Big Goal: To understand universal quantum groups (UQGs) \( \mathcal{O}_B \) associated to \( B \) (say from \( \mathbb{R} \)) and for Hopf algebras \( H \) that act on \( B \), \( \exists! \) Hopf algebra map \( \Pi : \mathcal{O}_B \to H \). 

Call a Hopf algebra \( \mathcal{O}_B \) a UQG associated to \( B \) if \( \mathcal{O}_B \) coacts on \( B \) (say from \( \mathbb{R} \)). 

If the relations of \( B \) are governed by a form \( X \to \mathbb{R} \), \( \Pi \) can consider UQG \( \mathcal{O}_\Phi \) that preserves this form. 

That is, \( \mathcal{O}_\Phi \) coacts universally on \( X \). 

Towards a 'nicer' UQG....

Late 1980s: Manin constructed a UQG \( \mathcal{O}_{A_1 B} \) associated to a pair of quadratic graded algs \( A = \mathbb{T} W \), \( B = \mathbb{T} V \). Nondeg. pairing \( A_1 \times B_1 \to \mathbb{W} \).
Namely if \( W = \bigoplus_{i} k y_{i} \) and \( V = \bigoplus_{j} k x_{j} \), then \( O_{A/B} \) acts on \( A \) and \( B \) simultaneously and universally as follows:

\[
\begin{align*}
A & \rightarrow O_{A/B} \otimes A \\
B & \rightarrow B \otimes O_{A/B}
\end{align*}
\]

\[
\begin{align*}
y_{i} & \mapsto \sum_{j} a_{ij} y_{j} \\
x_{i} & \mapsto \sum_{j} b_{ij} x_{j}
\end{align*}
\]

Here, \( a_{ij} \) are generators of \( O_{A/B} \).

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**Manin's question to Artin, Schelter, Tate (AST)** [paraphrased]

If \( A \) and \( B \) are homologically nice (\( \mathbb{N} \)-Koszul
- finite global \cdot Gorenstein condition ...) \( \), then is \( O_{A/B} \) also nice?

- ring-theoretically nice (\( \mathbb{N} \)-Gorenstein
- domain
- poly-algebra
- finite global)

(When do \( O_{A/B} \) and \( A \otimes B \) resemble each other algebraically?)

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**AST's answer:** Yes! for \( A = k \langle y_{1}, \ldots, y_{n} \rangle \), \( B = k \langle x_{1}, \ldots, x_{n} \rangle \) skew-polynomial algebra

\[
\begin{align*}
(y_{i} y_{j} = g_{ij} y_{j} y_{i}) \\
(x_{i} x_{j} = b_{ij} x_{j} x_{i})
\end{align*}
\]

for \( q_{(ij)} = \lambda \) for \( i \neq j \) and \( \lambda = -1 \)

Here, \( O_{A/B} \cong M_{A/B}[D^{-1}] \) a central localization of an iterated Ore extension in \( n^{2} \) variables

Central \( D \) is sometimes known as the "quantum determinant"
Although, Manin's question was posed initially for $A$ being 3-dim. subvarieties...

**Refined goal**

Address Manin's question for a wider class of homologically nice algebras: $N$-Koszul Artin-Schelter (AS) regular algebras $B$ of dimension $d$.

- \[ B = k \otimes B_1 \otimes B_2 \otimes \cdots \]
- \[ \text{gldim } B = d \]
- \[ \text{Ext}_B^i(k, B) \cong 0 \text{ for } i > d \]
- \[ A^\text{d-regular} \]

That is: understand the universal quantum symmetry of noncommutative projective spaces.

**Useful Fact [Dubois-Violette]** $N$-Koszul AS regular algebras are "super-potential algebras."

\[ B = B(t, N) = \text{T}V/\langle \partial^m - t \rangle \]

Say $\text{dim} V = n$. Take $m > N > 2$.

- \[ V^m \text{ is mixed superpotential} = \varphi \text{ cyclic elt for } \varphi \in \text{GL}(V) \]
- \[ \partial^m - t = \text{space of } (m-N) \text{ fold left partial derivatives of } t. \]

**Note:** $t$ can be identified with an elt of $(V^*)^m$ or a form: \[ (V^*)^m \rightarrow k \]

**Examples**

- **Chang:** All AS regular algos of dim 2 are $N$-homog. one-relator algos. $B = k[x_1, \ldots, x_n]/(r)$
- $m = N = \deg r$ if the $\mathfrak{g}$-twisted $$ substrate.
As regular algebras are only classified in the Noetherian (finite-rk) case
[by Artin-Schelter, Tate, van den Bergh]

Most generic quadratic family: 3-dim Sklyanin algebras

\[ S(a,b,c) = \mathbb{k}[x,y,z] / \left( \begin{array}{c}
axy + bzy + cxy = \delta_x t \\
a^2x + bzx + cy^2 = \delta_y t \\
a^2y + byx + cx^2 = \delta_z t
\end{array} \right) \]

Here, \( m = 3 \), \( N = 2 \), \( \tau abc = a(axy + yzx + zxy) + b(axy + zyx + yzx) + c(x^3 + y^3 + z^3) \).

