On the stability of some preservers

T. Oikhberg

Dept. of Mathematics
University of Illinois
Urbana IL 61801
oikhberg@illinois.edu

Abstract

We study some "almost preserver" problems on von Neumann algebra modules. More precisely, we study (1) maps which "almost preserve" the right or left annihilator; (2) the "almost band preservers" – that is, maps which "almost preserve corners", and (3) "almost centralizers," which almost preserve module actions. Under certain conditions, we show that maps of these types are automatically continuous, and can be approximated by maps which precisely preserve these relations; often, the operators from the latter class are multiplication operators.

Keywords: Preserver problem, C^*-algebra of real rank 0, von Neumann algebra, module, non-commutative function space

1. Introduction

Throughout the history of functional analysis and linear algebra, a lot of attention has been paid to the structure of *preservers* – that is, maps that preserve certain quantity or relation. The reader can consult [30] for a sample of preserver problems.

In this paper, we consider maps which "almost preserve" certain relations, trying to address three related questions:

- Are these "almost preservers" automatically continuous?
- Can the "almost preservers" be approximated by maps which precisely preserve the relation in question – in other words, is the class of preservers "stable"?
- Is there a simple representation for maps preserving certain quantity?

The earliest work on the "almost preserver" problem in the Banach algebra setting is due to B. E. Johnson [22, 23], who studied whether an "almost multiplicative" map is necessarily a small perturbation of a multiplicative one. Some

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recent work on maps almost preserving the product, or the Jordan product, is in [1] and [21].

Throughout this paper, we consider a C^*-algebra (often a von Neumann algebra) $\mathcal{A}$, and its left modules, right modules, or bimodules X and Y. If $\mathcal{A}$ has the identity $\mathbf{1}$, we assume that the modules are unital (for any $x \in X$, $\mathbf{1}x = x\mathbf{1} = x$). Further, we assume that multiplication is contractive: for any $a \in \mathcal{A}$ and $x \in X$, we have $\|ax\| \leq \|a\|\|x\|$ and $\|xa\| \leq \|a\|\|x\|$ (whenever these expressions make sense).

Clearly, if a C^*-algebra $\mathcal{B}$ contains $\mathcal{A}$ as a subalgebra, and the two share the same unit, then $\mathcal{B}$ is an $\mathcal{A}$-bimodule with the properties described above. Non-commutative symmetric function spaces provide another class of examples. We refer the reader to [15] or [29, Chapters 2-3]) for an introduction (see also [31] or [42] on non-commutative measure theory). Below, we outline the construction in the diffuse case.

Suppose a von Neumann algebra $\mathcal{A}$ is equipped with a normal faithful semifinite trace τ. Consider a symmetric function space $\mathcal{E}$ on $[0, \tau(\mathbf{1}))$ – that is, a Köthe function space (equipped with a norm $\|\cdot\|_{\mathcal{E}} = \|\cdot\|$), so that, if $x \in \mathcal{E}$, and a measurable function y satisfies $y^\dagger \leq x^\dagger$ ($f^\dagger$ denotes the decreasing rearrangement of f), then $y \in \mathcal{E}$, with $\|y\| \leq \|x\|$. We denote by $\mathfrak{M}(\mathcal{A}, \tau)$ (or simply $\mathfrak{M}(\mathcal{A})$) the $*$-algebra of τ-measurable operators. For $x \in \mathfrak{M}(\mathcal{A})$ denote by $\mu_x(\cdot)$ the *generalized singular value function* of x. Define the (noncommutative) *symmetric space* $\mathcal{E}(\mathcal{A})$ (of τ-measurable operators) as the subspace of $\mathfrak{M}(\mathcal{A})$, consisting of all x s.t. $\mu_x \in \mathcal{E}$. Then norm $\|x\|_{\mathcal{E}} := \|\mu_x\|_{\mathcal{E}}$ makes it into a Banach space. Any such space is automatically an $\mathcal{A}$-bimodule, and moreover, the embedding of $\mathcal{E}$ into $\mathfrak{M}(\mathcal{A})$ (the latter equipped with its measure topology) is continuous [15, Section 4].

A symmetric space $\mathcal{E}(\mathcal{A})$ is called *strongly* (*fully*) *symmetric* if, whenever $x, y \in \mathcal{E}(\mathcal{A})$ (resp. $x \in \mathcal{E}(\mathcal{A})$, $y \in \mathfrak{M}(\mathcal{A})$) satisfy $y \prec\prec x$ (here and below, $\prec\prec$ denotes the Hardy-Littlewood-Polya submajorization), then $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$ (with $y \in \mathcal{E}(\mathcal{A})$, in the case of full symmetry). Note that $\mathcal{A}$ itself is an example of a fully symmetric function space. The properties of being strongly or fully symmetric are known to pass from $\mathcal{E}$ to $\mathcal{E}(\mathcal{A})$.

We say that $\mathcal{E}(\mathcal{A})$ is *proper* if $\mathcal{E}$ does not contain $\chi_{[0,\infty)}$. If $\tau(\mathbf{1}) < \infty$, this condition is satisfied automatically; if $\tau(\mathbf{1}) = \infty$, we just mean that $\mathbf{1} \notin \mathcal{E}(\mathcal{A})$.

Standard operator algebraic notation and terminology will be used throughout the paper. Maps between normed spaces are assumed to be linear, but *a priori*, continuity is not assumed. As a matter of convention, we write $|z| = (z^*z)^{1/2}$. For a normed space Z, $\mathbf{B}(Z)$ stands for the closed unit ball of Z. The center of a C^*-algebra $\mathcal{A}$ is denoted by $\mathfrak{Z}(\mathcal{A})$. For an element $a \in \mathcal{A}$, we define the operators of *left* and *right multiplication* by a: $\mathbf{L}_a : x \mapsto ax$, $\mathbf{R}_a : x \mapsto xa$. For $a \in \mathfrak{Z}(\mathcal{A})$, $\mathbf{L}_a = \mathbf{R}_a$; we refer to both as the *multiplication operator* $\mathbf{M}_a$. We use the notation $\mathbf{P}(\mathcal{A})$ for the set of projections in $\mathcal{A}$, and write $\mathbf{P}_\tau(\mathcal{A}) = \{p \in \mathbf{P}(\mathcal{A}) : \tau(p) < \infty\}$. Further, $\mathbf{P}_f(\mathcal{A})$ is the set of finite projections in $\mathcal{A}$.

The paper is organized as follows: in Section 2 we consider almost annihilator preserving maps. We say that a linear map $T : X \to Y$ (X and Y are right

$\mathcal{A}$-modules, with $\mathcal{A}$ a real rank 0 C^*-algebra) is *ε-right annihilator preserving* if, for any $p \in \mathbf{P}(\mathcal{A})$ and $x \in X$, we have $\|p[Tx]\| \leq \varepsilon \|x\|$ whenever $px = 0$. We begin by showing (in Subsection 2.1) that, in many situations, any ε-right annihilator preserving map has to be continuous. In Subsection 2.2, we show that often, such maps are small perturbations of right multiplication maps.

In Section 3, motivated by the theory of Banach lattices (see e.g. [32, Chapter 3]), we introduce the class of (precisely) band-preserving maps: $T : X \to Y$ is ε-band-preserving (X, Y are $\mathcal{A}$-modules) if, for any $p \in \mathbf{P}(\mathcal{A})$ and $x \in X$, we have $p[Tx]p = Tx$ whenever $pxp = x$. Theorem 13 shows that, in many cases, any band-preserving map on X is implemented by multiplication $\mathbf{M}_a$ ($a \in \mathfrak{Z}(\mathcal{A})$); hence, in particular, such a map must be continuous.

In Section 4, we study the stability of band-preservers (introduced in Section 3): if a map T "almost satisfies" the band-preservation condition (that is, $\|Tx - p[Tx]p\| \leq \varepsilon \|x\|$ whenever $pxp = x$ – see Definition 3), can it be approximated by a band-preserving one? In particular, we give a positive answer for maps $T : X \to X$, where X is an order continuous strongly symmetric space, while $\mathcal{A}$ is separably acting and semifinite (Theorem 18).

Finally, in Section 5, we deal with almost centralizers – that is, the maps between $\mathcal{A}$-modules which "almost preserve" the module actions. For instance, an ε-right centralizers is a map $T : X \to Y$ so that $\|T(ax) - a[Tx]\| \leq \varepsilon \|a\| \|x\|$ holds for any $a \in \mathcal{A}$ and $x \in X$. In Theorem 22 we show (among other things) that, if $\mathcal{A}$ is a hyperfinite von Neumann algebra, and X, Y are reflexive, then any T as above is near a right centralizer (a 0-right centralizer).

2. Almost annihilator preservers

In this section we deal with operators which "almost preserve" annihilators. Suppose $\mathcal{A}$ is a C^*-algebra, and X is a left $\mathcal{A}$-module. The *right annihilator* of $a \in \mathcal{A}$ ($\mathcal{A}$ is a C^*-algebra) is $a^{[\mathrm{ra}]} = \{x \in X : ax = 0\}$. A map $T : \mathcal{A} \to \mathcal{A}$ is a *right annihilator preserver* (*RAP* for short) if, for any $a \in \mathcal{A}$, $T(a^{[\mathrm{ra}]}) \subset a^{[\mathrm{ra}]}$, or equivalently, $a[Tx] = 0$ whenever $ax = 0$. If $\mathcal{A}$ is a von Neumann algebra, it is easy to see that T is RAP if and only if, for any $p \in \mathbf{P}(\mathcal{A})$, and for any $x \in X$, we have $p[Tx] = 0$ whenever $px = 0$.

We use the latter reformulation of right annihilator preservation to study the stability of RAP maps. To make sense of this, however, we need to guarantee that projections are abundant in $\mathcal{A}$. Recall that a C^*-algebra $\mathcal{A}$ has *real rank* 0 if self-adjoint elements with finite spectrum are dense in the set of all self-adjoint elements. For more information about this class, see [7, Section V.3.2].

Definition 1. Suppose $\mathcal{A}$ is a real rank 0 C^*-algebra, and X, Y are left $\mathcal{A}$-modules. We say that a linear map $T : X \to Y$ is *ε-right annihilator preserving* (or a *ε-right annihilator preserver*, *ε-RAP* for short) if, for any $p \in \mathbf{P}(\mathcal{A})$ and $x \in \mathcal{E}$, we have $\|p[Tx]\| \leq \varepsilon \|x\|$ whenever $px = 0$.

Recall that a unital C^*-algebra $\mathcal{A}$ is called an *AW^* algebra* (or a *Ricart algebra*) if, for any $a \in \mathcal{A}$, there exists $p \in \mathbf{P}(\mathcal{A})$ so that $a^{[\mathrm{ra}]} = p^{[\mathrm{ra}]}$. In this case, any 0-RAP map (in the sense of Definition 1) is RAP. Such algebras have real rank 0, see e.g. [6] or [7, Section III.1.8].

We can define ε-left annihilator preserving (ε-LAP) maps in a similar fashion.

LAP and RAP maps have been studied intensively in the recent years. For instance, the proof of [16, Theorem 2.4] show that, if $\mathcal{A}$ is a von Neumann subalgebra of a von Neumann algebra X (with the same unit), then any RAP (or LAP) map $\mathcal{A} \to X$ is continuous. Some results on automatic continuity of RAP maps on $C*$-algebras can be found in [4].

Clearly, a right multiplication $\mathbf{R}_a : x \mapsto xa$ is right annihilator preserving. In certain cases, these are the only RAP maps. By [27], if $\mathcal{A}$ is a unital C^*-algebra, X is a unital bimodule (that is, $\mathbf{1}x = x\mathbf{1} = x \; \forall x \in X$), then any RAP $T \in B(\mathcal{A}, X)$ is implemented by right multiplication. Combining this theorem with the one from [16] (quoted above), we conclude that, if $\mathcal{A}$ is a von Neumann subalgebra of a von Neumann algebra X (with the same unit), then any RAP map from $\mathcal{A}$ to X is right multiplication by an element of X.

We therefore ask:

- Can every ε-left annihilator preserver be approximated by $\mathbf{R}_a$, for some a?
- Is an ε-left annihilator preserver automatically continuous?

Below we show that, in certain cases, the answer is positive.

2.1. Automatic continuity

To proceed, we need to introduce some notation. If $\mathcal{A}$ is an AW^*-algebra, we say that a left $\mathcal{A}$-module X has *Property (F)* if, for any net of commuting projections p_α in $\mathcal{A}$, increasing to $\mathbf{1}$, and any $x \in X$, we have $\|x\| = \lim_\alpha \|p_\alpha x\|$. This property holds, for instance, when X is a C^*-algebra containing $\mathcal{A}$, and sharing the same unit. To describe a class of symmetric function spaces with the Property (F), recall that such a space $\mathcal{E}(\mathcal{A}, \tau)$ (τ is a normal semi-finite faithful trace on a von Neumann algebra $\mathcal{A}$) is said to have the *Fatou norm* if, whenever (x_α) is a net in $\mathcal{E}_+$, increasing to $x \in \mathcal{E}$, then $\|x\| = \lim_\alpha \|x_\alpha\|$ (see e.g. [15, Section 5]). If $\mathcal{E}$ is a strongly (fully) symmetric function space on $[0, \tau(\mathbf{1}))$ with Fatou norm, then the same is true for $\mathcal{E}(\mathcal{A})$.

Lemma 1. *If the norm on a symmetric space $\mathcal{E}$ is Fatou, then $\lim_\alpha \|p_\alpha x p_\alpha\| = \|x\|$ whenever $x \in \mathcal{E}$, and $(p_\alpha) \subset \mathbf{P}(\mathcal{A})$ is a net of projections increasing to $\mathbf{1}$. In particular, $\mathcal{E}$ has Property (F).*

Proof. Suppose first $x \geq 0$. By [14, Proposition 1.3], $x^{1/2} p_\alpha x^{1/2} \nearrow x$, hence $\lim_\alpha \|x^{1/2} p_\alpha x^{1/2}\| = \|x\|$. However, for any z, $\mu_{z^*z} = \mu_{zz^*}$, hence $\|z^*z\|_{\mathcal{E}} = \|zz^*\|_{\mathcal{E}}$. Consequently,

$$\|x^{1/2} p_\alpha x^{1/2}\| = \|(p_\alpha x^{1/2})^*(p_\alpha x^{1/2})\| = \|(p_\alpha x^{1/2})(p_\alpha x^{1/2})^*\| = \|p_\alpha x p_\alpha\|,$$

hence $\lim_\alpha \|p_\alpha x p_\alpha\| = \|x\|$.

For the general $x \in \mathcal{E}$, write $x = u|x|$, where u is a partial isometry. By the above, $\lim_\alpha \|xp_\alpha\| = \lim_\alpha \||x|p_\alpha\| = \|x\|$ (that is, $\mathcal{E}$ has Property (F)). Similarly, $\lim_\alpha \|p_\alpha x\| = \|x\|$. It remains to show that, for any $\varepsilon > 0$, we have $\|p_\alpha x p_\alpha\| > \|x\| - \varepsilon$ whenever the index α is "large enough." First note that there exists α_1 so that $\|xp_\beta\| > \|x\| - \varepsilon/2$ for any $\beta \geq \alpha_1$. Similarly, there exists α_2 so that $\|p_\gamma x p_\beta\| > \|xp_\beta\| - \varepsilon/2 > \|x\| - \varepsilon$ for any $\gamma \geq \alpha_2$. Now, for $\alpha \geq \alpha_1, \alpha_2$, we have $\|p_\alpha x p_\alpha\| \geq \|p_\alpha x p_\beta\| > \|x\| - \varepsilon$.

Theorem 2. *Suppose a von Neumann algebra $\mathcal{A}$ is equipped with a faithful normal semifinite trace τ. Suppose, further, that Y has Property (F), and one of the two holds:*

1. *$X = \mathcal{A}$.*
2. *X is a proper strongly symmetric space.*

Then any ε-RAP map from X to Y is continuous.

For the proof, we need to establish a "non-commutative function space" counterpart of the classical automatic continuity result of J. Ringrose. In [36], he proved that an operator on a von Neumann algebra is continuous iff its restrictions to MASAs are continuous (later this was extended to C^*-algebras by J. Cuntz [11]). For our version, we need to describe the subspace of a symmetric function space $\mathcal{E}$ associated with a MASA $\mathcal{B}$. The following result (needed here and below) is apparently part of the operator algebra lore, but we aren't aware of any published proof.

