{n}$.*

Proof. By a small perturbation argument, we can find a finitely supported $x \in \mathbf{B}(X)$ so that

$$\|Ax\| > K = (1 + c|F|)C + 2c|F|,$$

where A is a t-greedy set for x of cardinality $n \in \mathbf{n}$, with $n > |F|$. Let $z = F^c x$. Then $\|z\| \leq \|x\| + c|F| \leq 1 + c|F|$.

Next show that the t-greedy algorithm for z can run in such a way that $\|\mathbf{G}_n^t z\| > C(1 + c|F|)\|z\|$. Once this is done, we immediately conclude that $y = z/\|z\|$ has the desired properties.

By the triangle inequality,

$$\big\|(A\backslash F)z\big\| = \big\|(A\backslash F)x\big\| \geq \|Ax\| - \|Fx\| > K - c|F|.$$

Clearly $A\backslash F$ is a t-greedy set for z, hence we can find a t-greedy set for z – call it B – so that $|B| = n$, and $A\backslash F \subset B$ (then $|B\backslash(A\backslash F)| = |F|$). Then

$$\|Bz\| \geq \|(A\backslash F)z\| - \|\sum_{i\in B\backslash(A\backslash F)} \langle e_i^*, z\rangle e_i\|.$$

However, $|\langle e_i^*, z\rangle| \leq |\langle e_i^*, x\rangle|$, hence

$$\|Bz\| \geq \|(A\backslash F)z\| - \sum_{i\in B\backslash(A\backslash F)} \|e_i^*\|\|e_i\| > K - 2c|F|.$$

By our choice of K, the right hand side equals $(1 + c|F|)C$, which means we are done. ☐

Finally, we mention a folklore result.

Lemma 2.6. *Suppose A is a t-greedy set for $x \in X$. Suppose, furthermore, that $B \subset A$ is such that*

$$\max_{i\in B} |\langle e_i^*, x\rangle| \leq \min_{i\in A\backslash B} |\langle e_i^*, x\rangle|.$$

Then $A\backslash B$ is a t-greedy set for x.

Proof of Theorem 2.1. (2) ⇒ (1): Observe that, for every k, B_k is a t-greedy set for x.

(1) ⇒ (3): We prove the contrapositive: if $\sup_{n\in\mathbf{n}} \|\mathbf{G}_n^t\| = \infty$, then the there exist $x \in X$ and $k_1, k_2, \ldots \in \mathbf{n}$ so that the sequence $(\|\mathbf{G}_{k_j}^t x\|)_{j\in\mathbb{N}}$ increases without a bound. More specifically, we find $k_1 < k_2 < \ldots$ in the sequence $\mathbf{n}$, and the elements $x_j = \sum_{i\in A_j} \alpha_i e_i$ so that:

1. $A_1, A_2 \ldots$ are finite disjoint sets, with $\sum_{s=1}^j |A_s| = k_j$.
2. For each j, $\beta_j \leq \inf_{i\in A_j} |\alpha_i| \leq \sup_{i\in A_j} |\alpha_i| \leq \gamma_j$, where $\gamma_1 \geq \beta_1 \geq \gamma_2 \geq \ldots$.
3. For each j, $\|x_j\| < 2^{-j}$, and $\|\mathbf{G}_{k_j-|A_1|-\ldots-|A_{j-1}|}^t x_j\| > 2^j$.

Once this is done, let $x = \sum_{j=1}^\infty x_j$, and $B_j = \cup_{s=1}^j A_s$. Then $\|x\| \leq 1$, and for any j, $A_j = B_j\backslash B_{j-1}$ is a t-greedy set for $B_{j-1}^c x$, and $|B_j| = k_j$. Further,

$$\|B_j x\| = \|x_1+\ldots+x_{j-1}+\mathbf{G}_{k_j-|A_1|-\ldots-|A_{j-1}|}^t x_j\| \geq \|\mathbf{G}_{k_j-|A_1|-\ldots-|A_{j-1}|}^t x_j\| - \sum_{\ell=1}^{j-1} \|x_\ell\| > 2^j - 1.$$

The selection proceeds recursively. Picking k_1 and x_1 is easy. Now suppose $k_1, \ldots, k_j$ and $x_1, \ldots, x_j$ have been chosen. By Lemma 2.5, we can find $k_{j+1} \in \mathbf{n}$, and x_{j+1}, supported on a set A_{j+1}, disjoint from $\cup_{s=1}^j A_s$, so that

$$\|x_{j+1}\| < \min\left\{2^{-(j+1)}, c^{-1}\beta_j\right\} \text{ and } \|\mathbf{G}_{k_{j+1}}^t x_{j+1}\| > 2^{j+1} + c\beta_j \sum_{s=1}^j |A_s|.$$

Writing $x_{j+1} = \sum_{i\in A_{j+1}} \alpha_i e_i$, we get

$$\gamma_{j+1} := \max_{i\in A_{j+1}} |\alpha_i| \leq c\|x_{j+1}\| \leq \beta_j.$$

Finally,

$$\left\|\mathbf{G}_{k_{j+1}-|A_1|-\ldots-|A_j|}^t x_{j+1}\right\| \geq \left\|\mathbf{G}_{k_{j+1}}^t x_{j+1}\right\| - c\sum_{s=1}^j |A_s|\gamma_{j+1} > 2^{j+1}.$$

(3) ⇒ (2): Suppose, for any $u \in X$, and any $k \in \mathbf{n}$, we have $\|u - \mathbf{G}_k^t u\| \leq C\|u\|$. Fix $\varepsilon > 0$, and $x \in X$, and show that $\|x - \mathbf{G}_k^t x\| < \varepsilon$ when $k \in \mathbf{n}$ is large enough. The case of a

finitely supported x is trivial. If x has infinite support, find a finitely supported $v \in X$, so that $x' = x - v$ has norm less than $\varepsilon/(2C)$. Let

$$F = \{i \in I : \langle e_i^*, x\rangle \neq \langle e_i^*, x'\rangle\} = \{i \in I : \langle e_i^*, v\rangle \neq 0\}.$$

and

$$G = \{i \in F : \langle e_i^*, x\rangle = 0\}.$$

Perturbing v slightly, we can and do assume that $\langle e_i^*, x'\rangle \neq 0$ whenever $\langle e_i^*, x\rangle \neq 0$.