Although, \( S(a,b,c) \) evade Groebner basis techniques,
we still think it is possible to study \( S(a_1,b_1,c_1), S(a_2,b_2,c_2) \) ....

First thing's first: Presentation of \( \Theta_{A,B} \).

Question: If \( A, B \) are \( N \)-koszul AS regular, generated in degree 1, does \( \Theta_{A,B} \) have a finite presentation?

Theorem: (CWWJ) Yes! If \( A = TW / I \), \( B = TV / J \) \( N \)-koszul AS reg w/ \( \dim V = \dim W = n \)

Then \( \Theta_{A,B} \) has \( n^2 + 1 \) generators w/ finitely many relations.

The usual generators \((u_{ij})_{i,j=1,...,n}\) \( \delta^A, \delta^B \) where

\( \delta_A \) are the “quantum determinants”

\( \delta_B \) are the quantum counterparts

Not necessarily centralelts

(though central in many cases)
In particular, fix basis of \( V \): \( v_1, \ldots, v_n \) and dual basis \( \theta_1, \ldots, \theta_n \) of \( W = V^* \).

If \( A = A(s, N) \) for \( s \in (V^*)^m \) : \( s : V^m \to k \)
\[
B = B(t, N) \text{ for } t \in V^m \text{ or } t : (V^*)^m \to k
\]

Let \( s \) _{i_1 \ldots i_m} = s(v_{i_1} \otimes \cdots \otimes v_{i_m}) \in k. \\
\( t \) _{i_1 \ldots i_m} = t(\theta_{i_1} \otimes \cdots \otimes \theta_{i_m})

We get that \( \Omega_{A, B} \) is generated by \(( U_{i,j} \) \_ \( i, j = 1 \ldots n, D_A^{\pm 1}, D_B^{\pm 1} \) with relations:

\[
\sum_{m=1}^{n} s_{i_1 \ldots i_m} U_{i_1 j_1} \cdots U_{i_m j_m} = s_{j_1 \ldots j_m} D_A \quad (1) \quad \forall 1 \leq j_1 \ldots j_m \leq n
\]

\[
\sum_{m=1}^{n} t_{i_1 \ldots i_m} U_{j_1 i_1} \cdots U_{j_m i_m} = t_{j_1 \ldots j_m} D_B^{-1} \quad (2) \quad \forall 1 \leq j_1 \ldots j_m \leq n
\]

\[
D_A D_A^{-1} = D_A^{-1} D_A = D_B D_B^{-1} = D_B^{-1} D_B = 1 \text{ on } A_{A, B}
\]

**Comment:** \( \Delta(U_{i,j}) = \sum_{d=1}^{n} u_{i,d} \otimes U_{d,j} \) \( \Delta(D_A^{\pm 1}) = D_A^{\pm 1} \otimes D_A^{\pm 1} \) \( \Delta(D_B^{-1}) = D_B^{-1} \otimes D_B^{-1} \)

**Comment:** \( E(U_{i,j}) = \delta_{i,j} \) \( E(D_A^{\pm 1}) = E(D_B^{-1}) = 1 \)

antipode: \( S(U_{i,j}) = \sum_{d=1}^{n} D_A^{-1} \tilde{s}_{i_1 \ldots i_{m-1}, j_1 \ldots j_{m-1}} U_{j_{m-1} i_{m-1} \ldots j_1 i_1} S_{j_1 \ldots j_{m-1}} \tilde{s}_{i_1 \ldots i_{m-1}, j_1 \ldots j_{m-1}} \quad (3) \text{ if } i_{m-1} j_{m-1} \neq j_1 i_1 \)

\[
= \sum_{i_{m-1} j_{m-1} = j_1 i_1} \tilde{t}_{i_1 \ldots i_{m-1}, j_1 \ldots j_{m-1}} U_{j_{m-1} i_{m-1} \ldots j_1 i_1} \tilde{s}_{j_1 \ldots j_{m-1}} D_B
\]

