November 9, 2018  On the quadratic dual of the Fomin–Kirillov algebras


In the field.

I'm interested in all things related to noncommutative algebra:
- representations
- symmetries
- and, of course, ring-theoretic & homological properties

In particular, I like to study noncomm algebras $A$ that are
- $N$-graded: $A = \bigoplus_{i \in \mathbb{Z}} A_i$, $A_i \cdot A_j \subseteq A_{i+j}$
- connected: $A_0 = k$, $k$-vs (more often $\dim A_i < \infty$)
- generated in degree 1 & quadratic: $A = T(A_1)/(R)$, $R \in A_0 \otimes A$
  
  which includes commutative poly-nil algebras
  & noncomm. graded algebras that behave like them.

So when I stumbled across the following noncomm algebras, which had many open questions attached to them, I was intrigued—

Defn: For $n \geq 2$, the Fomin–Kirillov algebra $[\text{En}]$ is an associative $k$-alg

generated by $\{X_{ij} \mid 1 \leq i < j \leq n\}$ of degree 1, subject to relations:

$X_{ij} = 0 \quad \forall i < j$

$X_{ij} X_{jk} - X_{jk} X_{ik} - X_{ik} X_{ij} = 0 \quad \forall i < j < k$

$X_{ij} X_{kj} - X_{jk} X_{ij} - X_{ij} X_{kj} = 0 \quad \forall i < j < k$

$X_{ij} X_{kl} - X_{kl} X_{ij} = 0 \quad \forall i < j < k < l$

introduced by Sergey Fomin & Anatol Kirillov in 1999 to study

the ordinary & quantum cohomology of flag manifolds. Fl.$n$. 
In particular, there's a nice collection of algebras of $E_n$, "Dunkl elements" that form a commutative subalgebra, say $F_n$, of $E_n$.

Theorem [FK, 97] $F_n$ is canonically $\cong$ to cohomology algebra of $FL_n$.

Soon after, Postnikov resolved a conjecture in [FK, 98] that the commutative algebra of $E_n$ (quantized $E_n$) is $\cong$ to quantum Cohom algebra of $FL_n$.

We'll come back to these commutative algebras at the end of the talk, if time permits. (Have more questions than answers pertaining to this)

Since then, the FK algebras have appeared in several fields: alg. combinatorics, number theory, noncommutative geometry, Hopf algebras, and more.

There are also several unresolved questions about their fundamental structure; the main one being:

Q: Is $E_n$ finite dimensional (or a $k$-algebra)?

A: $\dim_k E_n = 2^{n^2}$

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<th>$n$</th>
<th>$\dim_k E_n$</th>
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<tr>
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| $\infty$ | $?$ | But at least this lead to interesting directions...

At this point I'll present the main results of our work and will then discuss the "what" and "why care" after.
Recall that for a quadratic algebra $A = \frac{T(V)}{(R)}$, $V|_{k-V^o}$, $dV|_{V^o}$, $R|_{k-V^o}$.

Its quadratic dual is a $k$-algebra $A^* = \frac{T(V^*)}{(R^*)}$, $R^* = \text{orthogonal complement of } R = \{ f \in V^o | f(v) = 0 \text{ for } v \in R \}$.

**Example:** $S(V)^* = \Lambda(V)$, $\sqrt{1-k-V}$.

**Example:** Let $y_{ij} = x_{ij}$.

$$E_2 = \frac{k[x_{ij}]}{(x_{ij}^2)} \quad \Rightarrow \quad E_2^* = \frac{k[y_{ij}]}{(y_{ij}^2)}$$

$$E_3 = \frac{k \langle x_{ij}, x_{ij}, x_{ij} \rangle}{(x_{ij}^2, x_{ij}^2, x_{ij}^2)} \quad \Rightarrow \quad E_3^* = \frac{k \langle y_{ij}, y_{ij}, y_{ij} \rangle}{(y_{ij}^2, y_{ij}^2, y_{ij}^2)}$$