Let

$$\alpha = \min\big\{\min_{i\in F} |\langle e_i^*, x'\rangle|, \min_{i\in F\backslash G} |\langle e_i^*, x\rangle|\big\} \text{ and } \alpha' = \min\big\{t\alpha, \frac{\varepsilon}{2c|F|}\big\}.$$

Further, let $S = \{i \in I : |\langle e_i^*, x\rangle| \geq \alpha'\}$ (then $F\backslash G \subset S$), and $N = |F| + |S|$. We claim that, for any $k \in \mathbf{n} \cap (N, \infty)$, we have $\|x - \mathbf{G}_k^t x\| \leq \varepsilon$.

Indeed, suppose A is a t-greedy set for x, of cardinality k. Let $\beta = \min_{i\in A} |\langle e_i^*, x\rangle|$. Then $\sup_{i\notin A} |\langle e_i^*, x\rangle| \leq \beta/t$. As $|A| > |S|$, $\beta < \alpha' \leq t\alpha$. Moreover, for $i \in F\backslash G$, we have $|\langle e_i^*, x\rangle| \geq \alpha > \beta/t$, hence $F\backslash G \subset A$.

For $i \in F$, $|\langle e_i^*, x'\rangle| \geq \alpha > \beta/t$, while for $i \in A\backslash F$, $|\langle e_i^*, x'\rangle| = |\langle e_i^*, x\rangle| \geq \beta$. We noted above that $F\backslash G \subset A$, hence for $i \notin A \cup F = A \cup G$ we have $|\langle e_i^*, x'\rangle| = |\langle e_i^*, x\rangle| \leq \beta/t$. Thus, $A \cup G$ is a t-greedy set for x'.

Find $B \subset A$ so that $|B| = |G|$ and

$$\max_{i\in B} |\langle e_i^*, x\rangle| \leq \min_{i\in A\backslash B} |\langle e_i^*, x\rangle|.$$

Note that $B \cap S = \emptyset$ (in other words, $\max_{i\in B} |\langle e_i^*, x\rangle| < \alpha'$). Indeed, otherwise $|\langle e_i^*, x\rangle| \geq \alpha'$ for any $i \in A\backslash B$, or in other words, $A\backslash B \subset S$. Then

$$|A| = |G| + |A\backslash B| \leq |F| + |S| < |A|,$$

leading to a contradiction.

By Lemma 2.6, $(A\backslash B) \cup G$ is a t-greedy set of x' (of cardinality k). The fact $k \in \mathbf{n}$ implies

$$\big\|x' - \big[(A\backslash B) \cup G\big]x'\big\| \leq C\|x'\| < C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}.$$

Note that $x' - \big[(A\backslash B) \cup G\big]x' = x - (A\backslash B)x$, hence $\|x - (A\backslash B)x\| < \varepsilon/2$. As shown above, $\max_{i\in B} |\langle e_i^*, x\rangle| < \alpha' \leq \varepsilon/(2c|F|)$. By the triangle inequality,

$$\|x - Ax\| \leq \|x - (A\backslash B)x\| + \sum_{i\in B} |\langle e_i^*, x\rangle| \|e_i\| < \frac{\varepsilon}{2} + |B|c\frac{\varepsilon}{2c|F|} < \varepsilon. \quad \square$$

In [1] it was shown that, if a basis $(e_i) \subset X$ is such that $\|\mathbf{G}_n x\| \leq \|x\|$ for any $x \in X$ and $n \in \mathbb{N}$, then this basis is 1-suppression unconditional – that is, for any finite sets $A \subset B$, and any scalars (α_i), we have $\|\sum_{i\in B} \alpha_i e_i\| \leq \|\sum_{i\in A} \alpha_i e_i\|$. We prove the same for $\mathbf{n}$-QG bases.

Proposition 2.7. *Suppose $(e_i)_{i\in I}$ is a basis in an infinite dimensional Banach space X, and moreover, $\|\mathbf{G}_k x\| \leq \|x\|$ for any $k \in \mathbf{n}$, and any $x \in X$. Then (e_i) is 1-suppression unconditional – that is, $\|\sum_{i\in B} \alpha_i e_i\| \leq \|\sum_{i\in A} \alpha_i e_i\|$ whenever $B \subset A$ are finite sets.*

Proof. We proceed along the lines of [1]. We have to show that, whenever x and y have disjoint finite support, we have $\|x + y\| \geq \|x\|$. Write $x = \sum_{i\in A} \alpha_i e_i$, with $\alpha_i \neq 0$ for $i \in A$. If $|A| \in \mathbf{n}$, then the statement follows just as in [1]. Otherwise, find $n \in \mathbf{n}$ so that $n > |A|$, and find a set C, disjoint from both A and B, so that $|C| + |A| = n$. If $\gamma_i \neq 0$ for $i \in C$, then $\|x + \sum_{i\in C} \gamma_i e_i\| \leq \|x + \sum_{i\in C} \gamma_i e_i + y\|$. As the coefficients γ_i can be arbitrarily close to 0, we are done. $\square$

Remark 2.8. The original result of [1] works in both finite and infinite dimensional setting. In our proof, however, we overcome the sparsity of "good" greedy projections by "borrowing from infinity".

