The Least Spanning Area of a Knot and the Optimal Bounding Chain Problem

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Hyamfest, July 2011

Based on arXiv:1012.303

Slides available at http://dunfield.info

Knot: A smooth embedding of S^1 in a closed orientable 3-manifold Y.

Spanning surface: If $K = 0$ in $H_1(Y;\mathbb{Z})$, it is the boundary of an orientable embedded surface S.

Problem: Find the least genus $g(K)$ of such an S.

In the 1960s, Haken used normal surfaces to give an algorithm to compute $g(K)$. Here, Y is given as a simplicial complex $\mathcal{T}$, and K is a loop of edges in $\mathcal{T}^1$.

Knot Genus: *Given* $K \subset \mathcal{T}^1$ *and* $g_0 \in \mathbb{N}$*, is* $g(K) \leq g_0$*?*

Agol-Hass-Thurston (2002)

Knot Genus is ***NP****-complete.*

Decidable

Exp. time Is $\dim\left(Kh_*(K)\right) \leq 10$?

NP

Is there a hamiltonian cycle?

Are two graphs isomorphic?

Traveling salesman

P Polynomial time

Is a list sorted?

Is Δ_K monic?

Word prob. in a hyp. group

When Y is simple, e.g. S^3, then Knot Genus should be in **NP** $\cap$ **co-NP**, and might even be in **P**.

Least area: Y Riemannian, K null-homologous. By geometric measure theory, there exists a spanning surface of least area.

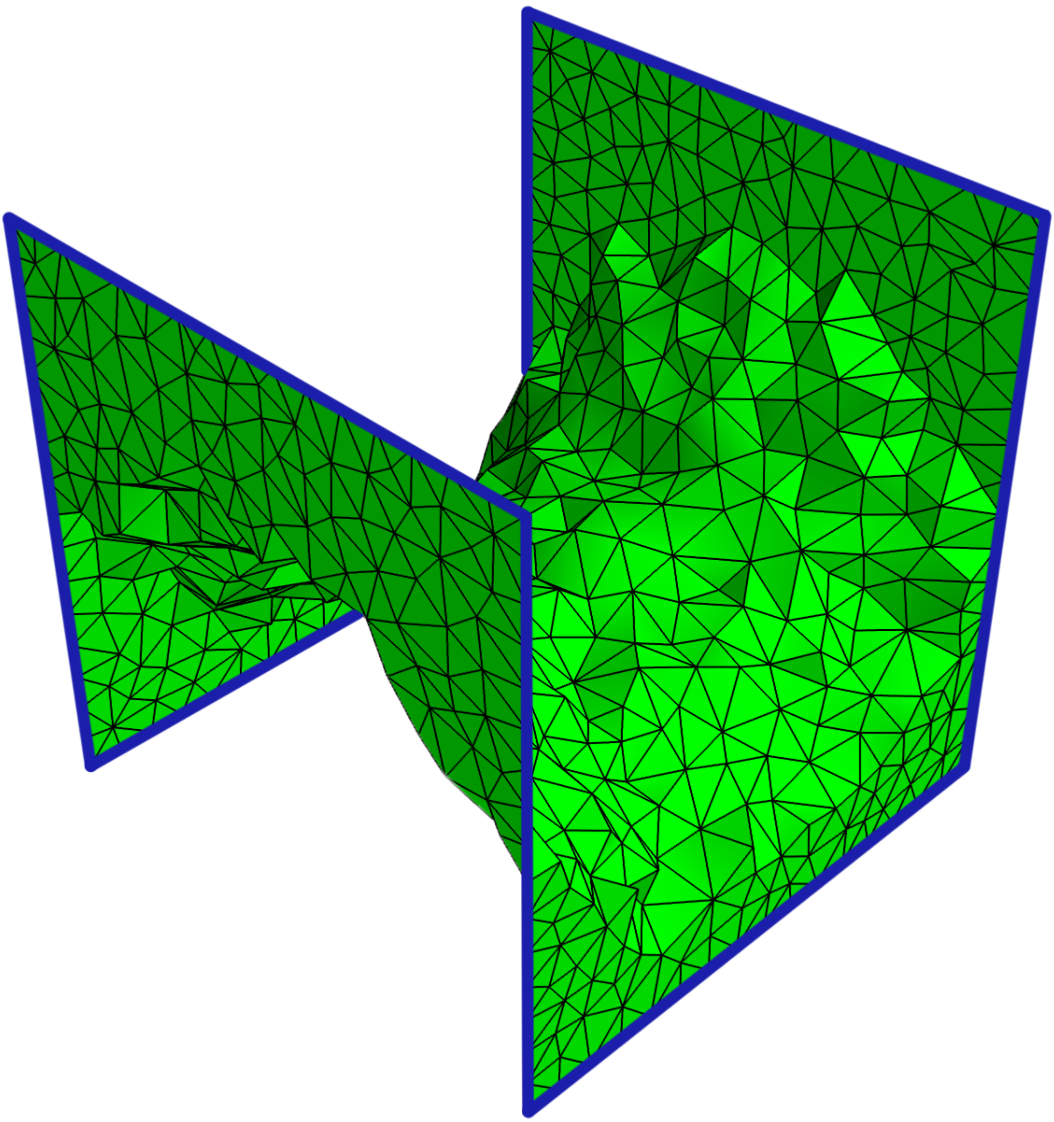
Discrete version: Assign each 2-simplex in $\mathcal{T}$ an area (in $\mathbb{N}$), consider spanning surfaces "built out of" 2-simplices of $\mathcal{T}$.

