On Quantum Groups assoc. to non-Noetherian regular alg of dim 2

Joint work w/ Xingting Wang orXiv:1503.09185

Motivation: Classic Symmetry (coordinate)
Symmetry of geometric objects is governed by group actions

\[ G \text{ action} \rightarrow X \text{ space/varietys} \]
\[ G \text{ group} \]
\[ M \text{ manifold} \]

√dualize

\[ O(6) \rightarrow O(X) \]
\[ \mathcal{C}(G) \rightarrow \mathcal{C}(M) \]

Commutative Hopf algebra
Functionals on object

Quantum Symmetry

Can't see object

√dualizing works well

\[ H \text{ coaction} \rightarrow A \]

Not necessarily commutative Hopf algebra

---what we aim to study---

A Hopf algebra (or quantum group) \( H \) is

- an associative algebra

\[ (H, \mu, 1) \text{ with } \mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) \]

- a coassociative coalgebra

\[ (H, \Delta, \varepsilon) \text{ with } (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \]

- with antipode \( S: H \rightarrow H \)

"inverse"

== Subject to compatibility conditions.==
$H$ acts on an algebra $A$ if $A$ is an $H$-comodule algebra:

- $A$ is an $H$-comodule with structure map $p: A \to A \otimes H$
- $m_A$ (mult) and $u_A$ (unit) are $H$-comodule maps

In other words, $A$ is an algebra in the category of $H$-comodules.
Given an algebra $A$ say a universal quantum group $Q_A$ that acts on $A$ preserving the grading of $A$.

If $Q(A)$ is a Hopf algebra so that:

$$A \otimes Q(A)$$

$$\xrightarrow{\text{id} \otimes \eta}$$

$$\xrightarrow{\text{diag}}$$

$$A \otimes H$$

$\Phi$ $H$-coactions on $A$.

$Q(A)$ is tough to study, so let's focus on a special case $Q_A(\text{GL})$, introduced by Manin for graded quadratic algebras (1988).

**Examples**

$$A$$

$q_{ij} = J$

$q_q = \frac{q^{ij} - q^{ji}}{q^{ij} - q^{ji}}$ (Jordanian deformation)

$q_q = \frac{q^{ij} q^{ji}}{(q^{ij} q^{ji} - q^{ij} q^{ji})}$ (1-parameter deformation)

$Q_A(\text{GL})$, with central "determinant" (discussed later)

$Q_A(\text{SL})$, with trivial "determinant"

$Q_A(\text{GL}(k))$, with trivial "determinant" (algebraic groups that act on $k^n$)

$Q_A(\text{GL}(k))$, with central "determinant" (algebraic groups that act on $k^n$)

$Q_A(\text{SL}(k))$, with trivial "determinant" (algebraic groups that act on $k^n$)

familiar quantum algebras
- Artin-Schelter reg. algos of dimension 2
  (cf. Toby's course)

familiar Hopf algebras (presentation later...)
(Just in case this is not yet defined in Toby's course.)

An [Artin-Schelter (AS) regular algebra of dimension \(d\)] is a connected graded \(k\)-algebra \(A = k \oplus A_1 \oplus A_2 \oplus \cdots\) of global dimension \(d \leq \infty\) that is AS-Gorenstein \([\text{Ext}_A^d(k,A) = k] \oplus \cdots\)
sharing nice homological properties with \(k[u_1, u_2, \ldots, u_d]\).

In fact, all of the (hopf) algebras above are AS-regular.
* Noetherian, domain, have polynomial growth. * Nice ring-theoretic properties

**PHILOSOPHY**

THE UNIV QUANTUM LINEAR GROUPS \(\mathcal{O}_q(GL)\).

SHOULD SHARE THE SAME RING-THEORETIC & HOMOLOGICAL PROPERTIES OF THE \(A\) COMODULE ALGEBRA A

(With additional conditions?)

Verified for many classes of Noetherian AS regular algebras A
(q of dimension 2, skew polynomial rings, etc.)

We investigate the case when \(A\) is not necessarily Noetherian.

