Computational complexity of problems in 3-dimensional topology

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slides at: http://dunfield.info/preprints/

[Novikov 1962] *For $n \geqslant 5$, there does not exist an algorithm which solves:*

IsSphere: Given a triangulated M^n is it homeomorphic to S^n?

Thm (Geometrization + many results) *There is an algorithm to decide if two compact 3-mflds are homeomorphic.*

Today: How hard are these 3-manifold questions? How quickly can we solve them?

Decision Problems: Yes or no answer.

SORTED: Given a list of integers, is it sorted?

SAT: Given $p_1, \ldots p_n \in \mathbb{F}_2[x_1, \ldots, x_k]$ is there $\mathbf{x} \in \mathbb{F}_2^k$ with $p_i(\mathbf{x}) = 0$ for all i?

UNKNOTTED: Given a planar diagram for K in S^3 is K the unknot?

INVERTIBLE: Given $A \in M_n(\mathbb{Z})$ does it have an inverse in $M_n(\mathbb{Z})$?

P: Decision problems which can be solved in polynomial time in the input size.

SORTED: $O(\text{length of list})$

INVERTIBLE: $O\left(n^{3.5} \log(\text{largest entry})^{1.1}\right)$

NP: Yes answers have proofs that can be checked in polynomial time.

SAT: Given $\mathbf{x} \in \mathbb{F}_2^k$, can check all $p_i(\mathbf{x}) = 0$ in linear time.

UNKNOTTED: A diagram of the unknot with c crossings can be unknotted in $O(c^{11})$ Reidemeister moves. [Lackenby 2013]

coNP: No answers can be checked in polynomial time.

UNKNOTTED: Yes, assuming the GRH [Kuperberg 2011].

Conj: UNKNOTTED is in **P**.

KnotGenus: Given a triangulation T, a knot $K \subset T^{(1)}$, and a $g \in \mathbb{Z}_{\geqslant 0}$, does K bound an orientable surface of genus $\leqslant g$?

[Agol-Hass-W.Thurston 2006]
KnotGenus is **NP**-complete.

Conj (AHT) If $b_1(T) = 0$, then **KnotGenus** is in **coNP**.

Is **KnotGenus** in **P** when $b_1 = 0$?

Is the homeomorphism problem for 3-manifolds in **NP**?

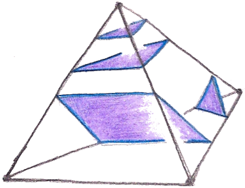
What about deciding hyperbolicity? or being an L-space?

[AHT] KnotArea is **NP**-complete.

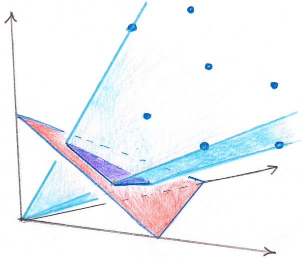
[Dunfield-Hirani 2011] KnotArea is in **P** when $b_1(T) = 0$.

Computing Khovanov homology and $\widehat{HFK}$ are in **EXPTIME**. Just computing the Jones polynomial is **#P**-hard, but the Alexander polynomial can computed in poly time.

Normal surfaces meet each tetrahedra in a standard way:



and correspond to lattice points in a finite polyhedral cone in $\mathbb{R}^{7t}$ where $t = \#T$:



[Haken 1961] There is a minimal genus surface bounding K in normal form whose vector is fundamental (e.g. on a vertex ray). Hence **KnotGenus** is decidable.

[Hass-Lagarias-Pippenger 1999]
A fund surface has coordinates $O\left(\exp t^2\right)$.

[AHT 2006] KnotGenus is in **NP**.

Certificate: A vector $\mathbf{x}$ in $\mathbb{Z}^{7t}$ with entries with a most $O(t^2)$ digits.

Check:

(1) That $\mathbf{x}$ represents a normal surface S.

(2) That $\chi(S) \leqslant 1 - 2g$.

(3) That S connected and orientable.