3. Examples of n-QG basis which are not quasi-greedy

Clearly, any quasi-greedy basis is **n**-QG, for any sequence **n**. Two examples below show that, in general, the converse fails. Our first example (Proposition 3.1) covers a large family of sequences **n**. The second one (Proposition 3.2) works for one specific **n**, but produces a basis which is superdemocratic.

We say that a sequence $\mathbf{n} = (n_1 < n_2 < \ldots)$ has *arbitrarily large gaps* if $\limsup_k n_{k+1}/n_k = \infty$.

Proposition 3.1. *If a sequence* **n** *has arbitrarily large gaps, then there exists a Banach space X with Schauder basis (e_i) which is* **n**-*t-QG for any $t \in (0,1]$, but not quasi-greedy.*

Proof. Write $\mathbf{n} = (n_1 < n_2 < \ldots)$, and find $k_1 < k_2 < \ldots$ the sequence $(n_{k_i+1}/n_{k_i})_{i\in\mathbb{N}}$ increases without a bound. For $i \in \mathbb{N}$ let

$$c_i = \Big(\frac{n_{k_i+1}}{n_{k_i}}\Big)^{1/4} \text{ and } m_i = \Big\lfloor \sqrt{n_{k_i+1} n_{k_i}} \Big\rfloor. \tag{3.1}$$

Let $\tilde{m}_i = \sum_{j<i} m_j$ (so that $\tilde{m}_1 = 0$, and $\tilde{m}_{i+1} = \tilde{m}_i + m_i$ for $i \geq 1$). To define the Banach space X with its basis $(e_j)_{j\in\mathbb{N}}$, set, for a finite sequence $(\alpha_j)_{j\in\mathbb{N}}$,

$$\Big\| \sum_j \alpha_j e_j \Big\| = \max\Big\{ \big\|(\alpha_j)_{j\in\mathbb{N}}\big\|_2, \sup_{i\in\mathbb{N}} \frac{c_i}{\sqrt{m_i}} \max_{1\leq \ell \leq m_i} \Big| \sum_{j=\tilde{m}_i+1}^{\tilde{m}_i+\ell} \alpha_j \Big| \Big\},$$

where $\big\|(\alpha_j)_{j\in\mathbb{N}}\big\|_2 = \big(\sum_{j=1}^\infty |\alpha_j|^2\big)^{1/2}$. Clearly (e_j) is a Schauder basis – in fact, its basis constant equals 1.

However, this basis is not quasi-greedy. Indeed, consider

$$x = x_i = \sum_{j=1}^{m_i} (-1)^j e_{\tilde{m}_i+j}.$$

Then $\|x\| = \sqrt{m_i}$. Now let $k = \lfloor m_i/2 \rfloor$, then an appropriate choice of a k-greedy set produces

$$\mathbf{G}_k x = \sum_{j=1}^k e_{\tilde{m}_i+2j},$$

and in this case,

$$\big\|\mathbf{G}_k x\big\| = \frac{c_i k}{\sqrt{m_i}} \sim c_i \sqrt{m_i}.$$

It remains to show that, for any norm one $x = \sum_j \alpha_j e_j$, any $n \in \mathbf{n}$, and any $t > 0$, we have $\|\mathbf{G}_n^t x\| \leq 2/t$. To this end, suppose S is a t-greedy set for x, of cardinality n. Clearly it suffices to show that, for any i, and any $\ell \in [1, m_i]$ we have

$$\frac{c_i}{\sqrt{m_i}} \Big| \sum_{j\in[\tilde{m}_i+1, \tilde{m}_i+\ell]\cap S} \alpha_j \Big| \leq \frac{2}{t}. \tag{3.2}$$

To this end, note that, by (3.1),

$$m_i c_i^2 = \Big(\frac{n_{k_i+1}}{n_{k_i}}\Big)^{1/2} \Big\lfloor \sqrt{n_{k_i+1} n_{k_i}} \Big\rfloor \leq n_{k_i+1} \text{ and } \frac{m_i}{c_i^2} = \Big\lfloor \sqrt{n_{k_i+1} n_{k_i}} \Big\rfloor \Big(\frac{n_{k_i}}{n_{k_i+1}}\Big)^{1/2} \geq n_{k_i} - 1.$$

Consequently, for any $n \in \mathbf{n}$, and any i either $n \geq m_i c_i^2$, or $n \leq m_i/c_i^2 + 1$.

If $n \leq m_i/c_i^2 + 1$, then, by Hölder Inequality,

$$\frac{c_i}{\sqrt{m_i}} \sum_{j \in [\tilde{m}_i+1, \tilde{m}_i+m_i] \cap S} |\alpha_j| \leq \frac{c_i}{\sqrt{m_i}} \sqrt{|S|} \big(\sum_{j \in S} |\alpha_j|^2 \big)^{1/2} \leq \frac{c_i}{\sqrt{m_i}} \sqrt{n} \leq 2,$$

giving us (3.2).