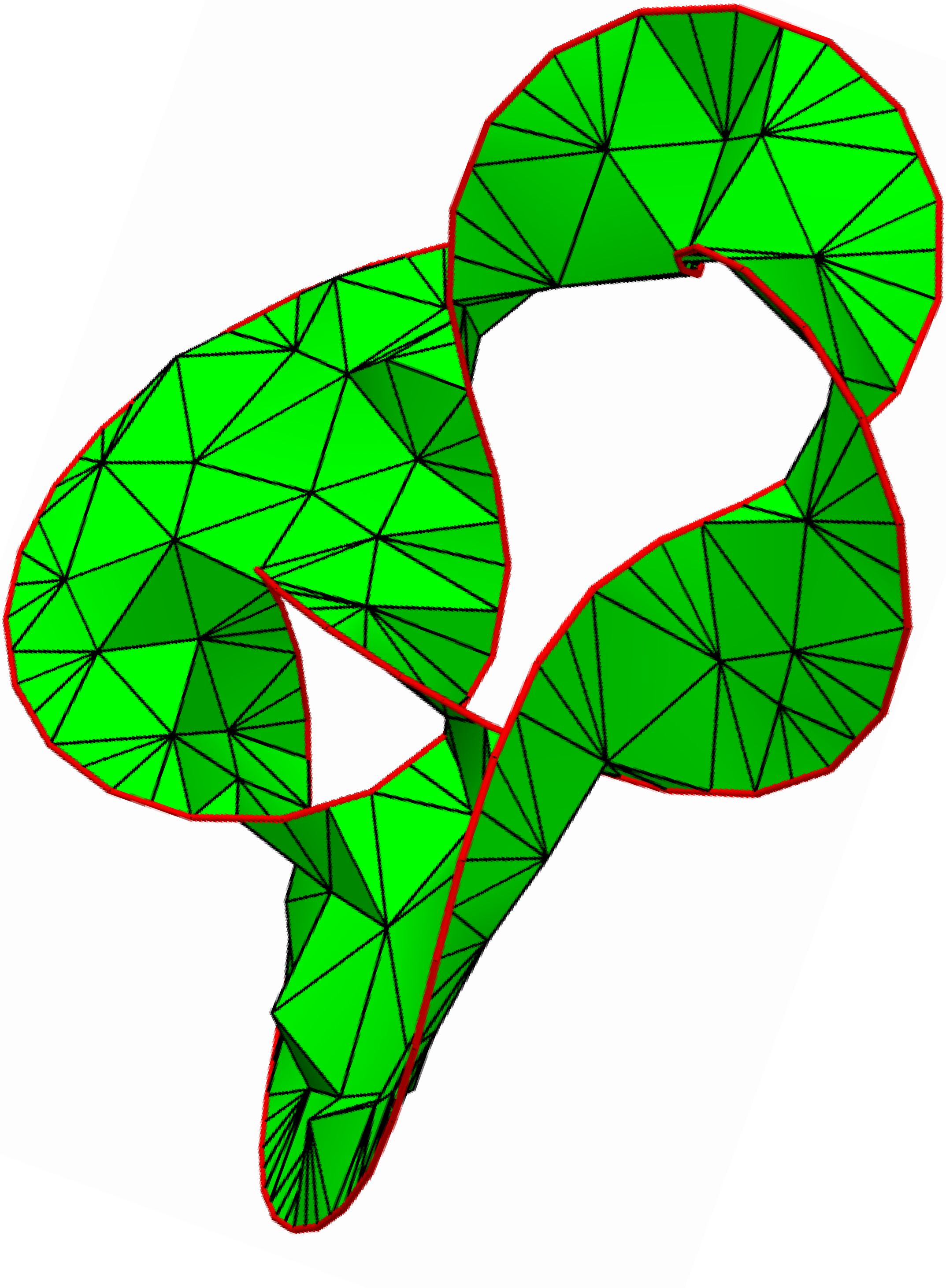
Least Spanning Area: *Given* $K \subset \mathcal{T}^1$ *and* $A_0 \in \mathbb{N}$*, is there a spanning surface with area* $\leq A_0$*?*

Agol-Hass-Thurston (2002)