(4) That ∂S is as advertised.

All can be done in time polynomial in t but need a very clever idea for (3) and (4).

[Kuperberg 2011] Assuming GRH, **UNKNOTTING** is in **coNP**.

Certificate: $\rho\colon\ \pi_1(S^3 - K) \to SL_2\mathbb{F}_p$ where $\log p$ is $O\big(\text{poly}(\text{crossings})\big)$.

Check: The following imply π_1 is not cyclic and so K is knotted.

(1) Relators for π_1 hold, so ρ is a rep.

(2) A pair of generators have noncommuting images.

Proof that such a rep exists uses algebraic geometry/number theory and:

[Kronheimer-Mrowka 2004]
When $K \subset S^3$ is nontrivial, there is a rep $\pi_1(S^3 - K) \to SU_2$ with nonabelian image.

"In theory, there is no difference between theory and practice. But, in practice, there is."

–Jan L. A. van de Snepscheut

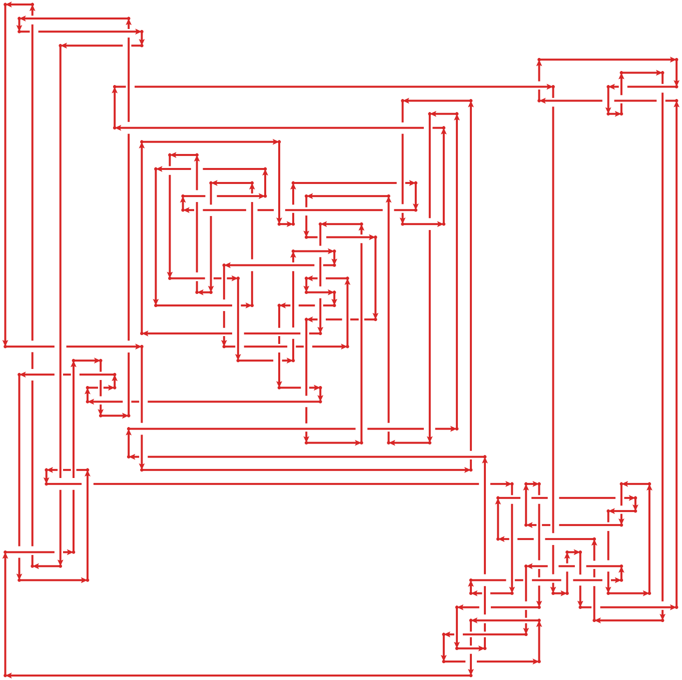
Mystery: In practice, many 3-mfld questions are easier than the best theoretical bounds indicate.

Q: How big a knot can we compute the genus for?

Q: Where do we even get big knots from?

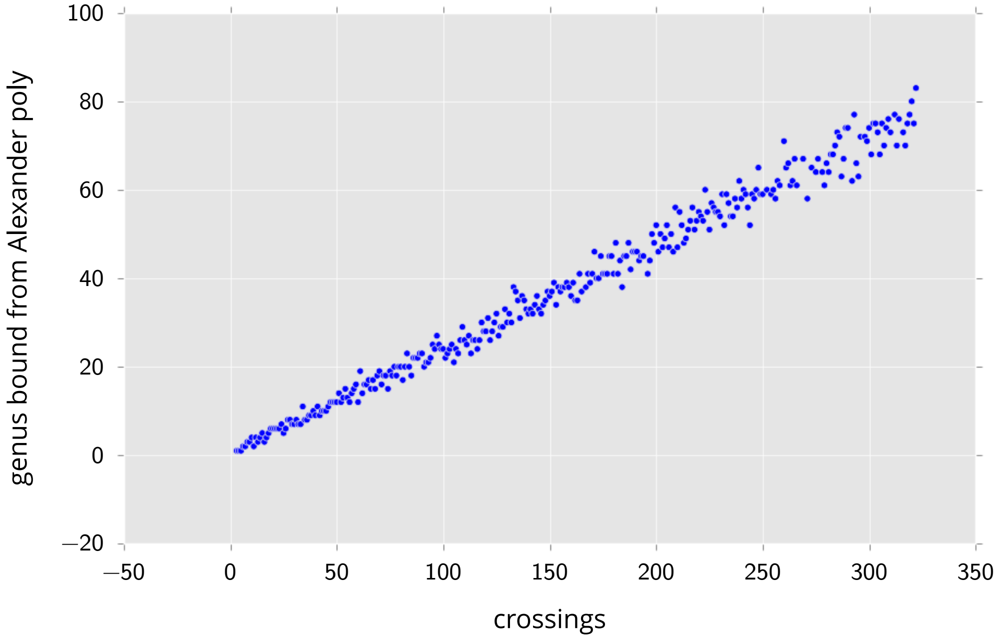
There are more 100 crossing prime knots than there are atoms in the Earth!