Next suppose $n \geq m_i c_i^2$. Let $\alpha = \min_{j \in S} |\alpha_j|$, then

$$1 = \|x\| \geq \Big(\sum_{j \in S} |\alpha_j|^2 \Big)^{1/2},$$

which implies $\alpha \leq n^{-1/2}$. For $1 \leq \ell \leq m_i$,

$$\frac{c_i}{\sqrt{m_i}} \Big| \sum_{j \in [\tilde{m}_i+1, \tilde{m}_i+\ell]} \alpha_j \Big| \leq \|x\| = 1.$$

For $j \notin S$, we have $|\alpha_j| \leq \alpha/t \leq n^{-1/2} t^{-1}$, hence

$$\frac{c_i}{\sqrt{m_i}} \Big| \sum_{j \in [\tilde{m}_i+1, \tilde{m}_i+\ell] \cap S} \alpha_j \Big| \leq \frac{c_i}{\sqrt{m_i}} \Big(\Big| \sum_{j \in [\tilde{m}_i+1, \tilde{m}_i+\ell]} \alpha_j \Big| + \sum_{j \in [\tilde{m}_i+1, \tilde{m}_i+\ell] \cap S^c} |\alpha_j| \Big)$$
$$\leq 1 + \frac{c_i}{\sqrt{m_i}} \cdot m_i \frac{1}{t\sqrt{n}} = 1 + \frac{1}{t} \frac{c_i \sqrt{m_i}}{\sqrt{n}} \leq \frac{2}{t},$$

establishing (3.2). □

Recall that a basis $(e_i)_{i \in I}$ is called *superdemocratic* if $\|\mathbf{1}_{\varepsilon A}\|$ depends (up to a constant) only on the cardinality of the finite set A, but not on its composition, or on the choice of signs ε: more precisely, there exists a constant K so that the inequality $\|\mathbf{1}_{\varepsilon A}\| \leq K \|\mathbf{1}_{\delta B}\|$ holds for any $\delta_i, \varepsilon_i = \pm 1$, and any $A, B \subset I$ of equal cardinality). In particular, any superdemocratic basis is unconditional with constant coefficients (UCC): there exists a constant $C \geq 1$ so that, for any finite set $A \subset I$, and any $\varepsilon_i = \pm 1$, we have

$$C^{-1} \Big\| \sum_i \varepsilon_i e_i \Big\| \leq \Big\| \sum_i e_i \Big\| \leq C \Big\| \sum_i \varepsilon_i e_i \Big\|.$$

It is well-known that any quasi-greedy basis is UCC.

Proposition 3.2. *There exist a sequence* $\mathbf{n}$*, and a superdemocratic Schauder basis* (e_i) *in a separable Banach space* X*, which is* $\mathbf{n}$*-*t*-QG for any* t*, but not quasi-greedy.*

Proof. Find the sequences $n_1 < n_2 < \ldots$, $N_1 < N_2 < \ldots$, and $\gamma_1 > \gamma_2 > \ldots > 0$ so that:

(1) For every k, $n_k \geq \sum_{j=1}^k 2^{2N_j+1}$.
(2) For every k, $\gamma_{k+1} \leq (\log n_k + 1)^{-1}$ (hence $\lim_k \gamma_k = 0$).
(3) $\limsup_k N_k \gamma_k = \infty$.

For instance, we can take $n_k = 8^{N_k}$, $\gamma_k = 1/\sqrt{N_k}$, $N_{k+1} = (4N_k)^2$.

For the construction we recall some notation of [2]. Denote by $\mathcal{D}_k$ the set of dyadic subintervals of $[0,1]$ of length 2^{-k}, and $\mathcal{D} = \cup_{k \geq 0} \mathcal{D}_k$. Denote by $\mathfrak{f}$ the completion of $c_{00}(\mathcal{D})$ with respect to the norm

$$\big\| (\alpha_I)_{I \in \mathcal{D}} \big\|_{\mathfrak{f}} = \Big\| \bigvee_I |\alpha_I| \, |I|^{-1} \chi_I \Big\|_{L_1}.$$

As noted in [2], the canonical basis of the space $\mathfrak{f}$ is unconditional, and the inequality $c_1|A| \leq \|\mathbf{1}_{\varepsilon A}\|_{\mathfrak{f}} \leq |A|$ holds for any finite set A (c_1 is an absolute constant). Consequently, there exists an absolute constant $c_2 \geq 1$ so that the inequality $c_2^{-1}\|(\alpha_I)_I\|_{\ell_{1\infty}} \leq \|(\alpha_I)_I\|_{\mathfrak{f}}$ holds for any finite set (α_I).

Now for every $n \in \mathbb{N}$, introduce a new order on $\{1, \ldots, 2n\}$:

$$1 \prec_n n+1 \prec_n 2 \prec_n n+2 \prec_n \ldots \prec_n n \prec_n 2n.$$

This gives rise to an order on $S_n = \cup_{k=1}^{2n} \mathcal{D}_k$: we say that $I \precsim_n J$ if either (i) $I \in \mathcal{D}_k$, $J \in \mathcal{D}_\ell$, $k \prec_n \ell$, or (ii) $I, J \in \mathcal{D}_k$, and I is "to the left" of J. For $(\alpha_I)_{I \in S_n}$ define its summing norm:

$$\big\|(\alpha_I)_{I \in S_n}\big\|_{\mathfrak{s}_n} = \max_{J \in S_n} \big| \sum_{I \precsim_n J} \alpha_I \big|.$$

Finally, set

$$\big\|(\alpha_I)_{I \in S_n}\big\|_{\max_n} = \max\big\{\big\|(\alpha_I)_{I \in S_n}\big\|_{\mathfrak{f}}, \gamma_n \big\|(\alpha_I)_{I \in S_n}\big\|_{\mathfrak{s}_n}\big\}.$$

For $k \in \mathbb{N}$ let X_k be the space of $(\alpha_I)_{I \in S_{N_k}}$, equipped with the norm $\|\cdot\|_{\max_{N_k}} = \|\cdot\|_k$ described above (note that $\dim X_k = |S_{N_k}| = 2^{2N_k+1} - 2$). Let $X = (\sum_k X_k)_1$, with the basis obtained by concatenating the canonical bases of the summands X_k. We claim that this basis is (i) superdemocratic, (ii) not QG, and (iii) $\mathbf{n}$-QG, with $\mathbf{n} = (n_1, n_2, \ldots)$.