Lemma 3. *Suppose x is a self-adjoint element affiliated with a von Neumann algebra $\mathcal{A}$. Then x is affiliated with the commutative von Neumann algebra generated by its spectral projections.*

Proof. By [25] (see the proof of Lemma 5.6.7), x is affiliated with the commutative von Neumann algebra $\mathcal{B}$, generated by $(x + \iota\mathbf{1})^{-1}$ and $(x - \iota\mathbf{1})^{-1}$. Moreover, these two operators are bounded, normal, and adjoints of each other. Therefore, $\mathcal{B}$ is generated by the spectral projections of (say) $(x + \iota\mathbf{1})^{-1}$. By functional calculus (see [25, Theorem 5.6.27]), these coincide with spectral projections of x itself.

Suppose τ is a faithful normal semifinite trace on a von Neumann algebra $\mathcal{A}$, and $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$ so that $\tau|_{\mathcal{B}}$ is semifinite, we say that $\mathcal{B}$ is *τ-semifinite*. We can now formulate an automatic continuity result for symmetric spaces, in the spirit of [36].

Proposition 4. *Suppose τ is a semi-finite trace on the von Neumann algebra $\mathcal{A}$, and $\mathcal{E}$ is a proper strongly symmetric space. If Y is a Banach space, and the linear map $T : \mathcal{E} \to Y$ is continuous on $\mathcal{E}[\mathcal{B}]$ whenever $\mathcal{B}$ is a τ-semifinite MASA in $\mathcal{A}$, then T is continuous.*

Proof. In [36, Section 2] (specifically, see Theorem 2.5 and Corollary 2.6), it is shown that, if a linear map $T : \mathcal{A} \to Y$ is continuous on every MASA in a von Neumann algebra $\mathcal{A}$, then it is continuous. The reasoning can be adapted to the case of operators on $\mathcal{E}$ in a fairly straightforward manner. The proof of Corollary 2.6 of [36] goes through without changes if we replace $\mathcal{A}$ by $\mathcal{E}$, while Lemmas 2.1-2.4 require only minimal adjustments. Only a very minor change (outlined below) is needed in the proof of [36, Theorem 2.5]. As in that theorem, we assume that $Y = \mathbb{C}$, and $T = f : \mathcal{E} \to \mathbb{C}$ is a linear functional.

We need a technical fact: if $p \in \mathbf{P}(\mathcal{A})$, and $\mathcal{B}$ is a τ-semifinite MASA in $p\mathcal{A}p$, then $\mathcal{A}$ contains a MASA $\tilde{\mathcal{B}}$, so that $\mathcal{B} \subset \tilde{\mathcal{B}}$. Indeed, find a maximal family of mutually orthogonal finite trace projections $(p_i)_{i \in I}$, which are dominated by $p^\perp$. Clearly $\sum_i p_i = p^\perp$. Then any MASA in $\mathcal{A}$ which contains both $\mathcal{B}$ and the projections p_i satisfies our conditions.

Keeping the notation of [36], let Q_0 be the maximal central projection for which $\mathcal{A}Q_0$ is abelian. Then f is bounded on $\mathcal{E}Q_0$. Suppose, for the sake of contradiction, that f is unbounded on $\mathcal{E}Q_0^\perp$. Find norm-one self-adjoint elements $A_1, A_2, \ldots \in \mathcal{E}Q_0^\perp$ with mutually orthogonal supports Q_i, so that $\lim_i |f(A_i)| = \infty$. In fact, by passing from A_i to its positive or negative part, we can assume that all the A_i's are positive. Let $Q = \sum_{i=1}^\infty Q_i$. For every $i \in \mathbb{N}$ and $c > 0$, the spectral projection $\chi_{(c,\infty)}(A_i)$ has finite trace, hence these projections generate a τ-semifinite commutative $*$-subalgebra $\mathcal{C} \subset Q\mathcal{A}Q$. By the "technical fact" from the preceding paragraph, there exists a τ-semifinite MASA $\mathcal{B} \subset \mathcal{A}$, which contains $\mathcal{A}Q_0$ and $\mathcal{C}$. By Lemma 3, A_i's belong to $\mathcal{E}[\mathcal{B}]$. Thus, f is unbounded on $\mathcal{E}[\mathcal{B}]$, giving the desired contradiction.

Proof of Theorem 2. We conduct the proof of Case (2); Case (1) is handled similarly, except that we can appeal to [36] directly, instead of Proposition 4.

Suppose, for the sake of contradiction, that $T : X \to Y$ is discontinuous. Then there exists a τ-semifinite MASA $\mathcal{B} \subset \mathcal{A}$ so that the restriction of T to $X[\mathcal{B}]$ is discontinuous. Denote by $\mathcal{P}$ the family of all $p \in \mathbf{P}(\mathcal{B})$ for which $T|_{p\mathcal{B}[X]}$ is continuous. We shall prove that $\mathbf{1} \in \mathcal{P}$. This is done in several steps.

Step 1. If $\mathcal{I} \subset \mathbf{P}(\mathcal{B})$ consists of mutually orthogonal projections, then there exists $C > 0$ so that $\|T|_{pX[\mathcal{B}]}\| > C$ for at most finitely many $p \in \mathcal{I}$.

Indeed, otherwise we can find (mutually orthogonal) $p_1, p_2, \ldots \in \mathcal{I}$ so that $\|T|_{p_1 X[\mathcal{B}]}\| > 4$, and $\|T|_{p_{i+1} X[\mathcal{B}]}\| > 4\|T|_{p_i X[\mathcal{B}]}\|$ for $i \in \mathbb{N}$. Find $x_i \in \mathbf{B}(p_i X[\mathcal{B}])$ so that $\|Tx_i\| > 4^i$. Let $x = \sum_j 2^{-j} x_j$, and $x_i' = x - 2^{-i} x_i = \sum_{j \neq i} 2^{-j} x_j$. Then the inequality

$$\|p_i[Tx]\| \geq \|p_i[T(2^{-i} x_i)]\| - \|p_i[Tx_i']\| \geq 2^i - 2\varepsilon$$

holds for any i, which is a contradiction. ⊞

Step 2. The following properties of $\mathcal{P}$ are easy to establish.

(i) If $p_1, \ldots, p_n \in \mathcal{P}$, then $\vee_k p_k \in \mathcal{P}$. (ii) If $q \leq p \in \mathcal{P}$, then $q \in \mathcal{P}$. ⊞

Step 3. Show that $\mathcal{P}$ is stable under joins: $\vee_{p \in \mathcal{P}} \in \mathcal{P}$.

To this end, establish first that $C := \sup_{p \in \mathcal{P}} \|T|_{pX[\mathcal{B}]}\| < \infty$. Indeed, $\mathcal{P}$ is an upward directed net. If the supremum above is infinite, we can find $p_1 < p_2 < \ldots$

in $\mathcal{P}$ so that

$$\|T|_{p_1 X[\mathcal{B}]}\| > 4 \text{ and } \forall j \ \|T|_{p_{j+1} X[\mathcal{B}]}\| > 4\|T|_{p_j X[\mathcal{B}]}\|.$$

Let $q_1 = p_1$, $q_j = p_{j+1} - p_j$ for $j \geq 1$. It is easy to see that, for any j, $\|T|_{q_j X[\mathcal{B}]}\| > 3^j$, which contradicts the result of Step 1.

Let $q = \vee_{p \in \mathcal{P}}$, and show that $\|T|_{qX[\mathcal{B}]}\| \leq C + \varepsilon$. Indeed, otherwise we can find $x \in \mathbf{B}(qX[\mathcal{B}])$ so that $\|Tx\| > C + \varepsilon$. For $p \in \mathcal{P}$ let $x_p = px$, and $y_p = (q-p)x$. As Y has Property (F), $\|p[Tx]\| > C + \varepsilon$ for p "large" enough. On the other hand,

$$\|p[Tx]\| \leq \|p[Tx_p]\| + \|p_i[Ty_p]\| \leq \|Tx_p\| + \varepsilon\|y_p\| \leq C + \varepsilon,$$

a contradiction. ⊞

Step 4. Above, we established that $q := \vee_{p \in \mathcal{P}} p \in \mathcal{P}$. To complete the proof, we need to show that $q = \mathbf{1}$, or equivalently, $q^\perp = 0$.

Note that, for any non-zero projection $r \leq q^\perp$, $T|_{rX[\mathcal{B}]}$ is unbounded. Consequently, $q^\perp$ has no minimal subprojections (if r is minimal, then $rX[\mathcal{B}]$ is 1-dimensional). Thus, if $q^\perp \neq 0$, we can find mutually orthogonal non-zero finite trace projections $r_1, r_2, \ldots \leq q^\perp$. By Step 1, all but finitely many of them belong to $\mathcal{P}$, yielding a contradiction.

We also mention the automatic continuity of operators on AW^*-algebras.

Proposition 5. *Suppose $\mathcal{A}$ is a AW^*-algebra, and Y is a left $\mathcal{A}$-module with Property (F). Then any ε-RAP map $T : \mathcal{A} \to Y$ is continuous.*

Proof. By [11], it suffices to show that T is continuous on any MASA. However, by [7, Section III.1.8], any such MASA is a von Neumann algebra. Now invoke the proof of Theorem 2.

Next we address the more general case of C^*-algebras of real rank 0.

Theorem 6. *Suppose $\mathcal{A}$ is a C^*-algebra of real rank 0 without minimal projections, and X is an $\mathcal{A}$-bimodule. Then any ε-RAP map $T : X \to \mathcal{A}$ is automatically continuous.*

For the proof of Theorem 6, introduce some notation. For $q \in \mathbf{P}(\mathcal{A})$, let $X^{[q]} = qX$. For $T : X \to \mathcal{A}$, we set $T^{[q]} : X^{[q]} \to \mathcal{A}^{[q]} : x \mapsto q[Tx]$. We also need a lemma, which will be established later.

Lemma 7. *Suppose that, in the conditions of Theorem 6, $T : X \to \mathcal{A}$ is unbounded. Then for any sequence $(C_i)_{i \in \mathbb{N}} \subset (0, \infty)$ there exist projections $\mathbf{1} = q_0 > q_1 > q_2 > \ldots$ in $\mathbf{P}(\mathcal{A})$ so that, for any $i \in \mathbb{N}$, $T^{[q_i]}$ is unbounded, and $\|T^{[q_{i-1} - q_i]}\| > C_i$.*

Proof of Theorem 6. Suppose, for the sake of contradiction, that a ε-RAP map $T : X \to \mathcal{A}$ is unbounded. Use Lemma 7 to obtain projections $\mathbf{1} = q_0 > q_1 > q_2 > \ldots$ in $\mathbf{P}(\mathcal{A})$ so that, for any $i \in \mathbb{N}$, $\|T^{[q_{i-1}-q_i]}\| > 2^i + \varepsilon$. For $i \in \mathbb{N}$ find $x_i \in X^{[q_{i-1}-q_i]}$ so that $\|x_i\| < 2^{-i}$ and $\|T^{[q_{i-1}-q_i]}x_i\| > 2^i + \varepsilon$. Let $x = \sum_{i=1}^{\infty} x_i$. For every i,

$$\|(q_{i-1} - q_i)[Tx]\| \geq \|(q_{i-1} - q_i)[Tx_i]\| - \|(q_{i-1} - q_i)[T(x - x_i)]\| > 2^i,$$

which is the desired contradiction.

Remark 1. In general, Theorem 6 is not valid without additional assumptions on X or $\mathcal{A}$ (such as the non-existence of minimal projection in $\mathcal{A}$). For instance, suppose $\mathcal{A} = B(\ell_2)$, and X is a "row" copy of ℓ_2 – that is, an $x \in X$ can be described as an infinite matrix, with $(x_1, x_2, \ldots)$ in the top row, and zeros in all subsequent rows. Thus, we can identify $x \in X$ with $\widetilde{x} \in \ell_2$. For a linear operator $\phi : \ell_2 \to \ell_2$, define $T : X \to X$ via $\widetilde{Tx} = \phi\widetilde{x}$. Clearly T is RAP; however, T is unbounded whenever ϕ is.

The following result (needed to prove Lemma 7) may be of independent interest.

Proposition 8. *Suppose $\mathcal{A}$ is a C^*-algebra of real rank 0 without minimal projections, $p \in \mathbf{P}(\mathcal{A})$, and $a \in \mathcal{A}$ satisfies $pa = a$. Then for any $\delta > 0$ there exists $q \in \mathbf{P}(\mathcal{A})$ so that $0 < q < p$, and $\min\{\|qa\|, \|q^\perp a\|\} > \|a\| - \delta$.*

Proof. Pick $\alpha \in (1/3, 1/2)$. Let $x = (aa^*)^\alpha$. By [7, II.3.2.1] (or [35, Proposition 1.4.5]), we can find $u \in p\mathcal{A}$ so that $uu^* = (a^*a)^{1/2-\alpha}$, and $a = xu$. Let $\beta = (1-2\alpha)/\alpha$ (then $0 < \beta < 1$). Pick $\sigma > 0$ so small that

$$(\|a\|^{2\alpha} - \sigma)^{2+\beta} - \rho > (\|a\| - \delta)^2, \text{ where } \rho = \|a\|^{2\alpha}\sigma^\beta + 2\|a\|^{2(1-\alpha)}\sigma.$$

As real rank zero passes to hereditary subalgebras, we can find $v = \sum_{j=1}^N \gamma_j r_j$ so that $\|v\| \leq \|a\|^{2\alpha}$, and $\|v - (aa^*)^\alpha\| < \sigma$. Here, $r_1, \ldots r_N$ are mutually orthogonal subprojections of q, and $\gamma_1 > \ldots > \gamma_N > 0$. Find a projection q s.t. $0 < q < r_1$. We claim that q has the desired properties – that is, $\min\{\|qa\|, \|q^\perp a\|\} > \|a\| - \delta$. Below we estimate $\|qa\|$, since $\|(r-q)a\|$ is handled similarly, and $\|q^\perp a\| \geq \|(r-q)a\|$.

We show that $\|qaa^*q\| \geq (\|a\| - \delta)^2$. Write $aa^* = (aa^*)^\alpha(aa^*)^{1-2\alpha}(aa^*)^\alpha$. Note that $(aa^*)^{1-2\alpha} = ((aa^*)^\alpha)^\beta$, hence, by [5, Theorem X.1.1] (see also [2] for a generalization), $\|(aa^*)^{1-2\alpha} - v^\beta\| < \sigma^\beta$. By the triangle inequality,

$$\begin{aligned}
&\|v^{2+\beta} - aa^*\| = \|vv^\beta v - (aa^*)^\alpha(aa^*)^{1-2\alpha}(aa^*)^\alpha\| \\
&\leq \|(v - (aa^*)^\alpha)v^\beta v\| + \|(aa^*)^\alpha(v^\beta - (aa^*)^{1-2\alpha})v\| \\
&+ \|(aa^*)^\alpha(aa^*)^{1-2\alpha}(v - (aa^*)^\alpha)\| \\
&\leq \|a\|^{2\alpha}\sigma^\beta + 2\|a\|^{1-\alpha}\sigma = \rho.
\end{aligned}$$

Consequently,

$$\|qaa^*q\| \geq \|qv^{2+\beta}q\| - \rho = \|v\|^{2+\beta} - \rho > (\|a\|^{2\alpha} - \sigma)^{2+\beta} - \rho > (\|a\| - \delta)^2,$$

by our choice of σ.

Proof of Lemma 7. Construct the projections q_i recursively. Suppose q_0, ..., q_{i-1} have already been chosen. To pick q_i, find $x \in \mathbf{B}(X^{[q_{i-1}]})$ so that $\|T^{[q_{i-1}]}x\| > C_i + 2\varepsilon$. By Proposition 8, there exists a projection $q < q_{i-1}$ so that

$$\min\left\{\|q[T^{[q_{i-1}]}x]\|, \|(q_{i-1} - q)[T^{[q_{i-1}]}x]\|\right\} > C_i + \varepsilon.$$

Let $y = qx$ and $z = (q_{i-1} - q)x$. Then $\|q^\perp[Ty]\| \leq \varepsilon$, hence $\|T^{[q]}y\| > C_i$. Similarly, $\|T^{[q_{i-1}-q]}z\| > C_i$. Thus, both $T^{[q]}$ and $T^{[q_{i-1}-q]}$ have norms exceeding C_i. Moreover, at least one of them is unbounded. Thus, either $q_i = q$ or $q_i = q_{i-1} - q$ works for us.