First, with global dimension 2:

[\text{Zhang}] The AS regular algebras of global dimension 2 are

\[ A(\underline{e}) = A(n, \underline{e}) = k(u_1, \ldots, u_n) / \left( \sum_{i=1}^n e_i u_i u_i^* \right) \]

\[ (\underline{e}) = (e_i) \in GL(k) \]

\(\underline{e}\) = \(e_{ij}\)

\(e_{ij}\) is a matrix.

eg. \(A((0,1)) = k[u, v] - A((1,0)) = k[u, v] - A((1,1)) = k[u, v] - A((0,0)) = k[u, v]\).
We consider Manin's QGs \( O_A(\mathbb{C}) \) (GL) in two special cases:

- with central quantum determinant
- with squared antipode of finite order

\[
\mathfrak{o}(E' \otimes E) = O_A(\mathbb{C}) (GL) \quad \text{a little tough to compute}
\]

which includes trivial determinant

\[
\mathfrak{o}(E^{-1}) = O_A(\mathbb{C}) (SL)
\]

e.g. is the univ. Hopf alg. that acts on \( A(\mathbb{C}) \) with trivial det.

\[
\mathfrak{o}(E' \otimes E') = O_A(\mathbb{C}) (GL / S^2)
\]

e.g. is the univ. involutive Hopf alg. that acts on \( A(\mathbb{C}) \)

Usually the determinant is tricky to compute for Hopf actions, but it's easy for Hopf actions on \( A(\mathbb{C}) \).

[Chen-Kirkman-W-Zhang] The determinant of an H-coaction on \( A(\mathbb{C}) \) = \( k \langle v \rangle / (r) \) is the (group-like) element \( D \in H \) such that \( p : A \to A \otimes H \)

\[
r \mapsto r \otimes r^{-1}
\]

\( D \) is central of \( F, hJ = 0 \) in \( H \). \( D \) is trivial if \( h = 1 \).

Advantages to special cases:

1. There are "homological identities" that relate the two cases.
   - see work of Chen-W-Zhang, Reyes-Nagasaki-Zhang,
     \([20, 452, \text{Theorem 8.1}] \quad (\text{conj } g_0 \otimes s^2 = (\text{conj } \text{transp } g_0 \otimes \text{Nakayama}) \text{ trivial when } A = \mathbb{C}g \)

2. Have a nice quantization of \( O_A(\mathbb{C}) \) (\( \tau \)) in these cases.

(more amenable to computations)
[Mrozinski (2014)] Take $E, F \in \text{GL}(k)$, for $n \geq 2$.

Take $[G(E, F)]$ to be the Hopf algebra generated by $A = (a_{ij})$, $D^\pm$

with relations:

$AE^\dagger A^\top E = D I = IF A^\dagger I F^{-1} A$, \quad $DD^{-1} = D^{-1} D = 1$

(see paper for coalgebra structure & antipode).

Get that $G(E^{-1}, F)$ coacts on $A(E)$ with determinant $D^{-1}$.

[Dubois-Violette and Lavoie (1990)] $B(E) = G(E E^{-1}) / (D - 1)$

(Notice: the dates; this is “quantum group of a nondegenerate form”)

Get that $B(E^{-1})$ coacts on $A(E)$ with determinant 1.

Now we show that (see previous page).

Computed $Q_{A(E)}(\text{GL})$, $Q_{A(E)}(\text{SL})$, $Q_{A(E)}(\text{GL}/\text{SL})$ explicitly for $E \in \text{GL}_2(k)$ at end of Section 2 of paper.

Ex. for $A = k[I_{2 \times 2}]$:

$Q_{k[I_{2 \times 2}]}(\text{GL}_2(k))$, $Q_{k[I_{2 \times 2}]}(\text{SL}_2(k))$, $Q_{k[I_{2 \times 2}]}(\text{GL}_2(k)/\text{SL}_2(k))$

(Takeuchi's two parameter deformation of $O(\text{GL}_2(k))$)

Now we have the following results and questions!
Results & Questions

E \in \text{GL}(k(E))

\text{Properties of } A = A(\text{K(E)})

\text{Our Results}

\begin{align*}
\text{OA}(SL) & \quad \text{OA}^c(\text{GL}) + \text{OA}(\text{GL}/S^2)
\end{align*}

\text{Questions}

\begin{align*}
\text{OA}^c(\text{GL}) & \quad \text{OA}(\text{GL}/S^2)
\end{align*}

\text{Global dimension 2}

\text{As Gorenstein}

\text{Homologically smooth of dim 2}

\begin{align*}
\text{A has minimal project resolution in A^e\otimes A^e of length 2.} \\
\text{A is rigid.
\text{Gorenstein}}
\end{align*}

\text{Calabi-Yau [ } \nu \text{ is even, } \nu \text{ is skew symmetric}

\text{Koszul}

\text{Associated graded Koszul ???}

\text{Noetherian } \Leftrightarrow n = 2

\text{finite Gelfand-Kirillov dim. } \Leftrightarrow n = 2

\text{domain}

\text{graded coherent}

\text{Hilb series } 1 - nt + it^2

\text{Questions}

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\text{Comments}

\text{Results based on work of Bichon:}

\text{he produced a G. Drinfeld resolution of the smash of } B(k(E))

\text{probably need an equally nice resolution of the counit } \alpha_k \text{ to get started...}
... Speaking of choice of \( \mathbf{E} \).