(i) X is superdemocratic – in fact, for any A we have $\|\mathbf{1}_{\varepsilon A}\| \sim |A|$. Indeed, we can write $A = \cup_k A_k$, where $A_k = A \cap S_k$. Then $\|\mathbf{1}_{\varepsilon A}\| = \sum_k \|\mathbf{1}_{\varepsilon A_k}\|$. Clearly $\|\mathbf{1}_{\varepsilon B}\|_{\mathfrak{s}_{N_k}} \leq |B|$. As observed in [2], $\|\mathbf{1}_{\varepsilon B}\|_{\mathfrak{f}} \sim |B|$.

(ii) The proof that X is not QG is essentially due to [2], with minor changes. We sketch an argument for the sake of completeness. We only need to find a sequence (x_k), and the corresponding greedy sets A_k, so that $\sup_k \|x_k\| < \infty$, while $\sup_k \|A_k x_k\| = \infty$. Consider

$$x_k = \sum_{i=1}^{N_k} 2^{-i} \mathbf{1}_{\mathcal{D}_i} - \sum_{i=N_k+1}^{2N_k} 2^{-i} \mathbf{1}_{\mathcal{D}_i}.$$

As in [2] we verify that $\|x_k\|_k \sim 1$. Further, note that $A_k = \cup_{i=1}^{N_k} \mathcal{D}_i$ is a greedy set of A_k, and

$$\|A_k x_k\|_k \geq \gamma_k \|A_k x_k\|_{\mathfrak{s}_{N_k}} = N_k \gamma_k.$$

Now recall that $\lim_k N_k \gamma_k = \infty$.

(iii) Fix a norm one $x = (x_j) \in X$ $(x_j \in X_j)$. We have to show that, for any k, $\|Ax\| \leq \kappa/t$ if A is a t-greedy set for x with $|A| = n_k$ (κ is an absolute constant).

As noted above, for any $(\alpha_I)_{I \in S_k}$ we have

$$c_2^{-1}\big\|(\alpha_I)_{I \in S_k}\big\|_{\ell_{1\infty}} \leq \big\|(\alpha_I)_{I \in S_k}\big\|_k \leq \big\|(\alpha_I)_{I \in S_k}\big\|_{\ell_1}.$$

Consequently, if we express $x \in X$ in its canonical basis, we obtain

$$c_2^{-1}\big\|x\big\|_{\ell_{1\infty}} \leq \big\|x\big\| \leq \big\|x\big\|_{\ell_1}.$$

Let $A_j = S_j \cap A$. Write $x = (\alpha_{I,k})_{k \in \mathbb{N}, I \in S_k}$. Let $\alpha = \min_{(I,k) \in A} |\alpha_{I,k}|$, then $\alpha \leq \|x\|_{1\infty}/n_k \leq c_2/n_k$. Further, for $(I,k) \notin A$, $|\alpha_{I,k}| \leq \alpha/t$.

For $j \leq k$,

$$\big\|A_j x_j\|_j \leq \big\|x_j\|_j + \frac{\alpha}{t}|A_j^c| \leq \|x_j\|_j + \frac{c_2}{tn_k}|A_j^c| \leq \|x_j\|_j + \frac{c_2}{tn_k} 2^{2N_j+1}.$$

For $j > k$,

$$\big\|A_j x_j\big\|_j = \max\Big\{\big\|A_j x_j\big\|_{\mathfrak{f}}, \gamma_j \big\|A_j x_j\big\|_{\mathfrak{s}_j}\Big\}.$$

We have

$$\big\|A_j x_j\big\|_{\mathfrak{s}_j} \leq \big\|A_j x_j\big\|_1 \leq \big(\log|A_j| + 1\big)\big\|A_j x_j\big\|_{1\infty}.$$

Recall that, for $j > k$,

$$\gamma_j \leq \gamma_{k+1} \leq \frac{1}{\log n_k + 1} \leq \frac{1}{\log |A_j| + 1},$$

hence

$$\|A_j x_j\|_j \leq \max\{\|A_j x_j\|_{\mathfrak{f}}, \|A_j x_j\|_{1\infty}\} \leq c_2 \|A_j x_j\|_{\mathfrak{f}}.$$

By the unconditionality of $\|\cdot\|_{\mathfrak{f}}$,

$$\|A_j x_j\|_{\mathfrak{f}} \leq c_3 \|x_j\|_{\mathfrak{f}} \leq c_3 \|x_j\|_j,$$

for some universal constant $c_3 \geq 1$. Consequently,

$$\begin{aligned}\|Ax\| &= \sum_{j \leq k} \|A_j x_j\|_j + \sum_{j>k} \|A_j x_j\|_j \leq \sum_{j \leq k} \Big(\|x_j\|_j + \frac{c_2}{t n_k} 2^{2N_j+1}\Big) + c_2 c_3 \sum_{j>k} \|x_j\|_j \\ &\leq c_2 c_3 \sum_j \|x_j\|_j + \frac{c_2}{t n_k} \sum_{j \leq k} 2^{2N_j+1} \leq c_2 c_3 + \frac{c_2}{t},\end{aligned}$$

since $\sum_{j \leq k} 2^{2N_j+1} \leq n_k$. We have shown that $\|\mathbf{G}^t_{n_k} x\| \leq 2c_2 c_3/t$ for any $x \in \mathbf{B}(X)$. □

4. Bases in $\mathbb{N}$

In the preceding section we gave examples of sequences $\mathbf{n}$ "with large gaps" for which there exist non-quasi-greedy $\mathbf{n}$-QG bases. In this section we show that, if the gaps of a sequence $\mathbf{n} \subset \mathbb{N}$ are "sufficiently small", then any $\mathbf{n}$-QG basis must necessarily be quasi-greedy (Proposition 4.1).