Remark 2. Suppose K is a compact Hausdorff space. It is known (see e.g. [7, Section V.3.2]) that $C(K)$ has real rank 0 iff K is zero-dimensional. In this case, $T : C(K) \to C(K)$ is ε-RAP (or, equivalently, ε-LAP) if and only if

$$\text{for any } k \in K, \text{ we have } |[Tx](k)| \wedge |y(k)| \leq \varepsilon\|x\| \tag{1}$$

whenever $|x| \wedge |y| = 0$.

Indeed, suppose first that T is ε-RAP, $|x| \wedge |y| = 0$, and prove that (1) holds. This is definitely true if $|y(k)| < \varepsilon\|x\|$. Otherwise, let $A = \overline{\{t \in K : x(t) \neq 0\}}$ and $B = \{t \in K : |y(t)| \geq \varepsilon\|x\|/2\}$. The sets A and B are closed, and disjoint (due to the continuity of x and y). By [37, Proposition 8.2.2], there exists a clopen set $C \subset K$, containing B and disjoint from A. In other words, there exists $q \in \mathbf{P}(\mathcal{B})$ so that $qx = 0$, and $q(k) = 1$. Then $|[Tx](k)| \leq \|q[Tx]\| \leq \varepsilon\|x\|$, establishing (1).

Conversely, suppose (1) holds. If $qx = 0$, then $q \wedge |x| = 0$, and

$$\|q[Tx]\| = \max_{k \in K} |[Tx](k)| \wedge |q(k)| \leq \varepsilon\|x\|,$$

which means that T is ε-RAP.

In the terminology of [34], (1) shows that T is ε-RAP iff T is ε-band preserving. By [34, Section 3], such a T is bounded, and moreover, there exists $\phi \in C(K)$ so that $\|T - \mathbf{M}_\phi\| \leq \varepsilon$.

2.2. Approximation by right multiplication

Above, we have shown that, in certain situations, ε-RAP maps are necessarily continuous. Here, we show that sometimes, they are actually close to multiplication maps.

Proposition 9. *Suppose $\mathcal{A}$ is a unital C^*-algebra of real rank 0, Y is a left $\mathcal{A}$-module, and $T \in B(\mathcal{A}, Y)$ is ε-RAP. Then $\|T - \mathbf{R}_a\| \leq 8\varepsilon$, where $a = T\mathbf{1}$.*

Recall that Theorem 2 and Proposition 5 give us several cases of automatic continuity of ε-RAP maps $T : \mathcal{A} \to Y$.

PROOF. First show that, for $p \in \mathbf{P}(\mathcal{A})$, we have $\|Tp - pa\| \leq 2\varepsilon$. We have $\mathbf{1} = p + p^\perp$, hence $a = Tp + Tp^\perp$. Multiplying by p on the left, we obtain $pa = p[Tp] + p[Tp^\perp]$, hence $\|pa - p[Tp]\| \leq \|p[Tp^\perp]\| \leq \varepsilon$ (we use the fact that T is ε-RAP). Further, $\|p^\perp[Tp]\| \leq \varepsilon$, hence $\|Tp - pa\| \leq \|pa - p[Tp]\| + \|p^\perp[Tp]\| \leq 2\varepsilon$.

Consequently, $\|Tu - ua\| \leq 4\varepsilon$ whenever u is a self-adjoint unitary. Approximating by elements with finite spectrum, and using an extreme point argument, we observe that $\|Tx - xa\| \leq 4\varepsilon\|x\|$ for any $x \in \mathcal{A}_{sa}$. Use polarization to complete the proof.

Corollary 10. *Suppose $\mathcal{A}$ is a von Neumann algebra, and $T : \mathcal{A}_* \to \mathcal{A}_*$ is ε-RAP. Then there exists $a \in \mathcal{A}$ so that $\|T - \mathbf{R}_a\| \leq 8\varepsilon$.*

PROOF. By Theorem 2, T is continuous. Show first that T^* is ε-LAP. Consider $x \in \mathcal{A}$ and $p \in \mathbf{P}(\mathcal{A})$ with $xp = 0$ (that is, $x = xp^\perp$), and prove that $\|[T^*x]p\| \leq \varepsilon\|x\|$. By duality,

$$\begin{aligned}\|[T^*x]p\| &= \sup\Big\{\big|\langle [T^*x]p, y\rangle\big| : y \in \mathcal{A}_*, \|y\| \leq 1\Big\} \\ &= \sup\Big\{\big|\langle x, T(py)\rangle\big| : y \in \mathcal{A}_*, \|y\| \leq 1\Big\},\end{aligned}$$

hence, as $x = xp^\perp$,

$$\begin{aligned}\|[T^*x]p\| &= \sup\Big\{\big|\langle xp^\perp, T(py)\rangle\big| : y \in \mathcal{A}_*, \|y\| \leq 1\Big\} \\ &= \sup\Big\{\big|\langle x, p^\perp T(py)\rangle\big| : y \in \mathcal{A}_*, \|y\| \leq 1\Big\}.\end{aligned}$$

As T is ε-RAP, we have $\|p^\perp T(py)\| \leq \varepsilon\|y\| \leq \varepsilon$.

Applying Proposition 9 to the ε-LAP map T^*, we obtain $a \in \mathcal{A}$ so that $\|T^* - \mathbf{L}_a\| \leq 8\varepsilon$. By duality, $\|T - \mathbf{R}_a\| \leq 8\varepsilon$.

Now suppose $\mathcal{E}$ is a symmetric separable sequence space. For Hilbert spaces H and K, we consider the Schatten space $\mathcal{S}_\mathcal{E}(H, K)$, consisting of all compact operators $x : H \to K$ whose sequence of singular numbers $\mathbf{s}(x)$ belongs to $\mathcal{E}$, with $\|x\|_{\mathcal{S}_\mathcal{E}} = \|\mathbf{s}(x)\|_\mathcal{E}$ (see e.g. [19], [39] for more information). For future use, note that, for any $(t_1, t_2, \ldots) \in \mathcal{E}$, we have $\lim_n \|(0, \ldots, 0, t_n, t_{n+1}, \ldots)\|_\mathcal{E} = 0$. Consequently, the sequence space $\mathcal{E}$ is *order continuous* (see [14] for the definition); this property passes to $\mathcal{S}_\mathcal{E}(H, K)$, hence, by [10], for any $x \in \mathcal{S}_\mathcal{E}(H, K)$, and any net of projections (p_α) which decreases to 0, we have $\lim_\alpha \|p_\alpha x\|_\mathcal{E} = 0$.

A linear map $T : \mathcal{S}_{\mathcal{E}_1}(H, K) \to \mathcal{S}_{\mathcal{E}_2}(H, K)$ is ε-RAP if, for any projection p acting on K, and any $x \in \mathcal{S}_{\mathcal{E}_1}(H, K)$ with $px = 0$, we have $\|p[Tx]\| \leq \varepsilon\|x\|$. If $v \in B(H)$, then the right multiplication operator $\mathbf{R}_v : x \mapsto xv$ is RAP (that is, 0-RAP). We prove that every ε-RAP bounded operator can be approximated by a right multiplication operator. Consider first the finite dimensional case.

Proposition 11. *Suppose $T : \mathcal{S}_{\mathcal{E}_1}(\ell_2^n, \ell_2^m) \to \mathcal{S}_{\mathcal{E}_2}(\ell_2^n, \ell_2^m)$ is ε-RAP. Then there exists $v \in M_n$ so that $\|T - \mathbf{R}_v\| \leq c\varepsilon$. Here $c = 4(1 - m^{-2})$ if m is even, and $c = 4$ if m is odd.*

We use the shorthand notation $\mathcal{S}_{\mathcal{E}}(\ell_2^n, \ell_2^m) = \mathcal{E}^{mn}$; the elements of this space can be viewed as $m \times n$ matrices. For $1 \leq i \leq m$ and $1 \leq j \leq n$, E_{ij} is a matrix unit – the matrix with 1 in the (i, j) position, and zero entries elsewhere. Further, we sometimes identify M_{mn} (or $\mathcal{E}^{mn}$) with the space $\mathbb{C}^{mn}$: a matrix $x = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ can be interpreted as a vector $\tilde{x}$ with entries $\tilde{x}_{(ij)} = x_{ij}$. In this setting, any $T \in B(\mathcal{E}_1^{mn}, \mathcal{E}_2^{mn})$ is identified (algebraically) with an $mn \times mn$ matrix, acting on $\mathbb{C}^{mn}$, and which we denote by $\widetilde{T}$. One can check that, for $a \in M_m$ ($b \in M_n$), $\widetilde{\mathbf{L}_a} = a \otimes I_n$ (resp. $\widetilde{\mathbf{R}_b} = I_m \otimes b$).

Proof. Let $\mathfrak{U}_m$ be the group of unitaries on ℓ_2^m, equipped with the normalized Haar measure μ. Consider the map $\pi : \mathfrak{U}_m \to B(\mathcal{E}_1^{mn}, \mathcal{E}_2^{mn})$ defined via $\pi(u) : T \mapsto \mathbf{L}_u T \mathbf{L}_{u^*}$ (that is, for $x \in \mathcal{E}_1^{mn}$, $[\pi(u)T]x = u[T(u^* x)]$, or $\widetilde{\pi(u)T} = (u \otimes I_n)\widetilde{T}(u^* \otimes I_n)$). Further, define

$$\Phi : B(\mathcal{E}_1^{mn}, \mathcal{E}_2^{mn}) \to B(\mathcal{E}_1^{mn}, \mathcal{E}_2^{mn}) : T \mapsto \int \pi(u)T \, d\mu(u).$$

Clearly $\pi(u)$ is an isometry for every u, hence Φ is a contraction. Note that

$$\widetilde{\Phi(T)} = \int (u \otimes I_n)\widetilde{T}(u^* \otimes I_n) \, d\mu(u) = (\phi \otimes I_n)(\widetilde{T}),$$

where, for $x \in M_m$, $\phi(x) = \int uxu^* \, d\mu$. By the proof of [33, Lemma 4.6], $\phi(x) = m^{-1}\mathrm{tr}(x)$. Thus, the map $\widetilde{T} \mapsto \widetilde{\Phi(T)}$ is a projection onto $\mathbf{1} \otimes M_n$ (here $\mathbf{1}$ is the identity on M_m). By the paragraph preceding the proof, $\mathbf{1} \otimes M_n = \{\widetilde{\mathbf{R}_b} : b \in M_n\}$. Therefore, Φ is a contractive projection from $B(\mathcal{E}_1^{mn}, \mathcal{E}_2^{mn})$ onto its subspace $\{\mathbf{R}_b : b \in M_n\}$.

Observe that T is ε-RAP if and only if, for any projection p acting on ℓ_2^m, we have $\|\mathbf{L}_{p^\perp} T \mathbf{L}_p\| \leq \varepsilon$. Now let $k = \lfloor m/2 \rfloor$, and denote by ν the normalized uniform measure of the Grassman manifold $\mathbf{G}_{n,k}$. Let $\Psi(T) = \int \mathbf{L}_{p^\perp} T \mathbf{L}_p \, d\nu$. By the triangle inequality, $\|\Psi(T)\| \leq \varepsilon$, whenever T is ε-RAP. However,

$$\widetilde{\Psi(T)} = \int \widetilde{\mathbf{L}_{p^\perp}} \widetilde{T} \widetilde{\mathbf{L}_p} \, d\nu(p) = \int (p^\perp \otimes I_n)\widetilde{T}(p \otimes I_n) \, d\nu(p) = (\psi \otimes I_n)(\widetilde{T}),$$

where, for $x \in M_m$, $\psi(x) = \int pxp^\perp \, d\nu(p)$. By [33, Lemma 4.6] and its proof, $\psi(I) = 0$, and $\psi(x) = cx$ when $\mathrm{tr}x = 0$ ($c = k(m-k)/(m^2-1)$). Consequently, $\Psi|_{\mathrm{ran}\,\Phi} = 0$, and $\Psi|_{\ker \Phi} = cI_{\ker \Phi}$. In other words, $\Psi = c(I - \Phi)$.

Now suppose T is ε-RAP. By the above, $\|\Psi(T)\| \leq \varepsilon$. Write $T = \Phi(T) + c^{-1}\Psi(T)$. There exists $b \in M_n$ so that $\Phi(T) = \mathbf{R}_b$, and $\|T - \mathbf{R}_b\| \leq c^{-1}\varepsilon$.

Theorem 12. *Suppose H and K are Hilbert spaces, and $\mathcal{E}_1, \mathcal{E}_2$ are separable symmetric sequence spaces.*

1. *If K is infinite dimensional, and $T : \mathcal{S}_{\mathcal{E}_1}(H,K) \to \mathcal{S}_{\mathcal{E}_2}(H,K)$ is ε-RAP, then T is bounded.*
2. *If $T \in B(\mathcal{S}_{\mathcal{E}_1}(H,K), \mathcal{S}_{\mathcal{E}_2}(H,K))$ is ε-RAP, then there exists $v \in B(H)$ so that $\|T - \mathbf{R}_v\| \leq 5\varepsilon$.*

By (1), in (2) the boundedness of T is automatic unless $\dim H = \infty > \dim K$.

Remark 3. By Proposition 11, the result also holds when H and K are both finite dimensional. However, if H is infinite dimensional, and K is finite dimensional, then a RAP map needn't be bounded: consider, for instance, the map $T : x \mapsto xv$, where $v : H \to H$ is an unbounded operator.

Remark 4. Suppose, for simplicity, that both H and K are infinite dimensional. It is known (see e.g. [5]) that, for any x, $\mathbf{s}(xv) \prec\!\!\prec \mathbf{s}(x)\mathbf{s}(v)$, where, as before, $\prec\!\!\prec$ denotes the Hardy-Littlewood-Polya submajorization. On the other hand, considering diagonal operators, one can check that $\mathbf{R}_v$ defines a bounded operator $\mathcal{S}_{\mathcal{E}_1}(H,K) \to \mathcal{S}_{\mathcal{E}_2}(H,K)$ iff $\mathbf{s}(x)\mathbf{s}(v) \in \mathcal{E}_2$ whenever $\mathbf{s}(x) \in \mathcal{E}_1$.

Proof of Theorem 12. Rescaling if necessary, we assume that $\|(1,0,\ldots)\|_{\mathcal{E}_1} = \|(1,0,\ldots)\|_{\mathcal{E}_2} = 1$. Denote by $\mathcal{F}(H)$ and $\mathcal{F}(K)$ the nets of finite dimensional subspaces of H and K respectively, ordered by inclusion. For $E \in \mathcal{F}(H)$ and $F \in \mathcal{F}(K)$, denote the corresponding orthogonal projections by p_E and p_F respectively. Define $T_{EF} \in B(\mathcal{S}_{\mathcal{E}_1}(E,F), \mathcal{S}_{\mathcal{E}_1}(E,F))$ via $T_{EF}x = p_F[Tx]p_E$ ($x \in \mathcal{S}_{\mathcal{E}_1}(E,F)$). Clearly T_{EF} is ε-RAP. By Proposition 11, there exists $v_{EF} \in B(E)$ so that $\|T_{EF} - \mathbf{R}_{v_{EF}}\| \leq 4\varepsilon$. Note that, if $x = p_F x p_E$, then $\|p_F^\perp[Tx]\| \leq \varepsilon\|x\|$, hence

$$\|T|_{\mathcal{S}_{\mathcal{E}_1}(E,F)} - \mathbf{R}_{v_{EF}}\| \leq 5\varepsilon. \tag{2}$$

We shall find an "accumulation point" v of the operators v_{EF}, and prove that $\mathbf{R}_v$ is "close to" T.

For $E \in \mathcal{F}(H)$ and $F \in \mathcal{F}(K)$, and $w \in B(E)$, define

$$|\!|\!| w |\!|\!|_F = \|\mathbf{R}_w : \mathcal{S}_{\mathcal{E}_1}(E,F) \to \mathcal{S}_{\mathcal{E}_2}(E,F)\| \text{ and } |\!|\!| w |\!|\!| = \sup_F |\!|\!| w |\!|\!|_F.$$

It is easy to see that $\|w\| \leq |\!|\!| w |\!|\!| \leq \dim E\,\|w\|$. Further, $|\!|\!| w |\!|\!| = |\!|\!| w |\!|\!|_F$ whenever $\dim F \geq \dim E$.