$$\begin{pmatrix}
  x_{ij}^2 & x_{ij}^2 & x_{ij}^2 \\
  x_{ij}^2 & x_{ij}^2 & x_{ij}^2 \\
  x_{ij}^2 & x_{ij}^2 & x_{ij}^2
\end{pmatrix}
\begin{pmatrix}
  y_{ij}^2 & y_{ij}^2 & y_{ij}^2 \\
  y_{ij}^2 & y_{ij}^2 & y_{ij}^2 \\
  y_{ij}^2 & y_{ij}^2 & y_{ij}^2
\end{pmatrix}
\begin{pmatrix}
  y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji} \\
  y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji} \\
  y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji} & y_{ij} + y_{ji} + y_{ij} y_{ji}
\end{pmatrix}$$

**Lemma:** By taking $y_{ij} = -y_{ij}$ for $i < j$, we get that in general:

$$E_n^* = \frac{k \langle y_{ij} \rangle_{1 \leq i < j \leq n}}{(y_{ij} y_{jk} + y_{jk} y_{ij} + y_{ij} y_{jk} (\text{if } jk \text{ is distinct}) \text{ or } 0 \text{ if } i = j)}$$

**Main Result (WZ):** The algebras $E_n^*$ satisfy the following conditions:

**Ring-theoretic**

1. Noetherian
2. Module finite over $k$
3. Self-injective over $k$
4. not prime (⇒ not a domain) if $n \geq 3$

**Homological**

5. $A_1$-Auslander-regular $\Leftrightarrow n = 2$
6. $A_3$-Auslander-Gorenstein $\Leftrightarrow n = 2, 3$
7. $A_3$-CM, CM $\Leftrightarrow n = 4, 3$
8. depth $\leq 1$ if $n > 2$

$A_3 = \text{Artin-Schelter}$, $A_3 = \text{Auslander}$
$CM = \text{Cohen-Macaulay}$
Why care about quadratic chains?

In the nice case, when $A = T(V)/(R)$ (connected $N$-graded quadratic) is Koszul $[= \text{the trivial } A\text{-module } k = A/\oplus i \geq 1 A_i \text{ has a linear resolution}]$, we get that $A! = \text{Ext}^*_{A^!}(1, 1) =: E(A^!)$. $A!$ carries a lot of cohomological information about $A$ if vice versa because $(A^!)^! = A$.

But what makes $E_n$ so difficult to study, in this context, is the fact that $E_n$ is not Koszul $[n \geq 3]$ [Roon].

Still, $E_n^!$ is useful cohomologically -

**Fact** For $A$ connected $N$-graded quadratic, not nec. Koszul,

get that $A^! = \bigoplus i \geq 1 \text{Ext}^i_{A^!}(1, 1)$,

the "diagonal" subset of $E(A^!)$ generated in degree 1.

**Loose Fact:** "The homological growth (e.g., gldim) of $E(A^!)$, and of $A^!$, is finite.

see "Koszul equivalence in Aoo-setting" by W. Palmer, W. Win, Zue (2008) for more details.

Speaking of growth, let's discuss ring-Theoretic properties of noncomm. (graded) algs -

1. **Noetherian condition** = ACC on left fin. right ideals = 7 buy us a lot
2. $E_n^!$ is a module/$\mathcal{Z}(E_n^!)$ at finite rank = 7 of leverage...
4. A is prime $\iff \forall a, b \in A \text{ get } aA + bA = 0$.
   
   Weaker than domain condition; still desirable.
   
   E.g. matrix rings $M_n(k)$ are not domains, but are prime.
   
   For $n \geq 3$: $y_{ij} \left( y_{ij} - y_{ik} \right) = 0$ for distinct $i, j, k$.
   