Recall some notation from additive combinatorics: for $S \subset \mathbb{N}$ and $k \in \mathbb{N}$, kS stands for the set of numbers $s_1 + \ldots + s_k$ ($s_1, \ldots, s_k \in S$ need not be distinct). We set $k * S = \cup_{\ell=1}^k \ell S$. A set $S \subset \mathbb{N}$ is called an *asymptotic basis* if there exist $k, m \in \mathbb{N}$ so that $\{m, m+1, \ldots\} \subset k * S$. We refer the reader to [8], [11], or [14] for information on bases in $\mathbb{N}$.

Numerous examples of asymptotic bases are known, for instance:

- The set of primes [14].
- Any set S of positive lower asymptotic density provided $\gcd\{|a-b| : a, b \in S\} = 1$ [10].
- $\{i^r : i \in \mathbb{N}\}$ (Hilbert's solution to Waring's Problem).
- $\{p(i) : i \in \mathbb{N}\}$, where p is an integer-valued polynomial with the property that no $d \geq 2$ divides $p(i)$ for any i. This is due to Kamke; we refer to [7] for the precise references, and more information.
- $\{\lfloor f(i) \rfloor : i \in \mathbb{N}\}$ is a basis, if f belongs to a Hardy field, increases no faster than a polynomial, and is "far" from a rational function [3].

Motivated by this, we say that $S \subset \mathbb{N}$ is a *crude basis* if there exist $k, m \in \mathbb{N}$ so that, for every $i \in \mathbb{N}$, $[im, (i+1)m] \cap k * S \neq \emptyset$. Crude bases are "stable under finite perturbations": it is easy to see that, if S is a crude basis, and F is a finite subset of S, then $S \backslash F$ is a crude basis as well.

It follows from [3, Lemma 2.3] that, if S is a crude basis, then there exists $d \in \mathbb{N}$ so that every element of S is divisible by d, and S/d is an asymptotic basis; moreover, $S \cup \{a\}$ is an asymptotic basis whenever a is co-prime with d. In particular, any subset of $\mathbb{N}$ with positive lower asymptotic density is a crude basis (this can also be deduced directly from [10]). Further (see e.g. [7]), $\{p(i) : i \in \mathbb{N}\}$ is a crude basis whenever p is an integer-valued polynomial.

Proposition 4.1. *If an increasing sequence* $\mathbf{n}$ *forms a crude basis in* $\mathbb{N}$*, then any* $\mathbf{n}$*-*t*-QG basis is actually* t*-QG.*

The following lemma allows us to "concatenate" greedy sets.

Lemma 4.2. *Suppose* A *is a* t*-greedy set for* $x \in X$*, and* $n_1 + \ldots + n_k = |A|$*. Then we can represent* A *as a disjoint union* $A = \cup_{s=1}^k A_s$*, where, for every* s*, the set* A_s *has cardinality* n_s*, and is a* t*-greedy set for* $(\cup_{j<s} A_j)^c x$*.*

Proof. For $s \leq k-1$, let A_s be a greedy set of $(A \backslash (\cup_{j<s} A_j))x$, of cardinality n_s. Then $A_k = A \backslash (\cup_{j<k} A_j)$ has cardinality n_k. From the construction it easily follows that A_s is a t-greedy set for $(\cup_{j<s} A_j)^c x$ for any s. □

Proof of Proposition 4.1. Suppose a basis $(e_i)_{i \in I}$ is $\mathbf{n}$-t-QG for some sequence $\mathbf{n} = (n_1 < n_2 < \ldots)$. Equivalently, there exists $C \geq 1$ so that $\|A^c x\| \leq C\|x\|$ whenever A is a t-greedy set for $x \in X$, and $|A| \in \mathbf{n}$ (Theorem 2.1).

Find k and m so that $S = k * \mathbf{n}$ meets $[im, (i+1)m]$ for any i. Then $\|A^c x\| \leq C^k \|x\|$ whenever A is a t-greedy set for x, and $|A| \in k * \mathbf{n}$. Indeed, by Lemma 4.2, we can find disjoint sets $A_1, \ldots, A_\ell$ ($\ell \leq k$) so that (i) $|A_s| \in \mathbf{n}$ for every s, and (ii) A_s is a t-greedy set for $(\cup_{j<s} A_j)^c x$. Then

$$A^c x = A_\ell^c (A_{\ell-1}^c (\ldots (A_2^c (A_1^c x)) \ldots)),$$

hence

$$\big\|A^c x\big\| \leq C \big\|A_{\ell-1}^c (\ldots (A_2^c (A_1^c x)) \ldots)\big\| \leq \ldots \leq C^{\ell-1} \big\|A_1^c x\big\| \leq C^\ell \|x\|.$$

Now suppose B is a t-greedy set for $x \in X$. Find $n \in k * \mathbf{n}$ so that $r = \big||B| - n\big| \leq m$. By Lemma 2.4, we can find a t-greedy set A of cardinality n, so that $|A \triangle B| \leq m$. Further,

$$\big\|B^c x\big\| \leq \big\|A^c x\big\| + |A \triangle B| \sup_{i \in I} \|e_i\| \|e_i^*\| \|x\| \leq K \|x\|,$$

where $K = C^k + m \sup_{i \in I} \|e_i\| \|e_i^*\|$. By [18] (cf. Theorem 2.1), the basis (e_i) is t-QG. □

5. Some classical bases are not $\mathbf{n}$-QG

It is known that the Haar basis in L_1, and the trigonometric basis in L_p ($p \neq 2$), are not QG. Below we show there is no sequence $\mathbf{n}$ making them $\mathbf{n}$-QG.