We are able to do computations because of the following results:

[Chang, Bichon, Kroizinski] For \( \mathbf{E} \in \text{GL}(k) \):

1. \( \mathbf{A}([\mathbf{E}]) \cong \mathbf{A}(\mathbf{P}^T \mathbf{E} \mathbf{P}) \) as algebras
2. \( \mathbf{B}([\mathbf{E}]) \cong \mathbf{B}(\mathbf{P}^T \mathbf{E} \mathbf{P}) \) as Hopf algebras.
3. If \( \mathbf{E}^T \mathbf{F} \mathbf{E} \mathbf{F}^T = \mathbf{A} \mathbf{I} \mathbf{I} \mathbf{F} \mathbf{A} \mathbf{F}^T \) for some \( \mathbf{A} \in \mathbb{E}^k \), then
   \[ \mathbf{A}(\mathbf{E}, \mathbf{E}) \cong \mathbf{A}(\mathbf{P}^T \mathbf{E} \mathbf{P}, \mathbf{P}^T \mathbf{E} \mathbf{P} \mathbf{E} \mathbf{P}) \cong \mathbf{A}(\mathbf{P}^T \mathbf{E} \mathbf{F} \mathbf{E}^{-1} \mathbf{P}, \mathbf{P}^{-1} \mathbf{E} \mathbf{F}^{-1} \mathbf{P} \mathbf{E} \mathbf{F}^{-1} \mathbf{P} \mathbf{E} \mathbf{F}^{-1} \mathbf{P}) \]
   as Hopf algebras.

That is, can replace \( \mathbf{E} \) with a matrix congruent to \( \mathbf{E} \).

[Horn-Seegerhale] Each \( \mathbf{E} \in \text{GL}(k) \) is congruent to a direct sum, uniquely determined up to permutation of summands, of the following matrices.

\[
\mathbf{J}_n = \begin{pmatrix}
0 & 1 & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

"Jordan-type"

\[
\mathbf{D}_2(q) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

"q-type"

E.g. \( n = 2 \) \( \mathbf{E} \) congruent to

\[ \mathbf{J}_1 \oplus \mathbf{J}_1 = (0, 1), \quad \mathbf{J}_2 = (0, -1), \quad \mathbf{D}_2(q) = (q, 0, 0)
\]

\( q \in k^{*}, -1 \).

For \( k = \mathbb{F}_2 \), \( k = \mathbb{F}_q \) \( 1 \). For \( k = \mathbb{F}_q \) the AS algebras up to isomorphism.
Over the other hand, have results on when Hopf coactions on a regular algebra factor through coactions of coassociative Hopf algebras.

An H-coaction on an algebra \( R \) is \textbf{faithful} if the coaction does not factor through the coaction of a proper Hopf subalgebra \( H' \neq H \).

That is, if Hopf subalgebra \( H' \leq H \) with

\[
\begin{array}{ccc}
R & \xrightarrow{\phi H} & R \otimes H' \\
\phi H & \downarrow & \downarrow \\
& & R \otimes H'
\end{array}
\]

being a commutative diagram.

Theorem \( R = A \), Koszul. An regular alg., with \( \dim_H R = n \).

\( H = \) Hopf alg. with antipode of finite order, coacting on \( R \) inner faithfully with determinant 1.

\( M = \) matrix corresponding to Nakagama automorphism of \( R \) \( (x, y) \mapsto yx \).

\( c(M) = \{ \text{IP } \in \text{Mat}_n(k) \mid \text{IP} = \text{IP} J \} \) (centralizer of \( c(M) \) ), subalg. of \text{Mat}_n(k).

\( c_p(M) = \bigcup_{i=1}^n c(M_i) \) (power centralizer of \( c(M) \)).

If either

1. \( c(M) \) commutative, \( H \) involutory, \( D \) central, or
2. \( c(M) \) commutative, \( D^m \) central for some \( m > 1 \),

then \( H \) is cocommutative.

\vspace{1cm}

\textbf{Corollary} Take \( R = A(E) \) with \( E \) generic (as specified in paper), \( H \) as above.

If (1) \( H \) involutory, \( D \) central, or (2) \( D^m \) central for some \( m > 1 \), then \( H \) is cocommutative.

\vspace{1cm}

Next: coactions on higher-dimensional regular algebras.