What does a random 3-manifold look like?

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slides and references at:
http://dunfield.info/preprints/

Pick at random:

$$\left\{\text{Connected closed orientable 3-manifolds}\right\}$$

What does this actually mean?

A point $(a, b) \in \mathbb{Z}^2$ has $\gcd(a, b) = 1$ with probability $\frac{6}{\pi^2} \approx 0.608$.

A random trivalent graph is connected with probability 1; the mean number of loops is also 1.

Random Heegaard splittings: Fix g and generators S for $\mathrm{MCG}(\Sigma_g)$. A random 3-manifold of Heegaard genus g and complexity N is

$$M = \text{HeegaardSplitting}(\phi)$$

where $\phi \in \mathrm{MCG}(\Sigma_g)$ is a randomly chosen word in S of length N.

[Dunfield-W. Thurston] As $N \to \infty$, the probability that $b_1(M) > 0$ tends to 0.

[Maher] As $N \to \infty$, the probability that M is hyperbolic tends to 1.

Limits as $g \to \infty$ often exist:

[Dunfield-W. Thurston]

$$\mathrm{Prob}\left\{ \dim H_1(M; \mathbb{F}_p) = 0 \right\} = \prod_{k=1}^{\infty} \frac{1}{1 + p^{-k}}$$

For $p = 2$ this is ≈ 0.419422.

The number of surjections of $\pi_1(M)$ onto a finite simple group Q is Poisson distributed with mean $\left|H_2(Q; \mathbb{Z})\right| \Big/ \left|\mathrm{Out}(Q)\right|$.

[Dunfield-Wong] Let Z be the $SO(3)$ TQFT of prime level $r \geqslant 5$. Then

$$\mathrm{Prob}\left\{ \left|Z(M)\right| \geqslant x \right\} = e^{-x^2}$$

Meta Problem 1: How is your favorite invariant distributed for a random 3-manifold (or random knot, link, etc.)? Experiment should be your friend here!

Meta Problem 2: Prove a conjecture holds with positive probability.

Conj. A random 3-manifold is not an L-space, has left-orderable π_1, has a taut foliation, and has a tight contact structure.

Probabilistic method: Prove existence by showing at a random object has the desired property.

[Lubotzky-Maher-Wu 2014] For all $k \in \mathbb{Z}$ and $g \geqslant 2$ there exists an $\mathbb{Z}$HS with Casson invariant k and Heegaard genus g.

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