Proposition 5.1. *Suppose* $\mathbf{n}$ *is a sequence of positive integers.*

1. *The normalized Haar basis in* $L_1(0,1)$ *is not* $\mathbf{n}$*-QG.*
2. *Suppose* $1 < p < \infty$*, and the weight* w *on the torus* $\mathbb{T} = (0, 2\pi)$ *is such that* $\int_{\mathbb{T}} w(t)\,dt$ *and* $\int_{\mathbb{T}} w(t)^{-1/(p-1)}\,dt$ *are finite. If the trigonometric basis in* $L_p(w)$ *is* $\mathbf{n}$*-QG, then* $p = 2$*, and* w *is equivalent to a constant function – that is, there exists* $C \geq 1$ *so that* $C^{-1} \leq w \leq C$ *almost everywhere.*

In (2), we equip $L_p(w)$ with the norm

$$\|f\|_{p,w} = \Big(\int \big|f(t)\big|^p w(t)\,dt \Big)^{1/p},$$

and consider the trigonometric basis $(e_j)_{j \in Z}$ (here $e_j(\omega) = \exp(\iota t j)$ for $t \in \mathbb{T} = (0, 2\pi)$, $\iota = \sqrt{-1}$). The biorthogonal system consists of functions $e_j^* = w^{-1} e_j / (2\pi)$. The integrability of w and of $w^{-1/(p-1)}$ guarantees that e_j and e_j^* belong to $L_p(w)$ and $L_q(w)$ ($1/p + 1/q = 1$), respectively. Note that basis (e_j) need not be Schauder – by [12], it is a Schauder basis in $L_p(w)$ iff w belongs to the class A_p.

The rest of the section is devoted to proving Proposition 5.1. First note a property of $\mathbf{n}$-QG bases, related to unconditionality with constant coefficients.

Lemma 5.2. *Suppose a basis $(e_i)_{i\in I}$ is $\mathbf{n}$-QG. Then there exists $C \geq 0$ with the property that, for any $n \in \mathbf{n}$, and any sets $A, B \subset I$ with $A \cap B = \emptyset$ and $|A| = |B| = n$, we have $C^{-1}\|\mathbf{1}_A - \mathbf{1}_B\| \leq \|\mathbf{1}_A + \mathbf{1}_B\| \leq C\|\mathbf{1}_A - \mathbf{1}_B\|$.*

Proof of Proposition 5.1(1). Consider $h_j = 2^{j-1}\big(\mathbf{1}_{(0,2^{-j})} - \mathbf{1}_{(2^{-j},2^{1-j})}\big)$ $(j \in \mathbb{N})$. For $n \in \mathbf{n}$ set $x = \sum_{k=1}^{2n} h_k$. It is easy to check that $\|x\| \sim 2$. One greedy projection maps x to $y = \sum_{k=1}^{n} h_{2k}$, and one can show $\|y\| \sim n/2$. By Theorem 2.1, the basis of normalized Haar functions cannot be $\mathbf{n}$-QG. □

Proof of Proposition 5.1(2). The proof follows the lines of [12]. We can and do assume that all the members of the sequence $\mathbf{n}$ are even. Indeed, for a sequence $\mathbf{n} = (n_1 < n_2 < \ldots)$, define $\mathbf{n}' = (n_1' < n_2' < \ldots)$, where $n_k' = n_k$ if n_k is even, $n_k' = n_k + 1$ otherwise (with duplications eliminated). By the results from Section 2, a basis is $\mathbf{n}$-QG iff it is $\mathbf{n}'$-QG.

For $N \in \mathbb{N}$ (later, we take N to be even) consider the Dirichlet kernel

$$D_N(t) = \sum_{j=0}^{N-1} e_j(t) = \frac{1 - \exp(\iota N t)}{1 - \exp(\iota t)},$$

then

$$|D_N(t)|^2 = \frac{1 - \cos(Nt)}{1 - \cos t} = \frac{\sin^2(Nt/2)}{\sin^2(t/2)}.$$

We also need a "shifted version": for $u \in \mathbb{T}$,

$$D_{N,u}(t) = D_N(t-u) = \sum_{j=0}^{N-1} e^{-\iota j u} e_j(t).$$

If N is even, denote by $\mathcal{S}$ the set of subsets of $\{0, 1, \ldots, N-1\}$ with cardinality $N/2$. For a function $f : \mathcal{S} \to \mathbb{C}$, we can define its average value:

$$\mathrm{Ave}_A f(A) = \binom{N}{N/2}^{-1} \sum_{A\in\mathcal{S}} f(A).$$

For $A \in \mathcal{S}$, let

$$D_{N,u,A}(t) = \sum_{j=0}^{N-1} \varepsilon_j e_j(t-u) = \sum_{j=0}^{N-1} \varepsilon_j e^{-\iota j u} e_j(t), \quad \text{where } \varepsilon_j = \begin{cases} 1 & j \in A \\ -1 & j \notin A \end{cases}.$$

Lemma 5.2 shows that, if the trigonometric basis is $\mathbf{n}$-QG, then there exists $C > 0$ so that

$$C^{-1} \leq \inf_{u\in\mathbb{T}, N/2\in\mathbf{n}} \frac{\|D_{N,u}\|_{p,w}}{\mathrm{Ave}_A\|D_{N,u,A}\|_{p,w}} \leq \sup_{u\in\mathbb{T}, N/2\in\mathbf{n}} \frac{\|D_{N,u}\|_{p,w}}{\mathrm{Ave}_A\|D_{N,u,A}\|_{p,w}} \leq C. \tag{5.1}$$

First estimate the denominator:

Lemma 5.3. *There exists a constant C, which depends on w, but not on u or N, so that $C^{-1}\sqrt{N} \leq \mathrm{Ave}_A\|D_{N,u,A}\|_{p,w} \leq C\sqrt{N}$.*

Proof. We make use of the restricted Khintchine inequality: for complex numbers $a_1, \ldots, a_N$,

$$\mathrm{Ave}_A\Big|\sum_{j\in A} a_j - \sum_{j\notin A} a_j\Big|^p \sim \Big(\sum_j |a_j|^2 - N^{-1}\big|\sum_j a_j\big|^2\Big)^{p/2}.$$

This inequality was established in [13] for real numbers. The complex case can be handled similarly.