Consider the more difficult case – that of $\dim K = \infty$ (T is not assumed to be bounded). Let $\mathcal{I}$ be the net (ordered by inclusion $\prec$) of all pairs $(E,F) \in \mathcal{F}(H) \times \mathcal{F}(K)$ so that $\dim F \geq \dim E$. If $(E,F), (E',F') \in \mathcal{I}$ satisfy $(E,F) \prec (E',F')$, then

$$|\!|\!| v_{EF} - v_{E'F'}|_E |\!|\!| \leq \|T|_{\mathcal{S}_{\mathcal{E}_1}(E,F)} - \mathbf{R}_{v_{EF}}\| + \|T|_{\mathcal{S}_{\mathcal{E}_1}(E',F')} - \mathbf{R}_{v_{E'F'}}\| \leq 10\varepsilon.$$

Now suppose F_1, F_2, E satisfy $\min\{\dim F_1, \dim F_2\} \geq \dim E$. Letting $F = F_1 + F_2$, we get

$$|\!|\!| v_{EF_1} - v_{EF_2} |\!|\!| \leq |\!|\!| v_{EF_1} - v_{EF} |\!|\!| + |\!|\!| v_{EF} - v_{EF_2} |\!|\!| \leq 20\varepsilon.$$

Show that $\sup_{(E,F)\in\mathcal{I}} \|v_{EF}\| < \infty$. Indeed, suppose, for the sake of contradiction, that $\sup_{(E,F)\in\mathcal{I}} \|v_{EF}\| = \infty$. Then there exists a sequence of pairs $(E_n, F_n) \in \mathcal{I}$ so that, for any n, $(E_n, F_n) \prec (E_{n+1}, F_{n+1})$, and $(4^n + 1 + 20\varepsilon)\|v_{E_n F_n}\| < \|v_{E_{n+1} F_{n+1}}\|$ (we further assume $\|v_{E_1 F_1}\| > 4$). Let $G_1 = E_1$, and $G_{n+1} = E_{n+1} \ominus E_n$ for $n \geq 1$. Let $u_n = v|_{G_n}$. Then

$$\|v_{E_{n+1}F_{n+1}}\| \leq \|v_{E_{n+1}F_{n+1}}|_{E_n}\| + \|u_n\| \leq \|v_{E_n F_n}\| + 20\varepsilon + \|u_n\|,$$

hence $\|u_n\| > 4^n \|v_{E_n F_n}\| > 4^n$.

Find mutually orthogonal subspaces $G'_n \subset K$, so that $\dim G'_n = \dim G_n$. Find $x_n \in \mathcal{S}_{\mathcal{E}_1}(G, G')$ so that $\|x_n\|_{\mathcal{E}_1} = 1$, and $\|x_n u_n\|_{\mathcal{E}_2} > 4^n + 21\varepsilon$. Let $x = \sum_n 2^{-n} x_n$. For $k \in \mathbb{N}$ let $y_k = x - 2^{-k} x_k = \sum_{n \neq k} 2^{-n} x_n$. Denoting by p_k the projection onto G'_k, we have

$$\|Tx\| \geq \|p_k[Tx]\| \geq \|T(2^{-k} x_k)\| - \|p_k[Ty_k]\| \geq 2^{-k}\|x_k u_k\| - 20\varepsilon - \varepsilon > 2^k.$$

This inequality holds for any k, yielding the desired contradiction.

Consequently, the restriction of T to the set of finite rank operators in $\mathcal{S}_{\mathcal{E}_1}(H, K)$ is bounded. To show that T is bounded, consider an infinite rank $x \in \mathcal{S}_{\mathcal{E}_1}(H, K)$ with singular value decomposition $x = \sum_{j=1}^\infty \lambda_j \xi_j \otimes \eta_j$, where (ξ_j) and (η_j) are orthonormal systems in K and H respectively, and $\lambda_j > 0$ are the singular numbers of x. For $k \geq 1$, set $x_k = \sum_{j=k+1}^\infty \lambda_j \xi_j \otimes \eta_j$, $y_k = \sum_{j=1}^k \lambda_j \xi_j \otimes \eta_j = x - x_k$, and let p_k be the orthogonal projection onto $\mathrm{span}[\xi_j : j \geq k+1]$.

Fix $c > 0$, and show that $\|Tx_k\| < c$ for k large enough. Clearly, $\lim_k \|x_k\| = 0$. As $\|p_k^\perp[Tx_k]\| \leq \varepsilon\|x_k\|$, we conclude that $\|p_k^\perp[Tx_k]\| < c/2$ when k is sufficiently large. To estimate $\|p_k[Tx_k]\|$, note that the sequence (p_k) converges to 0 in the strong operator topology, hence $\lim_k \|p_k[Tx]\| = 0$. Find $N \in \mathbb{N}$ so that $\|x_N\| < c/(8\varepsilon)$; then find $K \in \mathbb{N}$ so that $\max\{\|p_K[Tx]\|, \|p_K[Ty_N]\|\} < c/8$. If $k \geq \max\{K, N\}$, then

$$\|p_k[Ty_k]\| \leq \|p_k[Ty_N]\| + \|p_k[T(y_k - y_N)]\| \leq \frac{c}{8} + \varepsilon\|y_k - y_N\| \leq \frac{c}{8} + \varepsilon\|x_N\| < \frac{c}{4},$$

hence

$$\|p_k[Tx_k]\| \leq \|p_K[Tx]\| + \|p_k[Ty_k]\| \leq \frac{3}{8}c.$$

Thus, for k large enough, $\|Tx_k\| \leq \|p_k[Tx_k]\| + \|p_k^\perp[Tx_k]\| < c$.

It remains to show that T is "close to" a multiplication operator. Abusing the notation, we identify $v_{EF} : E \to E$ with the operator $J_F v_{EF} P_E : H \to K$, where $P_E : H \to E$ is the orthogonal projection, and $J_F : F \to K$ is the injection. The net $(v_{EF})_{(E,F)\in\mathcal{I}}$ is bounded in $B(H, K)$ in the operator norm, hence it has a subnet (call it $\mathcal{I}'$) which converges weak* to some $v \in B(H)$. By the continuity of T (established above), the inequality $\|T - \mathbf{R}_v\| \leq 5\varepsilon$ will follow once we establish that, for any finite rank $x \in \mathbf{B}(\mathcal{S}_{\mathcal{E}_1}(H, K))$, we have $\|Tx - xv\| < 5\varepsilon$. Note that, for any $\delta > 0$, we can find a finite rank $y \in \mathbf{B}(\mathcal{S}_{\mathcal{E}_2^*}(K, H))$ so that $\|Tx - xv\| < \mathrm{tr}(y(Tx - xv)) + \delta$. Let $\mathcal{I}_x$ be the subnet of $\mathcal{I}'$ consisting of all pairs (E, F) for which $(\ker x)^\perp \subset E$ and $\operatorname{ran} x \subset F$ (clearly $\mathcal{I}_x$ is cofinal with $\mathcal{I}$). As v is an accumulation point of the net $(v_{EF})_{(E,F)\in\mathcal{I}_x}$, $\operatorname{tr}(y(Tx - xv))$ is an accumulation point of $(\operatorname{tr}(y(Tx - xv_{EF})))_{(E,F)\in\mathcal{I}_x}$. However,

$$\left|\operatorname{tr}(y(Tx - xv_{EF}))\right| \le \|Tx - xv_{EF}\| \le 5\varepsilon,$$

which leads to $\|Tx - xv\| < 5\varepsilon + \delta$. As $\delta > 0$ is arbitrary, we are done.

The case when $\dim K < \infty$, $\dim H = \infty$, and T is bounded, is easier, and can be handled similarly. We outline the argument. As before, the inequality

$$\|v_{EF}\| \le \|v_{EF}\|_F \le \|T\| + 5\varepsilon$$

holds for any E and F. Let $\mathcal{I}$ be the net (ordered by inclusion $\prec$) of all pairs $(E, F) \in \mathcal{F}(H) \times \mathcal{F}(K)$. There is a subnet converging weak* to some $v \in B(H)$. As before, we can show that $\|T - \mathbf{R}_v\| \le 5\varepsilon$.

3. Band-preserving operators on non-commutative function spaces

Motivated by the theory of Banach lattices, we introduce the class of band-preserving operators. Recall that an operator T on a Banach lattice X is called *band-preserving* if, for every disjoint $x, y \in X$, Tx is disjoint from y. Band-preserving operators on Banach lattices have been studied extensively, see e.g. [32, Section 3.1]. For instance, such operators are automatically continuous, and resemble (in a certain sense) multiplication operators.

If X is a Köthe function space (such as an L_p space) on a σ-finite measure space (Ω, Σ, μ), then $T : X \to X$ is band-preserving iff, for every $S \in \Sigma$, and every x, we have $[Tx]\chi_S = 0$ whenever $x\chi_S = 0$. To reformulate this in terms of von Neumann algebras and their modules, observe that X is an $L_\infty(\mu)$-bimodule, and χ_S is a projection in $L_\infty(\mu)$. Using this as a template, we introduce the following definition:

Definition 2. Suppose X and Y are normed $\mathcal{A}$-bimodules. We say that $T : X \to Y$ is *band preserving* (*BP* for short) if, for any $p \in \mathbf{P}(\mathcal{A})$, we have $Tx = p[Tx]p$ whenever $x = pxp$.

Roughly, a BP map preserves "upper-left square corners".

If Y is an $\mathcal{A}$-bimodule, then, for any $a \in \mathfrak{Z}(\mathcal{A})$, the multiplication operator $\mathbf{M}_a : Y \to Y$ is band-preserving. We prove that, under rather mild condition, any band-preserving operator equals $\mathbf{M}_a$, for some $a \in \mathfrak{Z}(\mathcal{A})$.

Theorem 13. *Suppose $\mathcal{A}$ is a von Neumann algebra, and one of the two possibilities holds:*

1. *Y is a von Neumann algebra containing $\mathcal{A}$, so that $\mathcal{A}' \cap Y = \mathfrak{Z}(\mathcal{A})$.*

2. *Y is a proper strongly symmetric space on $(\mathcal{A}, \tau)$ with Fatou norm, where τ is a normal faithful semi-finite trace on $\mathcal{A}$.*

Then a linear map $T : Y \to Y$ is band-preserving if and only if $T = \mathbf{M}_a$, for some $a \in \mathfrak{Z}(\mathcal{A})$. Moreover, T is bounded, with $\|T\| = \|a\|$.

Note that, in Case 1, we have to impose some restrictions on the position of $\mathcal{A}$ inside of Y. For instance, if $\mathcal{A} = \mathbb{C}\mathbf{1} \subset Y = B(H)$ (H is a Hilbert space). Then clearly any map from $\mathcal{A}$ to Y is BP.

It is known [32, Section 3.1] that any band-preserving operator on a $C(K)$ space is implemented by multiplication by an element of $C(K)$. Furthermore ([34], cf. [43]), any band preserving operator on a Köthe space $\mathcal{E}(\mu)$ is given by multiplication by a member of $L_\infty(\mu)$.

Clearly, if $a \in \mathfrak{Z}(\mathcal{A})$, then $\mathbf{M}_a$ is band-preserving, and has norm $\|a\|$. We need to prove the converse.

Proof of Theorem 13. Case 1: $\mathcal{A}$ is a von Neumann subalgebra of Y, $\mathcal{A}' \cap Y = \mathfrak{Z}(\mathcal{A})$.
For a band-preserving $T : \mathcal{A} \to \mathcal{A}$, let $a = T\mathbf{1}$. We establish that $a \in \mathfrak{Z}(\mathcal{A})$, and $T = \mathbf{M}_a$. Pick $p \in \mathbf{P}(\mathcal{A})$, and let $p^\perp = \mathbf{1} - p$. Then $a = T(p + p^\perp) = Tp + Tp^\perp$, hence $ap = (Tp)p + (Tp^\perp)p = Tp$, and similarly, $pa = Tp$. Thus, $ap = pa$. By the density of projections, $a \in \mathcal{A}' \cap Y = \mathfrak{Z}(\mathcal{A})$.

Next show that T maps $\mathfrak{Z}(\mathcal{A})$ into itself. For $x \in \mathfrak{Z}(\mathcal{A})$ and $p \in \mathcal{A}$, write $x = px + p^\perp x$. Then $T(px) \in pYp$, and $T(p^\perp x) \in p^\perp Y p^\perp$. Consequently, $p(Tx) = p(T(px)) = (T(px))p = (Tx)p$. Thus, $Tx \in \mathcal{A}' \cap Y$. By our assumption, $\mathcal{A}' \cap Y = \mathfrak{Z}(\mathcal{A})$.

We know that $\mathfrak{Z}(\mathcal{A})$ is a dual $C(K)$ space. In particular, $\mathfrak{Z}(\mathcal{A})$ is Dedekind complete as a Banach lattice, hence (see [32, Section 1.2]) it has the Principal Projection Property. By the above, the restriction of T to $\mathfrak{Z}(\mathcal{A})$ commutes with any projection in $\mathfrak{Z}(\mathcal{A})$, hence, by [32, Proposition 3.1.3], $T : \mathfrak{Z}(\mathcal{A}) \to \mathfrak{Z}(\mathcal{A})$ is band preserving in the lattice sense. Thus, by [32, Section 3.1], $Tx = ax$ for any $x \in \mathfrak{Z}(\mathcal{A})$ (in particular, $T|_{\mathfrak{Z}(\mathcal{A})}$ is bounded).

Now fix $p \in \mathbf{P}(\mathcal{A})$, and consider T as an operator from $\mathcal{A}_p = p\mathcal{A}p$ to $Y_p = pYp$. Denote by H the Hilbert space on which the algebras $\mathcal{A}$ and Y are acting, let $H_p = p(H)$, and view $\mathcal{A}_p$ and Y_p as acting on H_p. By [25, Proposition 5.5.6], $\mathcal{A}_p' = p\mathcal{A}'p$. Applying the previous paragraph to $T|_{\mathcal{A}_p}$, we conclude that, for any $x \in \mathfrak{Z}(\mathcal{A})$, $T(px) = apx$.

To finish the proof, recall [20] that any element of $\mathcal{A}$ is a finite linear combination of terms of the form pz ($p \in \mathbf{P}(\mathcal{A})$ and $z \in \mathfrak{Z}(\mathcal{A})$).

Proof of Theorem 13. Case 2: Y is a non-commutative function space.
Suppose first τ is finite. Let $a = T\mathbf{1}$. Then $Tp = pap$ for any $p \in \mathbf{P}(\mathcal{A})$. Further, $a = pap + p^\perp a p^\perp$, hence a commutes with any p.

Next show that $a \in \mathcal{A}$. Indeed, otherwise $|a| = (a^*a)^{1/2}$ has mutually disjoint non-zero spectral projections $p_1, p_2, \ldots$ so that $|a|p_k \geq 4^k p_k$. Let $\gamma_k = 2^{-k}/\|p_k\|_Y$, and consider $x = \sum_{k=1}^\infty \gamma_k p_k$ (the sum converges in Y). Then, for any n, $Tp_n = p_n a p_n = a p_n$, hence

$$p_n[Tx]p_n = p_n\big[T(\gamma_n p_n)\big]p_n + p_n\big[T(\sum_{k\neq n} \gamma_k p_k)\big]p_n = \gamma_n a p_n,$$

hence $\|Tx\|_Y \geq \gamma_n \|ap_n\|_Y \geq 2^n$. Consequently, $\|Tx\|_Y \geq 2^n$ for any n, which is impossible.

By the above, $a \in \mathcal{A}$ commutes with all projections, hence it belongs to $\mathfrak{Z}(\mathcal{A})$. Applying [20] as in Case 1, we conclude that, for $x \in \mathcal{A}$, $Tx = ax$. Now consider a self-adjoint $x \in Y \backslash \mathcal{A}$. The spectral resolution of x yields a decreasing sequence of projections (p_n) so that $\lim_n \tau(p_n) = 0$, and $p_n^\perp x = x p_n^\perp \in \mathcal{A}$ for every n. Then $T(xp_n^\perp) = axp_n^\perp$, and $T(xp_n) = p_n[T(xp_n)]p_n$. Therefore, $[Tx - ax]p_n^\perp = 0$. As $p_n^\perp \nearrow \mathbf{1}$, we conclude that $Tx = ax$.