Here's a sneak peak of joint work with Malik Obeidin, based on one natural model of random knot.

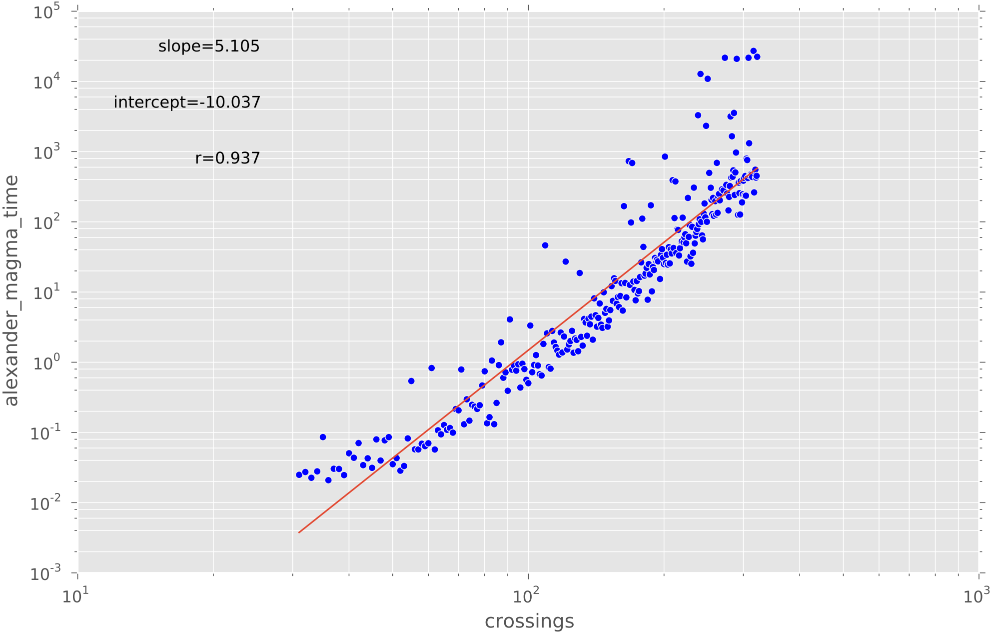


Personal best:

crossings: 126
genus: 27
fibered: No
time: 7 minutes
hyperbolic volume:
223.6132847441086613
tetrahedra: 243



