February 13, 2018  Noncommutative McKay correspondence [pre-talk]

(11UC-AG seminar)

A lot of what I do generalizes results in commutative algebra in Algebraic Geometry to the noncommutative/quantum setting.

**Classic Setting**

affine space $\mathbb{A}^n_c$

\[ \downarrow \]

coordinate ring of $\mathbb{A}^n_c$

= $\mathbb{C}[v_1, \ldots, v_n]$


**Noncommutative/Quantum Setting**

"quantum affine space"

\[ \downarrow \]

noncommutative graded algebra $\mathcal{A}$

that behaves ring-theoretically like $\mathbb{C}[v_1, \ldots, v_n]$

= coordinate ring of $\mathbb{A}^n_q$ space

group, and more generally

actions of Hopf algebras on $\mathcal{A}$

= "quantum symmetries"

parts of $\mathbb{A}^n_q$ remaining

fixed under symmetries

= invariant ring $\mathbb{C}[v_1]^G$

for $G$-action on $\mathbb{C}[v_1]$ 

\[ \text{largely unexplored} \]

= $\mathbb{A}^n_q$ invariant ring

\[ \text{well-studied} \]
In today's talk, the specific aim is to generalize the classic McKay correspondence to the noncommutative setting.

### Classic Setting

Several algebraic, geometric, rep-theoretic notions that correspond to ADE Dynkin diagrams.

- **Type A**: $A_n$ with $n$ vertices.
- **Type D**: $D_n$ with $n$ vertices.
- **Type E**: $E_6$, $E_7$, $E_8$.

#### ADE Diagrams

Let's consider $G = \langle (i,0) \rangle \leq SL_2(\mathbb{C}) = \mathbb{Z}_2$ acting on $A = \langle u, v \rangle \cong (u - v)/(u + v)$, by swapping gens $u \leftrightarrow v$.

This is a noncommutative version of the (finite) group $G$.

---

We'll see that in the noncommutative setting, one gets ADE diagrams + more diagrams.
There are five families of subgroups $G_2(SL_2(C))$
up to isomorphism; they act on $\mathbb{C}^n$;
and were studied extensively by F. Klein, 1884.

J. McKay (1980) used such $G$ to construct a
directed graph (quiver) for which its
isomorphism class corresponds

McKay quiver:

- **Vertices**: Corresponding to set of
  non-isomorphic simple modules of $G$

- **Arrows**: $V_i \rightarrow V_j$
  if $V_j$ is a fundamental
  representation of $G$

- **Fundamental representation**: $V_0 = Cx_0$, $g \cdot x_0 = x_0$
  $V_1 = Cx_1$, $g \cdot x_1 = -x_1$

- **Fundamental group**: $G = \langle g \mid g^2 = 1 \rangle \leq SL_2(C)$

- **Action of $G$**:
  - $V_0 \rightarrow V_0$
  - $V_1 \rightarrow V_1$

- **New in McKay's quiver**:

- **Yields**:
  - New ADE diagram
  - $\text{type } A_1$

- **Galois action**:
  - $G \cdot \mathbb{C}^n$ acts on $\mathbb{C}^n$ via $G$

- **Fundamental rep**: $V_0 \oplus V_1$

- **New arrows**:
  - $V_0 \rightarrow V_1$
  - $V_1 \rightarrow V_0$