Next handle the general case. For every $p \in \mathbf{P}_\tau(\mathcal{A})$, let $a_p = Tp$. By the above, $a_p \in \mathfrak{Z}(p\mathcal{A}p) = p\mathfrak{Z}(\mathcal{A})$, and $Tx = a_p x$ for every $x \in pYp$. If $q \geq p$, then $a_p = pa_q = a_q p$. We claim that $\sup_p \|a_p\|_{\mathcal{A}} < \infty$. Indeed, otherwise one can find $p_1 < p_2 < \ldots$ in $\mathbf{P}_\tau(\mathcal{A})$ so that $\|a_{p_1}\| > 1$, and $\|a_{p_{k+1}}\| > 4\|a_{p_k}\|$ for every k. Note that $a_{p_{k+1}} = a_{p_k} + (p_{k+1} - p_k)a_{p_{k+1}}(p_{k+1} - p_k)$. As $a_{p_{k+1}}$ commutes with $p_{k+1} - p_k$, the spectral resolution gives us a non-zero projection $r_k \leq p_{k+1} - p_k$ so that $|a_{p_{k+1}}|r_k = r_k|a_{p_{k+1}}| \geq 4^k r_k$. Let $\gamma_k = 2^{-k}/\|r_k\|_Y$, and consider $x = \sum_{k=1}^\infty \gamma_k r_k$. Then, for any n, $r_n[Tx]r_n = \gamma_k a_{p_{k+1}} r_k$, hence $\|Tx\| \geq \|r_n[Tx]r_n\| \geq 2^n$, which is impossible.

We view $\mathbf{P}_\tau(\mathcal{A})$ as a net, ordered by inclusion. By Goldstine Theorem, there exists a subnet $\mathcal{U} \subset \mathbf{P}_\tau(\mathcal{A})$ so that $\sigma(\mathcal{A}, \mathcal{A}_*) - \lim_{p \in \mathcal{U}} a_p = a$. To show that $\sigma(\mathcal{A}, \mathcal{A}_*) - \lim_{p \in \mathbf{P}_\tau(\mathcal{A})} a_p = a$, fix $b \in \mathcal{A}_+$ so that the support projection $s(b)$ has finite trace. Then for every $\varepsilon > 0$ there exists $p_0 \in \mathcal{U}$ so that, whenever $q \in \mathcal{U}$ satisfies $q \geq p_0$, we have $|\tau(b(a_q - a))| < \varepsilon$. Find $p_1 \in \mathcal{U}$ with $p_1 \geq p_0 \vee s(b)$. We show that, for any $p \in \mathbf{P}_\tau(\mathcal{A})$ with $p \geq p_1$, we have $|\tau(b(a_p - a))| < \varepsilon$. Indeed, $p_1 b p_1 = b$ and $p_1 a_p p_1 = a_{p_1}$, hence

$$\tau(b(a_p - a)) = \tau(p_1 b p_1 (a_p - a)) = \tau(p_1 b p_1 (a_p - a) p_1) = \tau(b(a_{p_1} - a)),$$

and therefore, $|\tau(b(a_p - a))| = |\tau(b(a_{p_1} - a))| < \varepsilon$. The linear span of b's as above is dense in $\mathcal{A}_*$, hence

$$\lim_{p \in \mathbf{P}_\tau(\mathcal{A})} \tau(x a_p) = \tau(xa) \text{ for every } x \in \mathcal{A}_*.$$

As $a_p = p a_q p$ for $q \geq p$, we have $\|a\| = \sup_{p \in \mathbf{P}_\tau(\mathcal{A})} \|a_p\|$. To show that $a \in \mathfrak{Z}(\mathcal{A})$, it suffices to prove that $ap = pa$ for any $p \in \mathbf{P}(\mathcal{A})$. We already know this holds when $\tau(p) < \infty$. If $\tau(p) = \infty$, consider $\mathbf{P}_\tau(p\mathcal{A}p) = \{q \in \mathbf{P}_\tau(\mathcal{A}) : q \leq p\}$ as a net, ordered by inclusion. This net converges weak* to its supremum – which is p, by the semi-finiteness of τ. As the commutant of a is weak* closed, we conclude that a commutes with p. Thus, $a \in \mathfrak{Z}(\mathcal{A})$.

It remains to prove that, for any positive $x \in Y$, $Tx = ax$. Suppose first that $x = pxp$, for some $p \in \mathbf{P}_\tau(\mathcal{A})$. Then, for any $q \in \mathbf{P}_\tau(\mathcal{A})$ with $q \geq p$, we have $T = a_q x$. For every y in the Köthe dual $Y^\times$, we have $xy \in \mathcal{A}_*$, hence

$$\tau\big((Tx)y\big) = \lim_{q \in \mathbf{P}_\tau(\mathcal{A})} \tau(a_q xy) = \tau(axy).$$

As $Y^\times$ norms Y (see e.g. [15, Lemma 30]), we conclude that $Tx = ax$.

By polarization, it suffices to show that $Tx = ax$ for any $x \in Y_+$. Assume the support projection of x (which we denote by p_∞) is infinite. Clearly $Tx =$

$p_\infty[Tx]p_\infty$. For any n, $p_n = \chi_{[1/n,\infty)}(x)$ has finite trace. As $p_n \nearrow p_\infty$, Lemma 1 shows that $\|Tx - ax\| = \lim_n \|p_n[Tx - ax]p_n\|$. Write $x = xp_n + x(p_\infty - p_n)$, then

$$Tx = p_n[T(xp_n)]p_n + (p_\infty - p_n)[T(x(p_\infty - p_n))](p_\infty - p_n),$$

hence $p_n[Tx - ax]p_n = T(xp_n) - axp_n = 0$. This gives $Tx = ax$.

4. Almost band preserving operators

In this section we consider almost BP maps. The results are similar to those of the previous section; however, sometimes the assumptions are more restrictive (for instance, in Theorem 18, the von Neumann algebra is separably acting).

Definition 3. Suppose $\mathcal{A}$ is a von Neumann algebra, and X, Y are $\mathcal{A}$-bimodules. An operator $T : X \to Y$ is called *ε-Band Preserving* (*ε-BP* for short) if, for any $x \in X$, we have $\|Tx - p[Tx]p\| \le \varepsilon\|x\|$ whenever $p \in \mathbf{P}(\mathcal{A})$ satisfies $pxp = x$.

Clearly, any 0-BP operator is BP (in the sense of the preceding section).

The proof of Theorem 2 yields:

Proposition 14. *Suppose $\mathcal{A}, X, Y$ are as in Theorem 2. Then any ε-BP map from $X \to Y$ is continuous.*

Next we show that, in some situations, a ε-BP map is "near" a multiplication by an element of the center.

4.1. Almost BP operators on von Neumann algebras

Proposition 15. *Suppose $\mathcal{A}$ is a von Neumann algebra, and $T \in B(\mathcal{A})$ is ε-BP. Then there exists $a \in \mathfrak{Z}(\mathcal{A})$ so that $\|T - \mathbf{M}_a\| \le 16\varepsilon$.*

Proof. Let $b = T\mathbf{1}$. For any $p \in \mathbf{P}(\mathcal{A})$, we have $Tp + Tp^\perp = b$, $\|p(Tp)p - Tp\| \le \varepsilon$, and $\|p^\perp(Tp^\perp)p^\perp - Tp^\perp\| \le \varepsilon$. Note that $\|p(Tp)p^\perp\| \le \|p(Tp)p - Tp\| \le \varepsilon$, and similarly, $\|p(Tp^\perp)p^\perp\| \le \varepsilon$. By the triangle inequality, $\|pbp^\perp\| \le 2\varepsilon$. Thus, there exists $a \in \mathfrak{Z}(\mathcal{A})$ so that $\|a - b\| \le 2\varepsilon$ ($\mathrm{dist}(b, \mathfrak{Z}(\mathcal{A})) \le 2\varepsilon$, by [38, Theorem 3.6]; but there is $a \in \mathfrak{Z}(\mathcal{A})$ realizing this distance, see e.g. [18] and [44]).

We have to show $\|T - \mathbf{M}_a\| \le 16\varepsilon$. By an extreme point argument, it suffices to prove that $\|Tp - ap\| \le 4\varepsilon$ for any $p \in \mathbf{P}(\mathcal{A})$. To establish the last inequality, first invoke the triangle inequality:

$$\|Tp - ap\| \le \|p(Tp)p - pbp\| + \|Tp - p(Tp)p\| + \|a - b\| \le \|p(Tp)p - pbp\| + 3\varepsilon.$$

As $Tp + Tp^\perp = b$, we have

$$\|p(Tp) - pbp\| \le \|p(Tp^\perp)p\| \le \|p(Tp^\perp)p - p^\perp(Tp^\perp)p^\perp\| \le \varepsilon.$$

The following simple observation will be used below.

Lemma 16. *If p is a projection in a von Neumann algebra $\mathcal{A}$, and Y is an $\mathcal{A}$-bimodule, then, for any $y \in Y$, $\|pyp^\perp + p^\perp yp\|_Y \le \|y\|_Y$.*

Proof. Observe that

$$pyp^\perp + p^\perp yp = \frac{1}{2}\big(y - (p - p^\perp)y(p - p^\perp)\big),$$

and use the triangle inequality, and the unitary invariance of $\|\cdot\|_Y$.

In the next proposition, we consider ε-BP maps from $\mathcal{A}$ to a non-commutative function space. As before, we assume $\mathcal{A}$ is equipped with a normal faithful semifinite trace τ.

We shall say that a projection $p \in \mathfrak{Z}(\mathcal{A})$ is *τ-purely infinite* if any projection $q \in \mathfrak{Z}(\mathcal{A})$ with $0 < q \leq p$ must satisfy $\tau(q) = \infty$. By Zorn's Lemma, we can find a maximal family $(p_i)_{i \in I}$ of τ-purely infinite mutually orthogonal projections in $\mathfrak{Z}(\mathcal{A})$. Then $z_p = \sum_i p_i$ is τ-purely infinite (indeed, if z_p has a non-zero subprojection q with $\tau(q) < \infty$, then $p_i \wedge q$ is a non-zero projection of finite trace, for some i). Moreover, if $p \in \mathfrak{Z}(\mathcal{A})$ is a subprojection of $z_s = \mathbf{1} - z_p$, then there exists $q \in \mathfrak{Z}(\mathcal{A})$ so that $0 \leq q \leq p$,a dn $\tau(q) < \infty$. Write $\mathfrak{Z}(\mathcal{A}) = \mathcal{B}_s \oplus \mathcal{B}_p$, where $\mathcal{B}_s = z_s \mathfrak{Z}(\mathcal{A})$ and $B_p = z_p \mathfrak{Z}(\mathcal{A})$ (the subscripts s and p stand for "semifinite" and "purely infinite" respectively).

Note that, if τ is finite, then $\mathcal{B}_p = \{0\}$ and $z_p = 0$. Conversely, if every central projection has infinite trace, then $\mathcal{B}_s = \{0\}$ and $z_s = 0$.

Proposition 17. *Suppose the von Neumann algebra $\mathcal{A}$ is equipped with a normal faithful semifinite trace τ, and $\mathcal{E} = \mathcal{E}(\mathcal{A}, \tau)$ is a proper strongly symmetric space. Then, for any ε-BP map $T \in B(\mathcal{A}, \mathcal{E})$, there exists $a \in \mathcal{E}(\mathcal{B}_s)$ so that, for any $x \in \mathcal{A}$, $\|Tx - ax\| \leq 24\varepsilon\|x\|$. Consequently, if $\mathcal{B}_s = \{0\}$, then $\|T\| \leq 24\varepsilon$.*

Proof. Let $b = T\mathbf{1}$. Write $b = f + \iota g$, where $f, g \in \mathcal{E}$ are the real and imaginary parts of b: $f = \Re b = (b + b^*)/2$, $g = \Im b = (b - b^*)/(2\iota)$. For $c \in \mathcal{E}$, define the inner derivation $D_c : \mathcal{A} \to \mathcal{E} : x \mapsto cx - xc$. First show that $\|D_f\|, \|D_g\| \leq 8\varepsilon$.

We tackle D_f, as D_g can be treated similarly. As in Proposition 15, observe that, for any $p \in \mathbf{P}(\mathcal{A})$, we have $Tp + Tp^\perp = b$, $\|p(Tp)p - Tp\|_{\mathcal{E}} \leq \varepsilon$, and $\|p^\perp(Tp^\perp)p^\perp - Tp^\perp\|_{\mathcal{E}} \leq \varepsilon$. Consequently, $\|p(\Re(Tp))p - \Re(Tp)\|_{\mathcal{E}} \leq \varepsilon$, and $\|p^\perp(\Re(Tp^\perp))p^\perp - \Re(Tp^\perp)\|_{\mathcal{E}} \leq \varepsilon$. Applying Lemma 16 to $y = p^\perp(\Re(Tp^\perp))p^\perp - \Re(Tp^\perp)$, we obtain

$$\big\|p(\Re(Tp^\perp))p^\perp + p^\perp(\Re(Tp^\perp))p\big\|_{\mathcal{E}} \leq \varepsilon.$$

Similarly, $\big\|p(\Re(Tp))p^\perp + p^\perp(\Re(Tp))p\big\|_{\mathcal{E}} \leq \varepsilon$. As $f = \Re(Tp) + \Re(Tp^\perp)$, the triangle inequality yields $\|pfp^\perp + p^\perp fp\|_{\mathcal{E}} \leq 2\varepsilon$ for any $p \in \mathbf{P}(\mathcal{A})$.

Now consider a self-adjoint unitary u, which can be written as $p - p^\perp$. Then $f - ufu = 2(pfp^\perp + p^\perp fp)$ has norm not exceeding 4ε, hence $\|D_f u\|_{\mathcal{E}} \leq 4\varepsilon$ for every u as above. By an extreme point argument, $\|D_b x\|_{\mathcal{E}} \leq 4\varepsilon\|x\|_{\mathcal{A}}$ whenever $x \in \mathcal{A}$ is self-adjoint. By polar decomposition, $\|D_f\| \leq 8\varepsilon$.

By [8, Corollary 11], there exist self-adjoint $f_0, g_0 \in \mathcal{E}$ so that $\|f_0\|_{\mathcal{E}}, \|g_0\|_{\mathcal{E}} \leq 8\varepsilon$, $D_f = D_{f_0}$, and $D_g = D_{g_0}$. We claim that $f - f_0, g - g_0 \in \mathcal{E}(\mathcal{B}_s)$. Note that $h = f - f_0$ commutes with $\mathcal{A}$, hence the same is true for the projection $\chi_I(h)$, for any finite interval $I \subset \mathbb{R}$. Thus, all spectral projections of h belong to $\mathfrak{Z}(\mathcal{A})$.

Then $\chi_S(h) \perp z_p$ whenever $0 \notin S$. Consequently $z_p h = h z_p = 0$. Further, if $0 \notin S$, then $\chi_S(h) \in \mathcal{B}_s$, thus, by Lemma 2, $h \in \mathcal{E}(\mathcal{B}_s)$.

We also note that $\|T - \mathbf{L}_b\| \leq 8\varepsilon$. Indeed, for a projection p, we have

$$b = Tp + Tp^{\perp} = (Tp)p + (Tp)p^{\perp} + (Tp^{\perp})p + (Tp^{\perp})p^{\perp},$$

hence

$$bp - (Tp)p = (b - Tp)p = (Tp^{\perp})p,$$

and therefore,

$$\|bp - Tp\|_{\mathcal{E}} \leq \|(Tp^{\perp})p\|_{\mathcal{E}} + \|(Tp)p^{\perp}\|_{\mathcal{E}} \leq 2\varepsilon.$$

By an extreme point argument, $\|T - \mathbf{L}_b\| \leq 8\varepsilon$.

Now set $a = (f - f_0) + \imath(g - g_0)$, then $\|a - b\|_{\mathcal{E}} \leq 16\varepsilon$. Then, for $x \in \mathbf{B}(\mathcal{A})$,

$$\|Tx - \mathbf{L}_a x\|_{\mathcal{E}} \leq \|Tx - \mathbf{L}_b x\|_{\mathcal{E}} + \|a - b\| \leq 24\varepsilon.$$

4.2. Almost BP operators on symmetric function spaces

Next we consider ε-BP operators on a fully symmetric $\mathcal{E}(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a separably acting von Neumann algebra, equipped with a normal faithful semi-finite trace τ.