   $(a \rightarrow b \rightarrow c, aA = 0)$ (smarter) $\not\rightarrow 0$ (cannot)
   
   Hierarchy: Domain $\Rightarrow$ Prime $\Rightarrow$ Semi-Domain ($\Rightarrow$ nilpotent ideals)

   Question: Is $E_n$ semi-Domain $\forall n \geq 2$?

3. (GK-dimension) If a connected, $N$-graded, locally-finite algebra $A$

   \[ (A = \bigoplus_{0} A_i, \dim A_i < \infty, A_0 = k) \]

   is defined by:

   \[ \text{GKdim}(A) = \limsup_{n \to \infty} \log \left( \sum_{i=0}^{n} \dim A_i \right) \]

   a very useful growth measure...

   * $\text{GKdim}(A) = 0 \iff \dim A < \infty$

   * $\text{GKdim}(k[x_1, \ldots, x_n]) = n$

   * Say $A$ is of polyn. growth if $\text{GKdim}(A) \in \mathbb{Z}^+$

   * A commutative $\Rightarrow$ $\text{GKdim}(A) = \text{Krull dim}(A)$. 
On homological properties of noncommutative graded algebras

This mirrors the hierarchy of nice homological properties of commutative local algebras $(R, \mathfrak{m})$. Assume $R$ Noetherian, $\text{Kdim}(R) = d < \infty$.

- Regular $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay
  - $\text{gl.dim} R = d < \infty$
  - $\text{inj.dim} R = d < \infty$
  - $\text{depth} = \text{max} \{ \text{length} \}$
  - $\text{depth} = \text{max} \{ \text{length} \}$
  - $\text{deg} \Rightarrow \text{reg} \Rightarrow \text{inj} \Rightarrow \mathfrak{m}^d \neq 0$
  - $\text{Ext}^i_R(\mathfrak{m}, R) \neq 0$, $i = d$

Then analogous hierarchy for noncommutative graded algebras $A = \bigoplus_{i \geq 0} A_i$

Take $\mathfrak{m} = \bigoplus_{i > 0} A_i$ (augmentation ideal). Assume $A$ Noetherian, $\text{Kdim}(A) = d < \infty$

- AS-regular $\Rightarrow$ AS-Gorenstein $\Rightarrow$ AS-CM
  - $\text{gl.dim} A = d < \infty$
  - $\text{inj.dim} A = d < \infty$
  - $\text{CM}$
  - Local coh. is concentrated in one degree
  - $\text{Coh.dim}(A) = 0$ unless $c = d$

Here, for $N \in A$-mod:

- $\text{grade} \, jA(N) = \inf \{ i \mid \text{Ext}^i_A(N, A) \neq 0 \}$
- $\text{depth} \, \text{depth}(N) = \inf \{ i \mid \text{Ext}^i_A(R, N) \neq 0 \}$

For all right $A$-modules $M$, $\text{Ext}^i_A(N, A)$
In summary, we understand a lot about $E_n$!

This may help provide insight into the structure of $E_n$.

In the proof of the Main Theorem, there were two important commutative subalgebras $E_n$ that were used:

$$E_n = \text{subalgebra of } E_n \text{ generated by } \{y_{ij} : a_{ij} \text{ is } j \text{th}\}$$

$$E_n = \langle k < a_{ij} > | i < j \text{th} \rangle / \langle (a_{ij} a_{jk} - a_{ij} a_{ik}) \forall i < j < k \rangle$$

With Peter Etingof, we showed that

- $E_n \cong E_n$
- computed its Hilbert series (combinatorial formula)
- showed that it's reduced and thus semiprime
  
- no non-trivial ideals

Question 1: Is $E_n^{\ell} = \text{the comm. subalg.}$?

More crucially—

Question 2: What is the relationship between

- the comm. subalg. $E_n^{\ell}$ of $E_n$.
- the (quantum) cohomology alg. of the flag manifold?
  (which pertains to Fomin-Shapiro-Stanley's original work)