where \( S, \tilde{t} \) are forms on \((V^*)^m \), \( V^m \) resp so that

\[
\sum_{d=1}^{n} \tilde{s}_{d_1 \ldots d_m, s_1 \ldots s_m} \delta_{d_1 \ldots d_m} = \sum_{e_1 \ldots e_m} \tilde{t}_{e_1 \ldots e_m, e_1 \ldots e_m} \delta_{e_1 \ldots e_m} = S_{d, e}
\]

\[ S(D_A^{\pm 1}) = D_A^{\mp 1}, S(D_B^{-1}) = D_B^{+1} \]
Ex. \( A = k[x,y] \), \( B = k[x,y] \) : get \( O_{A,B} = O(GL_n(k)) \) commutative Hopf algebra. -

where \( D_A = D_B = u_{11} u_{22} - u_{12} u_{21} \)

\( t_{11} = 0, t_{12} = 1 \Rightarrow \tilde{t}_{11} = 0, \tilde{t}_{12} = -1 \)

\( t_{21} = 1, t_{22} = 0 \Rightarrow \tilde{t}_{21} = 1, \tilde{t}_{22} = 0 \)

\( s_{ij} \leftrightarrow \tilde{s}_{ij} \) defined \( \sim \) similarity to

\((\dagger) \) yields the relation: \( u_{11} u_{22} - u_{12} u_{21} = 0 \) for \((j_1, j_2) = (1, 1) \)

\( (i_1, i_2) = (1, 2) \)

\( u_{11} u_{22} - u_{12} u_{21} = D_B^{-1} \)

for \((j_1, j_2) = (1, 2) \)

\( (i_1, i_2) = (1, 2) \)

get usual \& relations of \( O(GL_n(k)) \)

Also get \( \text{for} v \in V = n \)

\( O_{SV^n}, S(V) = O(GL_n(k)) \)

by similar computations.

Don't like computations, you say?

Prop [cww] \( O_{A,B} \) is realized as a pushout of the one-sided \( U_qG_0 \) \( O_A, O_B \).

Take \( F = \) free Hopf algebra that coacts on \( V \) universally.

Get \( \xymatrix{ F \ar[r] & O_A \ar[r] & \text{Hopf algebra maps} \ar[d] \ar[r] & O_B } \)

get induced \( \phi \)-coact on \( V \)

from \( \phi \)-coaction on \( V \)

\( O_A, O_B \)

Hopf algebra maps so that \( \xymatrix{ F \ar[r] & O_A \ar[r] & O_{A,B} } \)

& Hopf alg \( \phi \) that coact on \( A \rightarrow B \)

\( \Rightarrow \xymatrix{ F \ar[r] & O_A \ar[r] & O_{A,B} \ar[r] & O_B } \)
Homological co-determinant of the $A,B$-coaction on $A$ is the $D_A$ $D_B$.

(as defined by Karkamk-Kuzmanovich-Zhang)

In fact:

Theorem [CW] If $B = B(t,N)$ is $N$-Koszul AS-regular, then
the homological co-determinant $D$ of a Hopf algebra coaction on $B$
written as $D = \begin{cases} p(t) = D^t & \text{for right } H\text{-coaction,} \\ p(t) = D \otimes t & \text{for left } H\text{-coaction.} \end{cases}$

* have 'SL' versions / trivial homological co-ideal / trivial quantized versions ($D = 1, H$)

for everything above.

* Our construction recovers several other UQG's in the literature

* 2-sided GL(n)-like UQG of LAST for $A,B$ skew-prime rings
  (a special case of)

* 2-sided GL(n)-like UQG of Takeuchi for $q_{ij} = q$ $q_{ij} = q'$ for $i \neq j$

* 1-sided SL(2)-like UQG of Dubois-Violette & Laurene (or 2-sided)
  (a generalization of...)

* 2-sided GL(2)-like UQG of Mozejko

* 1-sided SL(n)-like UQG of Bichan

Preserving one pre-regular form.
We also highlight genuinely new UQFs associated to $\mathcal{N}$-Koszul algebras $A, B$:

- 3-dim. Sklyanin alg. $n=3$, $d=3$, $\mathcal{N}=2$, $m=3$
- 4-dim. Sklyanin alg. $n=4$, $d=4$, $\mathcal{N}=2$, $m=4$
- Yang-Mills alg. $n \geq 2$, $d=3$, $\mathcal{N}=3$, $m=4$

Next up: Studying the ring-theoretic & homological properties

& (co)representative-theoretic properties of $\mathcal{O}_{A,B}$

(work in progress w/ Chirvani & Wang — )