Theorem 18. *Suppose $\mathcal{A}$ is a separably acting von Neumann algebra, equipped with a faithful normal semifinite trace τ. Suppose, furthermore, that $\mathcal{E}$ is a proper fully symmetric function space on $[0, \tau(\mathbf{1}))$ with Fatou norm. If $T : \mathcal{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ is ε-BP, then there exists $c \in \mathfrak{Z}(\mathcal{A})$ so that $\|T - \mathbf{M}_c\| \leq 216\varepsilon$ (when $\mathcal{A}$ is finite we can take the right hand side to be 212ε).*

We postpone the proof, since it requires several lemmas. First describe the action of ε-BP maps on copies of M_2 (the algebra of 2×2 matrices). We say that $(e_{ij})_{i,j=1}^n \subset \mathcal{A}$ are *matrix units in* $\mathcal{E}(\mathcal{A})$ if there exists an isometric $*$-isomorphism $\pi : M_n \to \mathcal{A}$ so that, for $1 \leq i, j \leq n$, we have $e_{ij} = \pi(E_{ij})$, where $(E_{ij})_{i,j=1}^n \subset M_n$ are the canonical matrix units (matrices with 1 in the (i, j) position, zeroes elsewhere).

Lemma 19. *Suppose T is a ε-BP operator on $\mathcal{E}(\mathcal{A})$, where $\mathcal{A}$ a von Neumann algebra equipped with a normal faithful semifinite trace τ. Suppose $(e_{ij})_{i,j=1}^2 \subset \mathcal{A} \cap \mathcal{E}(\mathcal{A})$ are matrix units. Then $\|Te_{11} - e_{12}(Te_{22})e_{21}\|_{\mathcal{E}} \leq 14\varepsilon\|e_{11}\|_{\mathcal{E}}$.*

Proof. By scaling τ we can assume that $\|e_{11}\|_{\mathcal{E}} = 1$. For $i = 1, 2$ set $x_{ii} = e_{ii}(Te_{ii})e_{ii}$. For $i \neq j$ and $k, \ell \in \{1, 2\}$, set $x_{ij}^{k\ell} = e_{kk}(Te_{ij})e_{\ell\ell}$. We have

$$\|Te_{ii} - x_{ii}\|_{\mathcal{E}} \leq \varepsilon, \tag{3}$$

and

$$\|Te_{ij} - \sum_{k,\ell} x_{ij}^{k\ell}\|_{\mathcal{E}} = \|Te_{ij} - \sum_{k=1}^{2} e_{kk}[T(e_{ij})] \sum_{\ell=1}^{2} e_{\ell\ell}\|_{\mathcal{E}} \leq \varepsilon \text{ for } i \neq j. \tag{4}$$

Consider now the projections

$$q_\pm = \frac{1}{2}\big(e_{11} \pm e_{21}\big)\big(e_{11} \pm e_{12}\big) = \frac{1}{2}\big(e_{11} + e_{22} \pm e_{12} \pm e_{22}\big).$$

(equivalent to $e_{11} \sim e_{22}$). We have $q_+ = (e_{11} + e_{22})q_+(e_{11} + e_{22})$, hence

$$\|Tq_+ - (e_{11} + e_{22})(Tq_+)(e_{11} + e_{22})\|_{\mathcal{E}} \leq \varepsilon.$$

But

$$2(e_{11} + e_{22})(Tq_+)(e_{11} + e_{22}) = \sum_{k=1}^{2} e_{kk}\big(\sum_{i,j} Te_{ij}\big)\sum_{\ell=1}^{2} e_{\ell\ell}.$$

By (3) and (4),

$$\Big\|2Tq_+ - \big(\sum_i x_{ii} + \sum_{i\neq j}\sum_{k,\ell} x_{ij}^{k\ell}\big)\Big\|_{\mathcal{E}} \leq 4\varepsilon.$$

Multiplying from both sides by e_{11} gives us:

$$\Big\|e_{11}(2Tq_+)e_{11} - \big(x_{11} + x_{12}^{11} + x_{21}^{11}\big)\Big\|_{\mathcal{E}} \leq 4\varepsilon, \tag{5}$$

Similarly, multiplying from both sides by e_{22} yields:

$$\Big\|e_{22}(2Tq_+)e_{22} - \big(x_{22} + x_{12}^{22} + x_{21}^{22}\big)\Big\|_{\mathcal{E}} \leq 4\varepsilon.$$

On the other hand,

$$\|(e_{11} + e_{22})Tq_+(e_{11} + e_{22}) - q_+(Tq_+)q_+\|_{\mathcal{E}} \leq \|Tq_+ - q_+(Tq_+)q_+\|_{\mathcal{E}} \leq \varepsilon.$$

It is easy to check that, for $y = \sum_{i,j=1}^{2} y_{ij}$, with $y_{ij} = e_{ii}ye_{jj}$, we have

$$4e_{11}q_\pm yq_\pm e_{11} = y_{11} \pm y_{12}e_{21} \pm e_{12}y_{21} + e_{12}y_{22}e_{21}.$$

Plugging in q_+ and $y = (e_{11} + e_{22})(Tq_+)(e_{11} + e_{22})$, we obtain

$$4e_{11}q_+(Tq_+)q_+e_{11} =$$
$$\big(x_{11} + x_{12}^{11} + x_{21}^{11}\big) + e_{12}\big(x_{22} + x_{12}^{22} + x_{21}^{22}\big)e_{21} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21},$$

hence the $\mathcal{E}$-norm of

$$4e_{11}(Tq_+)e_{11} - \Big(\big(x_{11} + x_{12}^{11} + x_{21}^{11}\big) +$$
$$e_{12}\big(x_{22} + x_{12}^{22} + x_{21}^{22}\big)e_{21} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21}\Big)$$

does not exceed 4ε. By (5), the $\mathcal{E}$-norm of

$$2\big(x_{11} + x_{12}^{11} + x_{21}^{11}\big) - \Big(\big(x_{11} + x_{12}^{11} + x_{21}^{11}\big) +$$
$$e_{12}\big(x_{22} + x_{12}^{22} + x_{21}^{22}\big)e_{21} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21}\Big)$$

does not exceed 12ε. Similarly, the $\mathcal{E}$-norm of

$$2\big(x_{22} + x_{12}^{22} + x_{21}^{22}\big) - \Big(e_{12}\big(x_{22} + x_{12}^{11} + x_{21}^{11}\big)e_{21} + \big(x_{22} + x_{12}^{22} + x_{21}^{22}\big) + \big(x_{12}^{21} + x_{21}^{21}\big)e_{12} + e_{21}\big(x_{12}^{12} + x_{21}^{12}\big)\Big)$$

does not exceed 12ε.

An application of the same procedure to q_- instead of q_+ shows that

$$2\big(x_{11} - x_{12}^{11} - x_{21}^{11}\big) - \Big(\big(x_{11} - x_{12}^{11} - x_{21}^{11}\big) + e_{12}\big(x_{22} - x_{12}^{22} - x_{21}^{22}\big)e_{21} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21}\Big)$$

and

$$2\big(x_{22} - x_{12}^{22} - x_{21}^{22}\big) - \Big(e_{12}\big(x_{22} - x_{12}^{11} - x_{21}^{11}\big)e_{21} + \big(x_{22} - x_{12}^{22} - x_{21}^{22}\big) + \big(x_{12}^{21} + x_{21}^{21}\big)e_{12} + e_{21}\big(x_{12}^{12} + x_{21}^{12}\big)\Big)$$

have $\mathcal{E}$-norms not exceeding 12ε.

Averaging the centered expressions with $x_{11} - x_{12}^{11} - x_{21}^{11}$ and $x_{11} + x_{12}^{11} + x_{21}^{11}$ in their left hand sides, we conclude that

$$\Big\|2x_{11} - \Big(x_{11} + e_{12}x_{22}e_{21} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21}\Big)\Big\|_{\mathcal{E}} \leq 12\varepsilon. \quad (6)$$

Similarly,

$$\Big\|2x_{22} - \Big(x_{22} + e_{21}x_{11}e_{12} + \big(x_{12}^{21} + x_{21}^{21}\big)e_{12} + e_{21}\big(x_{12}^{12} + x_{21}^{12}\big)\Big)\Big\|_{\mathcal{E}} \leq 12\varepsilon.$$

Multiplying by e_{12} on the left, and e_{21} on the right, we conclude that

$$\Big\|2e_{12}x_{22}e_{21} - \Big(e_{12}x_{22}e_{21} + x_{11} + e_{12}\big(x_{12}^{21} + x_{21}^{21}\big) + \big(x_{12}^{12} + x_{21}^{12}\big)e_{21}\Big)\Big\|_{\mathcal{E}} \leq 12\varepsilon. \quad (7)$$

Applying the triangle inequality to (6) and (7), we obtain $\|x_{11} - e_{12}x_{22}e_{21}\|_{\mathcal{E}} \leq 12\varepsilon$. By the triangle inequality,

$$\|Te_{11} - e_{12}(Te_{22})e_{21}\|_{\mathcal{E}} \leq \|Te_{11} - x_{11}\|_{\mathcal{E}} + \|x_{11} - e_{12}x_{22}e_{21}\|_{\mathcal{E}} + \|Te_{22} - x_{22}\|_{\mathcal{E}} \leq 14\varepsilon,$$

completing the proof.

The following lemma may be known, but we have not seen it stated explicitly. Some related results are contained in [24].

Lemma 20. *Suppose $\mathcal{B}$ is a MASA in a finite separably acting von Neumann algebra $\mathcal{A}$. Then for any $p \in \mathbf{P}(\mathcal{A})$ there exists $q \in \mathbf{P}(\mathcal{B})$ so that $p \sim q$.*

Proof. By [41, Lemma III.1.20], $\mathcal{B}$ can be generated by a single (automatically normal) element, call it b. By [17, Theorem 1], there exists a projection $q \sim p$ which commutes with b, hence belongs to $\mathcal{B}$.

To proceed, we need to describe conditional expectations onto MASAs. Denote by $\mathbf{P^c}(\mathcal{B})$ the set of all finite families $P = (p_1, \ldots, p_n)$, with $p_1, \ldots, p_n \in \mathbf{P}(\mathcal{B})$ mutually orthogonal, and adding up to $\mathbf{1}$. For such P define $\Phi_P : \mathcal{A} \to \mathcal{A}$ by setting $\Phi_P x = \sum_{k=1}^n p_k x p_k$. Note that

$$\Phi_P x = 2^{-n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \big(\sum_k \varepsilon_k p_k \big) x \big(\sum_k \varepsilon_k p_k \big),$$

hence Φ_P is contractive with respect to the norm $\mathcal{F}(\mathcal{A}, \tau)$ whenever $\mathcal{F}$ is a symmetric space.

In [3] one finds a construction of a conditional expectation Φ from $\mathcal{A}$ onto $\mathcal{B}$ so that, for any $x \in \mathcal{A}$, $\Phi(x) = \mathrm{w}^* - \lim_\alpha \Phi_{P_\alpha}(x)$ (the set $\mathbf{P^c}(\mathcal{B})$ is ordered by refinement; (P_α) is a subnet). In addition, if $\mathcal{A}$ is equipped with a normal faithful semifinite trace τ, and $\mathcal{B}$ is τ-semifinite, then Φ is normal (and faithful). One can check that Φ maps $\mathcal{A} \cap L^1(\mathcal{A})$ to $\mathcal{B} \cap L^1(\mathcal{B})$, and is contractive with respect to the $\| \cdot \|_1$ norm. Thus, Φ determines a contractive conditional expectation $L_1(\mathcal{A}) \to L_1(\mathcal{B})$. Consequently, Φ defines a contractive map from $\mathcal{A} + L_1(\mathcal{A})$ to $\mathcal{B} + L_1(\mathcal{B})$. By [13], if $\mathcal{E}$ is fully symmetric, then $\mathcal{E}(\mathcal{A})$ is an exact interpolation space for the pair $(\mathcal{A}, L^1(\mathcal{A}))$, hence Φ acts as a contraction from $\mathcal{E}(\mathcal{A})$ to $\mathcal{E}(\mathcal{B})$.

Note also that the equality $\Phi(x) = p[\Phi(pxp)]p + p^\perp[\Phi(p^\perp x p^\perp)]p^\perp$ holds for any $x \in \mathcal{E}$ and $p \in \mathbf{P}(\mathcal{B})$. Indeed, if α is so large that P_α refines x, then $\Phi_{P_\alpha}(x) = p[\Phi_{P_\alpha}(pxp)]p + p^\perp[\Phi_{P_\alpha}(p^\perp x p^\perp)]p^\perp$. Then pass to the limit.

We also need to investigate an interplay of conditional expectations and duality. Henceforth, $\mathcal{B}$ is a τ-semifinite MASA in $\mathcal{A}$. Suppose, furthermore, that $\mathcal{E}$ is a symmetric space on $[0, \tau(\mathbf{1}))$, with Fatou norm. By [14, Section 5], $\mathcal{E}(\mathcal{A}, \tau)$ embeds isometrically into its Köthe bidual $\mathcal{E}^{\times\times}(\mathcal{A}, \tau)$ – that is, $\mathcal{E}^\times(\mathcal{A}, \tau)$ norms $\mathcal{E}(\mathcal{A}, \tau)$ isometrically. Let $\mathcal{A}_0$ be the family of all elements of $\mathcal{A}$ whose (left or right) support projection has finite trace. One can see (as in [14, Proposition 5.3], that, for $x \in \mathcal{E}(\mathcal{A})$,

$$\|x\|_{\mathcal{E}^{\times\times}(\mathcal{A})} = \sup \big\{ |\tau(xy)| : y \in \mathcal{A}_0, \|y\|_{\mathcal{E}^\times(\mathcal{A})} \leq 1 \big\}.$$

We claim that, for any x and y as above, $\Phi(yx) = y\Phi(x)$. One way to see this consists of considering maps $\Phi_P^{(y)} : x \mapsto y \sum_{i=1}^n p_i x p_i$ (with $P = (p_1, \ldots, p_n) \in \mathbf{P^c}(\mathcal{B})$), and finding the weak* limit of the net $(\Phi_{P_\alpha}^{(y)}(x))$.

For future use, we need:

Lemma 21. *Suppose $\mathcal{A}$, $\mathcal{B}$, and Φ are as above, $\mathcal{E}$ is a fully symmetric function space on $[0, \tau(\mathbf{1}))$ with Fatou norm, and $T : \mathcal{E}(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ is ε-BP. Consider $T_{\mathcal{B}} = \Phi \circ T|_{\mathcal{E}(\mathcal{B})}$ as an operator on $\mathcal{E}(\mathcal{B})$. Then $\|T_{\mathcal{B}} - T|_{\mathcal{B}}\| \leq 4\varepsilon$, and $T_{\mathcal{B}}$ is ε-BP.*

PROOF. Suppose $p \in \mathbf{P}(\mathcal{B})$, and show that $\|\Phi Tx - p[\Phi Tx]p\| \le \varepsilon \|x\|$. If $P \in \mathbf{P^c}(\mathcal{B})$ refines p, then $\Phi_P Tx = p[\Phi_P Tx]p + p^\perp [\Phi_P Tx] p^\perp$, hence

$$\|\Phi_P Tx - p[\Phi_P Tx]p\| = \|p^\perp [\Phi_P Tx] p^\perp\| \le \|p^\perp [Tx] p^\perp\| \le \|Tx - p[Tx]p\| \le \varepsilon \|x\|. \tag{8}$$

To estimate $\|\Phi Tx - p[\Phi Tx]p\|$ from above, fix $\delta > 0$, and find $y \in \mathcal{A}_0$ so that $\|y\|_{\mathcal{E}} = 1$, and $\|\Phi Tx - p[\Phi Tx]p\| < |\tau(y(\Phi Tx - p[\Phi Tx]p))| + \delta$. As noted above,

$$\begin{aligned} \big|\tau(y(\Phi Tx - p[\Phi Tx]p))\big| &= \lim_\alpha \big|\tau(y(\Phi_{P_\alpha} Tx - p[\Phi_{P_\alpha} Tx]p))\big| \\ &\le \liminf_\alpha \big\|\Phi_{P_\alpha} Tx - p[\Phi_{P_\alpha} Tx]p\big\| \le \varepsilon. \end{aligned}$$

Thus, $\|\Phi Tx - p[\Phi Tx]p\| < \varepsilon + \delta$. Now recall that δ can be arbitrarily small.