It follows that

$$\mathrm{Ave}_A \big\| D_{N,u,A} \big\|_{p,w}^p = \int_{\mathbb{T}} \mathrm{Ave}_A \Big| \sum_j \varepsilon_j e_j(t-u) \Big|^p w(t)\, dt \sim \int_{\mathbb{T}} \Big(N - \frac{1 - \cos N(t-u)}{N(1-\cos(t-u))} \Big)^{p/2} w(t)\, dt.$$

Clearly the right hand side doesn't exceed $N^{p/2} \displaystyle\int w(t)\, dt \sim N^{p/2}$. On the other hand,

$$\frac{1-\cos Ns}{1-\cos s} \leq 4 \quad \text{when} \quad s \notin \Big[0, \frac{\pi}{6}\Big) \cup \Big(\frac{5\pi}{6}, 2\pi\Big].$$

Thus, for $N \geq 8$,

$$\int_{\mathbb{T}} \Big(N - \frac{1 - \cos N(t-u)}{N(1-\cos(t-u))} \Big)^{p/2} w(t)\, dt \geq \frac{N^{p/2}}{2} \int_{|t-u| \geq \pi/6} w(t)\, dt \geq cN^{p/2},$$

where c is a constant (depending only on w). □

Now recall some facts from classical harmonic analysis. Suppose 0 is a Lebesgue point of a non-negative $g \in L_1(\mathbb{T})$. Then (see e.g. [9, Section 1.3]):

(1) The sequence of functions $\Big(\dfrac{\sin(Nt/2)}{\sqrt{N}\sin(t/2)}\Big)^2$ is an approximate identity, hence

$$\lim_N \int_{\mathbb{T}} \Big(\frac{\sin(Nt/2)}{\sqrt{N}\sin(t/2)}\Big)^2 g(t)\, dt = g(0).$$

From this it easily follows that:

(2) For $p < 2$,

$$\lim_N \int_{\mathbb{T}} \Big(\frac{\sin(Nt/2)}{\sqrt{N}\sin(t/2)}\Big)^p g(t)\, dt = 0.$$

(3) For $p > 2$,

$$\lim_N \int_{\mathbb{T}} \Big(\frac{\sin(Nt/2)}{\sqrt{N}\sin(t/2)}\Big)^p g(t)\, dt = \infty$$

provided $g(0) \neq 0$.

Suppose, for the sake of contradiction, that the trigonometric system is $\mathbf{n}$-QG. Combining (5.1) with Lemma 5.3, we conclude the existence of $K > 0$ so that the inequality $K^{-1}\sqrt{N} \leq \|D_{N,u}\|_{p,w} \leq K\sqrt{N}$ holds for any $u \in \mathbb{T}$, and for any N so that $N/2 \in \mathbf{n}$. Applying the three limit statements above to $w(\cdot + u)$ instead of $g(\cdot)$, and recalling that almost every $u \in \mathbb{T}$ is a Lebesgue point of w, we conclude:

(1) If $p = 2$, then, whenever u is a Lebesgue point for w, we have

$$K^{-2} \leq \lim_{N\to\infty, N/2\in\mathbf{n}} \frac{1}{N} \|D_{N,u}\|_{2,w}^2 = w(u) \leq K^2.$$

Therefore, w is equivalent to a constant.

(2) If $1 < p < 2$, then

$$\lim_{N\to\infty, N/2\in\mathbf{n}} \frac{1}{N^{p/2}} \|D_{N,u}\|_{p,w}^p = 0$$

if u is a Lebesgue point of w. This contradicts $K\sqrt{N} \leq \|D_{N,u}\|_{p,w}$.

(3) Similarly, for $p > 2$,

$$\lim_{N\to\infty, N/2\in\mathbf{n}} \frac{1}{N^{p/2}} \|D_{N,u}\|_{p,w}^p = \infty$$

whenever u is a Lebesgue point with $w(u) > 0$, contradicting $\|D_{N,u}\|_{p,w} \leq K\sqrt{N}$. □

6. Questions

Question 6.1. Section 3 contains examples of non-quasi-greedy bases, which are $\mathbf{n}$-QG bases for some "very lacunary" sequences $\mathbf{n}$. Do we have similar examples featuring sequences $\mathbf{n}$ with smaller (but still large enough – see Section 4) gaps? Can we have $\mathbf{n}$ increase exponentially, or even slower?

Question 6.2. Per [4], $C[0,1]$ has no quasi-greedy bases. Can $C[0,1]$ have a $\mathbf{n}$-QG basis?

Question 6.3. Per [6], $L_1(0,1)$ has no uniformly bounded quasi-greedy Markushevich basis. Can $L_1(0,1)$ have a uniformly bounded $\mathbf{n}$-QG basis?

Question 6.4. Suppose $0 < t < s \leq 1$. From the definition, any $\mathbf{n}$-t-QG basis is also $\mathbf{n}$-s-QG. Under what conditions is the converse implication true? In particular, is every $\mathbf{n}$-QG basis necessarily $\mathbf{n}$-t-QG for any $t \in (0,1]$?

Question 6.5. Can an "adaptive" placement of gaps offer an advantage? More specifically: for a basis $(e_i)_{i\in I} \subset X$, and $x \in X$, can we find a sequence $\mathbf{n}(x) = (n_1(x) < n_2(x) < \ldots) \subset \mathbb{N}$ so that the sequence $\lim_k \mathbf{G}_{n_k(x)} x = x$? If yes, can the selection of such a sequence be accomplished in computationally simple manner?

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