100
80
60
40
20
0
−20
−50
0
50
100
150
200
250
300
350
genus bound from Alexander poly
crossings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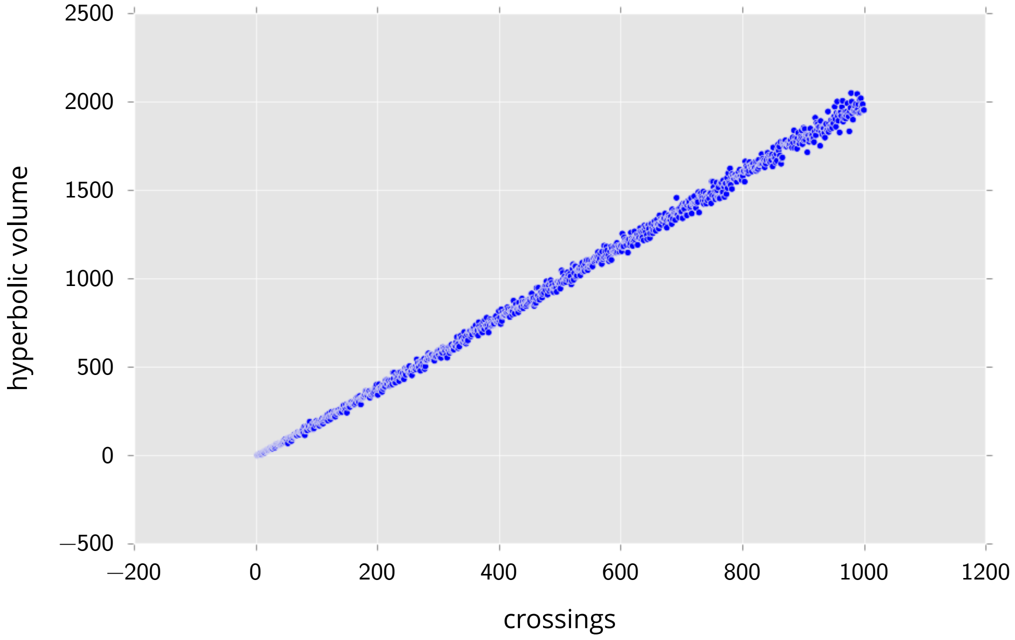
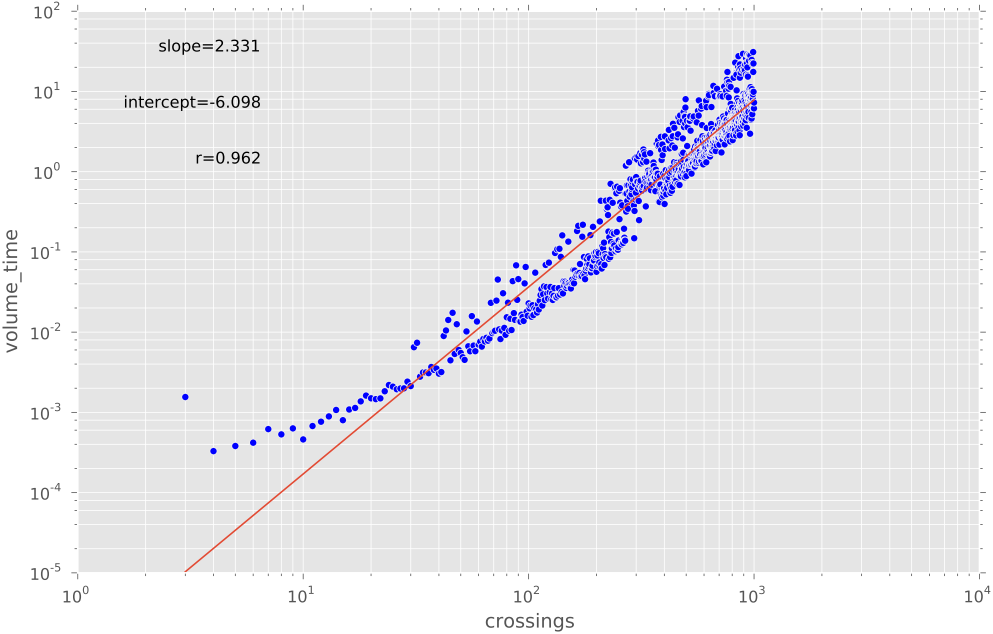
intercept=-10.037
r=0.937
alexander_magma_time
crossings
10^5
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^1
10^2
10^3



2500
2000
1500
1000
500
0
−500
−200
0
200
400
600
800
1000
1200
crossings
hyperbolic volume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intercept=-6.098
r=0.962
volume_time
crossings
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}