Next suppose $x \in \mathbf{B}(\mathcal{E}(\mathcal{B}))$, and show that $\|T_{\mathcal{B}} x - Tx\| \le 4\varepsilon$. Fix $P = (p_1, \ldots, p_n) \in \mathbf{P^c}(\mathcal{B})$. For $S \subset \{1, \ldots, n\}$, set $y_S = \sum_{i \in S} p_i x$, $z_S = \sum_{i \notin S} p_i x$, $r_S = \sum_{i \in S} p_i$, and $r_S^\perp = \sum_{i \notin S} p_i$. Then $x = y_S + z_S$, hence

$$\begin{aligned} &\sum_{i \in S, j \notin S} p_i [Tx] p_j + \sum_{i \notin S, j \in S} p_i [Tx] p_j \\ &= \big(r_S [T y_S] r_S^\perp + r_S^\perp [T y_S] r_S\big) + \big(r_S [T z_S] r_S^\perp + r_S^\perp [T z_S] r_S\big) \end{aligned} \tag{9}$$

By Lemma 16,

$$\big\|r_S [T y_S] r_S^\perp + r_S^\perp [T y_S] r_S\big\| \le \|T y_S - r_S [T y_S] r_S\| \le \varepsilon \|y_S\| \le \varepsilon,$$

and similarly, $\|r_S [T z_S] r_S^\perp + r_S^\perp [T z_S] r_S\| \le \varepsilon$. Therefore, $\| \sum_{i \in S, j \notin S} p_i [Tx] p_j + \sum_{i \notin S, j \in S} p_i [Tx] p_j\| \le 2\varepsilon$. By (9),

$$Tx - \Phi_P Tx = 2^{1-n} \sum_{S \subset \{1, \ldots, n\}} \big(r_S [T z_S] r_S^\perp + r_S^\perp [T z_S] r_S\big).$$

hence, by the triangle inequality, $\|Tx - \Phi_P Tx\| \le 4\varepsilon$. As before, we can "pass to the limit" to obtain $\|Tx - \Phi Tx\| \le 4\varepsilon$.

PROOF OF THEOREM 18 WHEN $\mathcal{A}$ IS FINITE. **Step 1.** *For every τ-semifinite MASA $\mathcal{B} \subset \mathcal{A}$, there exists $c = c_{\mathcal{B}} \in \mathfrak{Z}(\mathcal{A})$ so that $\|(T - \mathbf{L}_c)|_{\mathcal{E}(\mathcal{B})}\| \le 26\varepsilon$. Further, T is continuous.*

Let Φ be a conditional expectation from $\mathcal{E}(\mathcal{A})$ onto $\mathcal{E}(\mathcal{B})$ (constructed above), and consider $T_{\mathcal{B}} = \Phi \circ T|_{\mathcal{E}(\mathcal{B})}$ as an operator on $\mathcal{E}(\mathcal{B})$. By Lemma 21, $\|T_{\mathcal{B}} - T|_{\mathcal{B}}\| \le 4\varepsilon$, and $T_{\mathcal{B}}$ is ε-BP as an operator on $\mathcal{E}(\mathcal{B})$.

As $\mathcal{B}$ is a separably acting abelian von Neumann algebra, it can be identified with $L^\infty(\Omega, \mu)$, for some σ-finite measure μ. Then $\mathcal{E}(\mathcal{B})$ can be viewed as the Köthe function space $\mathcal{E}(\Omega, \mu)$. By [34, Section 3.1], $T_{\mathcal{B}}$ is necessarily continuous, and is close to a multiplication operator: there exists $a = a_{\mathcal{B}} \in \mathcal{B}$ so that $\|T_{\mathcal{B}} - \mathbf{L}_{a,\mathcal{B}}\| \le 4\varepsilon$. Here $\mathbf{L}_{a,\mathcal{B}}$ is the operator of multiplication by a, acting on $\mathcal{B}$. This implies that T is continuous on $\mathcal{E}(\mathcal{B})$ for any τ-semifinite MASA $\mathcal{B}$, and consequently, by Proposition 4, T is continuous on $\mathcal{E}(\mathcal{A})$.

Next show that there exists a $c = c_{\mathcal{B}} \in \mathfrak{Z}(\mathcal{A})$ so that $\|T_{\mathcal{B}} - \mathbf{L}_{c,\mathcal{B}}\| \leq 26\varepsilon$. Indeed, by [18, Lemma 1], there exists $c \in \mathfrak{Z}(\mathcal{A})$ so that, for any $e \in \mathbf{P}(\mathfrak{Z}(\mathcal{A}))$,

$$\|(a-c)e\| = \inf_{z \in \mathfrak{Z}(\mathcal{A})} \|(a-z)e\|. \tag{10}$$

It suffices to show that $\|a - c\| \leq 22\varepsilon$.

Passing from a to $a - c$, we can assume that $c = 0$. Suppose, for the sake of contradiction, that $\|a\| > \lambda > 22\varepsilon$. Pick $\delta \in (0, (\lambda - 22\varepsilon)/2)$. As a is normal, we can find $\mu \in \mathbb{C}$ with $|\mu| > \lambda + \delta$ so that the spectral projection $\chi_S(a) \neq 0$, where $S = \{\zeta \in \mathbb{C} : |\mu - \zeta| < \delta\}$. Any spectral projection of a belongs to $\mathcal{B}$, hence we can find a non-zero $p \in \mathbf{P}_\tau(\mathcal{B})$, with $p \leq \chi_S(a)$.

Let $e = \mathbf{z}(p)$ be the central cover of p. We have $\|ae\| \geq \|ap\| > \lambda$, hence $\|(a - \mu e)e\| > \lambda$ (otherwise $\|(a - \mu e)e\| < \|ae\|$, contradicting (10)). Consequently, we can find $\mu' \in \mathbb{C}$ so that $|\mu' - \mu| > \lambda$, and $\chi_{S'}(ae) \neq 0$, where $S' = \{\zeta \in \mathbb{C} : |\zeta - \mu'| < \delta\}$. Further, we can find $p' \in \mathbf{P}_\tau(\mathcal{B})$ satisfying $0 < p' \leq \chi_{S'}(ae)$. Note that $S \cap S' = \emptyset$, hence p and p' are mutually orthogonal. Further, $e \geq \mathbf{z}(p')$, hence, by [26, Proposition 6.1.8], there exist equivalent non-zero projections $q_1 \leq p$ and $q_2 \leq p'$ (which are necessarily mutually orthogonal).

Note that, for any $r \in \mathbf{P}(\mathcal{B})$, $r\mathcal{B}$ is a MASA in $r\mathcal{A}r$. Applying Lemma 20 to the MASAs $p\mathcal{B} \subset p\mathcal{A}p$ and $p'\mathcal{B} \subset p'\mathcal{A}p'$, we find $p_1, p_2 \in \mathbf{P}(\mathcal{B})$ so that $p_1 \sim q_1 \leq p$ and $p_2 \sim q_2 \leq p'$. Then there exists a partial isometry $u \in \mathcal{A}$ so that $uu^* = p_2$ and $u^*u = p_1$.

Lemma 19, applied to $e_{11} = p_1$, $e_{22} = p_2$, $e_{21} = u$, and $e_{12} = u^*$, yields $\|Tp_1 - u^*(Tp_2)u\|_{\mathcal{E}} \leq 14\varepsilon\gamma$, where $\gamma = \|p_1\|_{\mathcal{E}}$. On the other hand, by our choice of a, $\|Tp_j - ap_j\|_{\mathcal{E}} \leq 4\gamma$ $(j = 1, 2)$. Further, for any $\xi \in \operatorname{ran} p$ we have $\|(a - \mu)\xi\| \leq \delta\|\xi\|$, hence $\|ap_1 - \mu p_1\|_{\mathcal{E}} \leq \delta\gamma$. Similarly, $\|ap_1 - \mu' p_1\|_{\mathcal{E}} \leq \delta\gamma$. By the triangle inequality,

$$\begin{aligned}\big\|Tp_1 - u^*(Tp_2)u\big\|_{\mathcal{E}} &\geq \|ap_1 - u^*ap_2u\|_{\mathcal{E}} - 8\varepsilon\gamma \\ &\geq \|\mu p_1 - \mu' p_1\|_{\mathcal{E}} - (2\delta + 8\varepsilon)\gamma > (\lambda - 2\delta - 8\varepsilon)\gamma > 14\varepsilon\gamma.\end{aligned}$$

This yields a contradiction, since δ can be arbitrarily small. ⊞

Step 2. *If $c = c_{\mathcal{B}}$ is as in Step 1, and $x = x^* \in \mathcal{E}(\mathcal{A})$, then $\|Tx - cx\| \leq 106\varepsilon\|x\|$.*

Passing from T to $T - \mathbf{L}_c$ if necessary, we can assume that $c = 0$. Note that x is affiliated with a τ-semifinite MASA, which we call $\mathcal{B}'$. Indeed, let p be the support projection of x. If $p = \mathbf{1}$, just take the MASA generated by all spectral projections of x. If $p < \mathbf{1}$, find a MASA containing the spectral projections of x, and a maximal family of mutually orthogonal finite trace projections dominated by $p^\perp$.

By Step 1, there exists $c' \in \mathfrak{Z}(\mathcal{A})$ so that $\|(T - \mathbf{L}_{c'})|_{\mathcal{E}(\mathcal{B}')}\|_{\mathcal{E}} \leq 26\varepsilon$. We need to show that $\|c'\| \leq 80\varepsilon$. Indeed, otherwise there exists $\lambda \in \mathbb{C}$ with $|\lambda| > 80\varepsilon$, and $\delta \in (0, \|c'\| - 80\varepsilon)$, so that $e = \chi_S(c') \neq 0$ (here $S = \{\zeta \in \mathbb{C} : |\zeta - \lambda| < \delta\}$). If e has a finite trace central subprojection f, then the desired contradiction is at hand. Indeed, f belongs to both $\mathcal{B}$ and $\mathcal{B}'$. The inclusion $f \in \mathcal{B}$ implies

$\|Tf\|_{\mathcal{E}} \leq 26\varepsilon\|f\|_{\mathcal{E}}$, while $f \in \mathcal{B}'$ gives us

$$\|Tf\|_{\mathcal{E}} \geq \|c'f\|_{\mathcal{E}} - 26\varepsilon\|f\|_{\mathcal{E}} > (\lambda - \delta - 26\varepsilon)\|f\|_{\mathcal{E}}.$$

So suppose e has no finite trace central subprojections. Find $p \in \mathbf{P}(\mathcal{B})$ so that $0 < \tau(p) < \infty$, and $p \leq e$. Applying Lemma 20 to p in the MASA $\mathcal{B}'e \subset \mathcal{A}e$, we obtain $q \in \mathbf{P}(\mathcal{B}')$ so that $q \sim p$.

If $q \perp p$, then we are done. Indeed, by Lemma 19, $\|Tq\|_{\mathcal{E}} \leq \|Tp\|_{\mathcal{E}} + 14\varepsilon\|p\|_{\mathcal{E}}$. However, $\|Tp\|_{\mathcal{E}} \leq 26\varepsilon\|p\|_{\mathcal{E}}$, while

$$\|Tq\|_{\mathcal{E}} \geq \|c'q\|_{\mathcal{E}} - 26\varepsilon\|q\|_{\mathcal{E}} \geq (|\lambda| - 26\varepsilon)\|p\|_{\mathcal{E}}.$$

In the general case, recall that $f := \mathbf{z}(p) = \mathbf{z}(q)$ has infinite trace. Thus, $f \geq p \vee q$. Note that $\mathbf{z}(f - p \vee q)$ cannot be disjoint from $\mathbf{z}(p) = \mathbf{z}(q)$. By [26, Proposition 6.1.8], $f - p \vee q$ contains a non-zero projection r, equivalent to $p_1 \leq p$, and hence to $q_1 \leq q$ (here we once again use the equivalence of p and q). Applying Lemma 20 to $p_1 \in p\mathcal{B} \subset p\mathcal{A}p$, we find $p_2 \in \mathbf{P}(\mathcal{B})$ so that $p_1 \sim p_2 \leq p$. Similarly, find $q_2 \in \mathbf{P}(\mathcal{B}')$ so that $q_1 \sim q_2 \leq q$. By Lemma 19,

$$\|Tq_2\|_{\mathcal{E}} \leq \|Tr\|_{\mathcal{E}} + 14\varepsilon\|r\|_{\mathcal{E}} \leq \|Tp_1\|_{\mathcal{E}} + 28\varepsilon\|r\|_{\mathcal{E}} \leq 54\varepsilon\|r\|_{\mathcal{E}}.$$

However,

$$\|Tq_2\|_{\mathcal{E}} \geq \|c'q_2\|_{\mathcal{E}} - 26\varepsilon\|r\|_{\mathcal{E}} > (|\lambda| - 26\varepsilon)\|r\|_{\mathcal{E}},$$

leading to a contradiction. ⊞

To conclude the proof, note that $\|T - \mathbf{L}_c\| \leq 212\varepsilon$ due to polar decomposition.

Proof of Theorem 18 when $\mathcal{A}$ is semi-finite. We view $\mathbf{P}_f(\mathcal{A})$ (the set of all finite projections in $\mathcal{A}$) as a net, ordered by inclusion. Identify $p\mathcal{E}(\mathcal{A})p$ with $\mathcal{E}(p\mathcal{A}p)$.

For any $p \in \mathbf{P}_f(\mathcal{A})$, set (for brevity) $\mathcal{A}_p = p\mathcal{A}p$, and consider the map $T_p : \mathcal{E}(\mathcal{A}_p) \to \mathcal{E}(\mathcal{A}_p) : x \mapsto p(Tx)p$. Then $\|T|_{\mathcal{E}(\mathcal{A}_p)} - T_p\| \leq \varepsilon$, and moreover, T_p is ε-BP. Thus, by the proof in the finite case, there exists $c_p \in \mathfrak{Z}(\mathcal{A}_p)$ so that $\|(T_p - \mathbf{L}_{c_p})|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| \leq 106\varepsilon\|x\|_p$ (here $\mathcal{E}(\mathcal{B})_{sa}$ denotes the selfadjoint part of $\mathcal{E}(\mathcal{B})$). But $\mathfrak{Z}(\mathcal{A}_p) = p\mathfrak{Z}(\mathcal{A})$. Moreover (see the proof of [25, Proposition 5.5.6]), for any p there exists $d_p \in \mathfrak{Z}(\mathcal{A})$ so that $\|c_p\| = \|d_p\|$, and $c_p = pd_p$.

Show first that $\sup_{p \in \mathbf{P}_f(\mathcal{A})} \|T_p\| < \infty$. Observe that, if $0 < p < q$ are finite projections, then

$$\begin{aligned}\|c_p - pc_q\| &= \|\mathbf{L}_{c_p - pc_q}|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| \\ &\leq \|(T_p - \mathbf{L}_{c_p})|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| + \|(T_p - \mathbf{L}_{pc_q})|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| \\ &\leq \|(T_p - \mathbf{L}_{c_p})|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| + \|(T_r - \mathbf{L}_{c_r})|_{\mathcal{E}(\mathcal{A}_p)_{sa}}\| \leq 212\varepsilon.\end{aligned} \tag{11}$$

If, in this situation, $\|T_q\| > \|T_p\| + 400\varepsilon$, then $\|T_{q-p}\| \geq \|T_q\| - 500\varepsilon$. Indeed, $c_q = pc_q + (q-p)c_q = pc_qp + (q-p)c_q(q-p)$, hence $\|c_q\| = \max\{\|pc_q\|, \|(q-p)c_q\|\}$. By (11), $\|pc_q\| \leq \|c_p\| + 212\varepsilon$, hence $\|c_q\| = \|(q-p)c_q\|$. For $x \in \mathcal{A}_{q-p}$, we have

$$\|T_{q-p}x - (q-p)c_qx\|_{\mathcal{E}} = \|(q-p)(T_qx - c_qx)\|_{\mathcal{E}} \leq \|T_qx - c_qx\|_{\mathcal{E}} \leq 212\varepsilon.$$

Thus,

$$\|T_{q-p}\| \geq \|\mathbf{L}_{(q-p)c_q}\| - 194\varepsilon = \|(q-p)c_q\| - 212\varepsilon = \|c_q\| - 212\varepsilon \geq \|T_q\| - 2\cdot 212\varepsilon.$$

Suppose now, for the sake of contradiction, that $\sup_{p\in\mathbf{P}_f(\mathcal{A})} \|T_p\| = \infty$. By the above, we can find a sequence of mutually orthogonal finite projections (p_k), so that $\|T_{p_1}\| > 4$, and $\|T_{p_n}\| > 4\|T_{p_{n-1}}\|$ for $n > 1$. For each n find $x_n \in \mathcal{E}(\mathcal{A})$ so that $p_n x_n p_n = x_n$, $\|x_n\|_{\mathcal{E}} < 2^{-n}$, and $\|p_n(Tx_n)p_n\|_{\mathcal{E}} > 2^n$. Set $x = \sum_n x_n$, and $y_n = x - x_n$. Then for every n,

$$\|Tx\|_{\mathcal{E}} \geq \|p_n(Tx)p_n\|_{\mathcal{E}} \geq \|p_n(Tx_n)p_n\|_{\mathcal{E}} - \|p_n(Ty_n)p_n\|_{\mathcal{E}} > 2^n - \varepsilon.$$

Taking the supremum on the right hand side, we obtain a contradiction.

As noted above, for every $p \in \mathbf{P}_f(\mathcal{A})$, we can find $d_p \in \mathfrak{Z}(\mathcal{A})$ so that $c_p = pd_p$, and $\|d_p\| = \|c_p\|$. Consequently, $\sup_p \|d_p\| = \sup_p \|c_p\| < \infty$. Let $d \in \mathfrak{Z}(\mathcal{A})$ be a weak* cluster point of the net (d_p). We claim that, if $x \in \mathbf{B}(\mathcal{E}(\mathcal{A})_{sa})$ is such that $pxp = x$ for some finite projection x, then $\|Tx - dx\|_{\mathcal{E}} \leq 107\varepsilon$. Indeed, for any finite projection $q \geq p$,

$$\|Tx - d_q x\|_{\mathcal{E}} \leq \|T_q x - d_q x\|_{\mathcal{E}} + \|Tx - q(Tx)q\|_{\mathcal{E}} \leq 107\varepsilon$$

For $\delta > 0$, find $y \in \mathcal{B}_0 \cap \mathbf{B}(\mathcal{E}^\times(\mathcal{B}))$ so that $\tau((Tx - dx)y) + \delta > \|Tx - dx\|$. Then $\tau(dxy)$ is a cluster point of the net $\tau(d_q xy)$. However, for $q \geq p$, $|\tau((d_q x - Tx)y)| \leq \|d_q x - Tx\| \leq 107\varepsilon$. Thus, $\|Tx - dx\| < 107\varepsilon + \delta$, and δ can be arbitrarily small.

To finish the proof, it suffices to show that $\|Tx - dx\| \leq 108\varepsilon$ for any $x \in \mathbf{B}(\mathcal{E}(\mathcal{A})_{sa})$. Suppose, for the sake of contradiction, that, for some $x \in \mathbf{B}(\mathcal{E}(\mathcal{A})_{sa})$, we have $\|Tx - dx\| > 108\varepsilon$. Denote the support projection of x by p_∞, and set $p_n = \chi_{[1/n,\infty)}(|x|)$. As T is ε-BP, we have $\|p_\infty[Tx]p_\infty - Tx\| \leq \varepsilon$, hence $\|p_\infty[Tx]p_\infty - dx\| > 107\varepsilon$. Note that $p_n \nearrow p_\infty$, hence, by Lemma 1, $\|p_n[Tx - dx]p_n\| > 107\varepsilon$ for n large enough. Now write $x = x_n + y_n$, where $x_n = p_n x p_n$ and $y_n = (p_\infty - p_n)x(p_\infty - p_n)$. Then $\|p_n[Tx_n - dxp_n]p_n\| \leq 106\varepsilon$, and $\|p_n[Ty_n]p_n\| \leq \varepsilon$. Now the triangle inequality provides the desired contradiction: as $Tx = Tx_n + Ty_n$, we have $\|p_n[Tx - dx]p_n\| \leq 107\varepsilon$.

5. Almost centralizers

Definition 4. Suppose X and Y are left $\mathcal{A}$-modules. We say that $T : X \to Y$ is a *ε-right centralizer* if for any $x \in \mathbf{B}(X)$ and $a \in \mathbf{B}(\mathcal{A})$, we have $\|T(ax) - a[Tx]\| \leq \varepsilon$. One defines ε-left centralizers similarly. If X and Y are $\mathcal{A}$-bimodules, we say that $T : X \to Y$ is a ε-centralizer if is both ε-left and ε-right centralizer. We use the term "(left or right) centralizer" for a 0-(left ot right) centralizer.

By way of example, consider $X = Y = B(\ell_2)$, and $\mathcal{A} = \ell_\infty$ (viewed as the algebra of diagonal operators). Then the Schur multipliers are precisely the weak* to weak* continuous centralizers.

Imitating [35, Section 3.12], one can show that, if $\mathcal{A}$ is a C^*-algebra, then any right (or left) centralizer $T : \mathcal{A} \to Y$ is continuous, and moreover, it is implemented by a multiplication operator: there exists a unique $y \in Y^{**}$ so that, for any $a \in \mathcal{A}$, $Ta = ay$. If $\mathcal{A}$ is unital, then $y = T\mathbf{1}$. Some generalizations can be found in [9].

Note also (although it is not used in our paper) that, by [40], under certain conditions any centralizer is automatically completely bounded.

Note that any ε-right (left) centralizer is ε-right (resp. left) annihilator preserving. Moreover, any ε-centralizer is ε-BP, as well as ε-RAP and ε-LAP. Thus, the continuity, approximation, and multiplicative representation results from Sections 2 and 4 apply here.

In this section we show that, under certain conditions, every ε-(left or right) centralizer can be approximated by a (left or right) centralizer.

Below we deal with hyperfinite von Neumann algebras and their modules. Recall (see e.g. [7, III.3.4]) that a von Neumann algebra $\mathcal{A}$ is called *hyperfinite* if it contains an upward directed net of finite dimensional $*$-subalgebras $\mathcal{A}_i$ so that $\overline{\cup_i \mathcal{A}_i}^{\mathrm{w}^*} = \mathcal{A}$. The subalgebras $\mathcal{A}_i$ can be assumed to be unital. Further, if $\mathcal{A}$ has separable predual, then, instead of an "upward directed net", we can find an increasing weak*-dense sequence of subalgebras.

Theorem 22. *Suppose $\mathcal{A}$ is a hyperfinite von Neumann algebra, X and Y are normal dual left $\mathcal{A}$-modules, and $T : X \to Y$ is a ε-right centralizer. Suppose, furthermore, that one of the following two conditions holds:*

1. *X and Y are reflexive.*
2. *$\mathcal{A}$ is separably acting, T is weak* to weak* continuous, and the predual X_* is weakly sequentially complete.*

Then there exists a right centralizer R so that $\|T - R\| \leq \varepsilon$. If T is weak* to weak* continuous, then the same is true for R. Moreover, if X, Y are $\mathcal{A}$-bimodules, and T is a δ-nearly left centralizer, then R is a δ-left centralizer as well.

Examples of weakly sequentially complete spaces can be found in [12, Section 8.3]. For instance, the predual of any von Neumann algebra is weakly sequentially complete, hence part (2) of the theorem applies if X is a von Neumann algebra containing $\mathcal{A}$.

In the two-sided case, a similar result can be obtained:

Theorem 23. *Suppose $\mathcal{A}$ is a hyperfinite von Neumann algebra, X and Y are normal dual $\mathcal{A}$-modules, and $T : X \to Y$ is a ε-centralizer. Suppose, furthermore, that one of the following two conditions holds:*

1. *X and Y are reflexive.*
2. *$\mathcal{A}$ is separably acting, T is weak* to weak* continuous, and the predual X_* is weakly sequentially complete.*

Then there exists a centralizer R so that $\|T - R\| \leq \varepsilon$. If T is weak* to weak* continuous, then the same is true for R.

To prove these results, start with:

Lemma 24. *Suppose $\mathcal{A}$ is a finite dimensional von Neumann algebra, X, Y are left $\mathcal{A}$-modules, and $T : X \to Y$ is a ε-right centralizer. Then there exists a right centralizer R so that $\|T - R\| \leq \varepsilon$. If X, Y are $\mathcal{A}$-bimodules, and T is a δ-left centralizer, then R is a δ-left centralizer. If T is norm continuous, then R has the same property. If, moreover, X and Y are normal dual modules, and T is weak* to weak* continuous, then the same is true for R.*

Proof. Denote by $\mathcal{U}(\mathcal{A})$ the unitary group of $\mathcal{A}$, equipped with its normalized Haar measure μ. Define $R \in X \to Y$ by setting, for $x \in X$, $Rx = \int_{\mathcal{U}(\mathcal{A})} uT(u^*x)\, d\mu(u)$. Then for any $v \in \mathcal{U}(\mathcal{A})$,

$$v[Rx] = \int (vu)T(u^*x)\, d\mu(u) = \int (vu)T((uv)^*vx)\, d\mu(uv) = R(vx),$$

hence R is a right centralizer. Further, for $x \in \mathbf{B}(X)$,

$$\|Rx - Tx\| \leq \sup_{u \in \mathcal{U}(\mathcal{A})} \|Tx - uT(u^*x)\| = \sup_{u \in \mathcal{U}(\mathcal{A})} \|u^*[Tx] - T(u^*x)\| \leq \varepsilon,$$

hence $\|R - T\| \leq \varepsilon$. Moreover, if T is a δ-left centralizer, then for any $x \in \mathbf{B}(X)$ and $a \in \mathbf{B}(\mathcal{A})$, we have

$$\begin{aligned} \|R(xa) - [Rx]a\| &= \Big\| \int_{\mathcal{U}(\mathcal{A})} \big(uT(u^*xa) - u(Tu^*x)a\big)\, d\mu(u) \Big\| \\ &\leq \sup_{u \in \mathcal{U}(\mathcal{A})} \big\|T(u^*xa) - [T(u^*x)]a\big\| \leq \delta, \end{aligned}$$

hence R is a δ-left centralizer.

As $T - R$ is bounded (in fact, it has norm not exceeding ε), we conclude that T is bounded if and only if R is. Now suppose T is weak* to weak* continuous, and prove that the same holds for R. In other words, we need to show that, for any net (x_α) which converges to x in the weak* topology of X, and any f in the predual $\mathcal{E}_{2*}$, we have $\langle f, Rx_\alpha \rangle \to_\alpha \langle f, Rx \rangle$. From the definition of R,

$$\langle f, Rx_\alpha \rangle = \int_{\mathcal{U}(\mathcal{A})} \big\langle f, u[T(u^*x_\alpha)] \big\rangle\, d\mu(u) = \int_{\mathcal{U}(\mathcal{A})} \big\langle [T_*(fu)]u^*, x_\alpha \big\rangle\, d\mu(u),$$

where T_* is the pre-adjoint operator of T. Clearly

$$\lim_\alpha \big\langle [T_*(fu)]u^*, x_\alpha \big\rangle = \big\langle [T_*(fu)]u^*, x \big\rangle$$

for any u. As computing R involves integration over the compact group $\mathcal{U}(\mathcal{A})$, we conclude that $\langle f, Rx_\alpha \rangle \to \langle f, Rx \rangle$.

Proof of Theorem 22. Find an increasing net of finite dimensional unital subalgebras $\mathcal{A}_i \subset \mathcal{A}$ (i belongs to a certain index set I), so that $\mathcal{A}$ is the weak* closure of $\cup_i \mathcal{A}_i$.

For each i, there exists (by Lemma 24) a left $\mathcal{A}_i$-module map $R_i : X \to Y$ so that $U_i = T - R_i$ has norm not exceeding ε. Now equip $\varepsilon \mathbf{B}(Y)$ with its weak* topology, and consider $\mathcal{F} = \big(\varepsilon \mathbf{B}(Y)\big)^{\mathbf{B}(X)}$, equipped its product topology (this makes $\mathcal{F}$ into a compact set). In $\mathcal{F}$ consider a net ϕ_i, defined via $\phi_i(x) = U_i x$ ($x \in \mathbf{B}(X)$). As $\mathcal{F}$ is compact, the net (ϕ_i) has a convergent subnet – that is, there exists a subnet J so that $(U_j x)_{j \in J}$ converges weak* for every $x \in \mathbf{B}(X)$. Thus, there exists a map $U : X \to Y$ so that $U_j x \to Ux$ weak* for any x. By the uniqueness of the weak* limit, U is linear. Moreover, $\|U\| \leq \limsup_j \|U_j\| \leq \varepsilon$. Now set $R = T - U$, then $Rx = \text{weak}^* - \lim_{j \in J} R_j x$. We have to show that R is a right centralizer – that is, $R(ax) = a[Rx]$ for any $x \in X$ and $a \in \mathcal{A}$. Clearly the above identity holds for $a \in \cup_i \mathcal{A}_i$. To handle the general a, we consider the cases (1) and (2) of our theorem separately.

(1) X and Y are reflexive. For a general $a \in \mathcal{A}$, take a bounded net $(a_\alpha) \subset \cup_i \mathcal{A}_i$ weak*-convergent to a. Then $a_\alpha x$ converges to ax in the weak* (that is, weak) topology of X, hence $R(a_\alpha x) \to R(ax)$ weakly. On the other hand, $a_\alpha [Rx] \to a[Rx]$ weakly.

Now suppose T is a δ-left centralizer. Then, for any i, R_i is a δ-left centralizer. By the weak* continuity of multiplication, the same is true for R.

(2) $\mathcal{A}$ is separably acting, X_* is weakly sequentially complete, and T is weak* to weak* continuous. There exists an increasing sequence of unital subalgebras $(\mathcal{A}_i)_{i \in \mathbb{N}}$, whose union is weak* dense in $\mathcal{A}$. Then $(R_j)_{j \in J}$ as above is also a sequence (R_j arises from $\mathcal{A}_{i_j}$). It is easy to see that R_j is weak* to weak* continuous for any j, in other words, $R_j = R_{j*}^*$. To show that R is weak* to weak* continuous as well, note that, for any $x \in X$ and $f \in Y_*$, the scalars $\langle f, R_j x \rangle = \langle R_{j*} f, x \rangle$ converge to $\langle f, Rx \rangle = \langle R^* f, x \rangle$. By the weak sequential completeness of X_*, $R^* f \in X_*$ (this is the $\sigma(X^*, X)$ limit of $(R_{j*} f) \subset X_*$), hence $R = (R^*|_{X_*})^*$ is weak* to weak* continuous.

Consequently, R is a right centralizer. Indeed, for $b \in \mathcal{A}_i$ we have $b[R_j x] = R_j(bx)$ for $j \in J$ large enough, hence $b[Rx] = R(bx)$. For a generic $a \in \mathcal{A}$, find a sequence $a_j \in \mathcal{A}_{i_j}$ converging to $\mathcal{A}$ in the weak* topology. Then for $x \in X$,

$$a[Rx] = \mathrm{w}^* - \lim_j a_j [Rx] = \mathrm{w}^* - \lim_j R(a_j x) = R\big(\mathrm{w}^* - \lim(a_j x)\big) = R(ax).$$

In a similar manner, we prove that, if T is a δ-left centralizer, then the same is true for R.

The proof of Theorem 23 proceeds along similar lines. The only difference is that Lemma 24 has to be replaced by:

Lemma 25. *Suppose $\mathcal{A}$ is a finite dimensional von Neumann algebra, X, Y are $\mathcal{A}$-bimodules, and $T : X \to Y$ is a ε-centralizer. Then there exists a bimodule map R so that $\|T - R\| \leq \varepsilon$. If T is norm (resp. weak*) continuous, then R has the same property.*

Proof. Once more assume first that $\mathcal{A}$ is finite. For $x \in X$ set

$$Rx = \int_{\mathcal{U}(\mathcal{A})} \int_{\mathcal{U}(\mathcal{A})} u[T(u^* x v^*)]v \, d\mu(u) \, d\mu(v)$$

Note that

$$Tx = \int_{\mathcal{U}(\mathcal{A})} \int_{\mathcal{U}(\mathcal{A})} u\big(u^*[Tx]v^*\big)v \, d\mu(u) \, d\mu(v)$$

hence

$$\|Rx - Tx\| \leq \sup_{u,v \in \mathcal{U}(\mathcal{A})} \big\|u^*[Tx]v^* - T(u^* x v^*)\big\| \leq \varepsilon.$$

which leads to $\|R - T\| \leq \varepsilon$. As in the proof of Lemma 24, we can check that for any unitaries $a, b \in \mathcal{A}$, we have $a[Rx]b = R(axb)$, hence R is an $\mathcal{A}$-bimodule map. The rest of the proof follows the lines of Lemma 24 as well.

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