*Least Spanning Area is **NP**-complete.*

Thm (D-H) *When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*





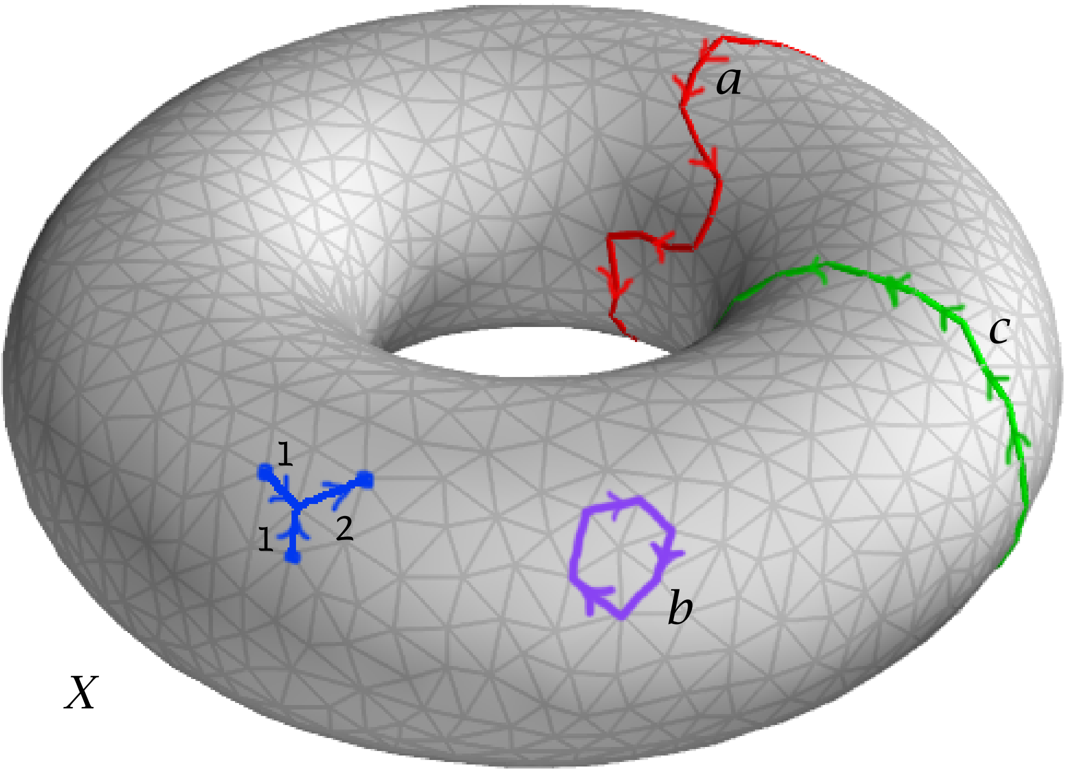
Algorithm uses linear programming.

Thm (D-H) *When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*

Approach:

1. Consider the related Optimal Bounding Chain Problem, where S is a union of 2-simplices of $\mathcal{T}$ but perhaps not a surface.
2. Reduce to an instance of the Optimal Homologous Chain Problem that can be solved in polynomial time. [Dey-H-Krishnamoorthy 2010]
3. Desingularize the result using two topological tools.

Homology: X a finite simplicial complex, with $C_n(X;\mathbb{Z})$ the free abelian group with basis the n-simplices of X.



Boundary map: $\partial_n\colon\ C_n(X;\mathbb{Z}) \to C_{n-1}(X;\mathbb{Z})$

Homology:

$$H_n(X;\mathbb{Z}) = {}^{\ker(\partial_n)}\big/_{\mathrm{image}(\partial_{n+1})}$$

Assign a "volume" to each n-simplex in X, which gives $C_n(X;\mathbb{Z})$ an ℓ^1-norm.

$$\|c\|_1 = \sum |a_i| \mathrm{Vol}(\sigma_i) \quad \text{where } c = \sum a_i \sigma_i$$

Optimal Homologous Chain Problem (OHCP)
Given $a \in C_n(X;\mathbb{Z})$ find $c = a + \partial_{n+1} x$ with $\|c\|_1$ minimal.

Optimal Bounding Chain Problem (OBCP)
Given $b \in C_{n-1}(X;\mathbb{Z})$ which is 0 in $H_{n-1}(X;\mathbb{Z})$, find $c \in C_n(X;\mathbb{Z})$ with $b = \partial_n c$ and $\|c\|_1$ minimal.

Thm (D-H) *OHCP and OBCP are **NP**-hard.*

OHCP with mod 2 coefficients is **NP**-hard by [Chen-Freedman 2010].

Dey-H-Krishnamoorthy (2010) *When X is relatively torsion-free in dimension n, then the OHCP for $C_n(X;\mathbb{Z})$ can be solved in polynomial time.*

Key: Orientable $(n+1)$-manifolds are relatively torsion-free.

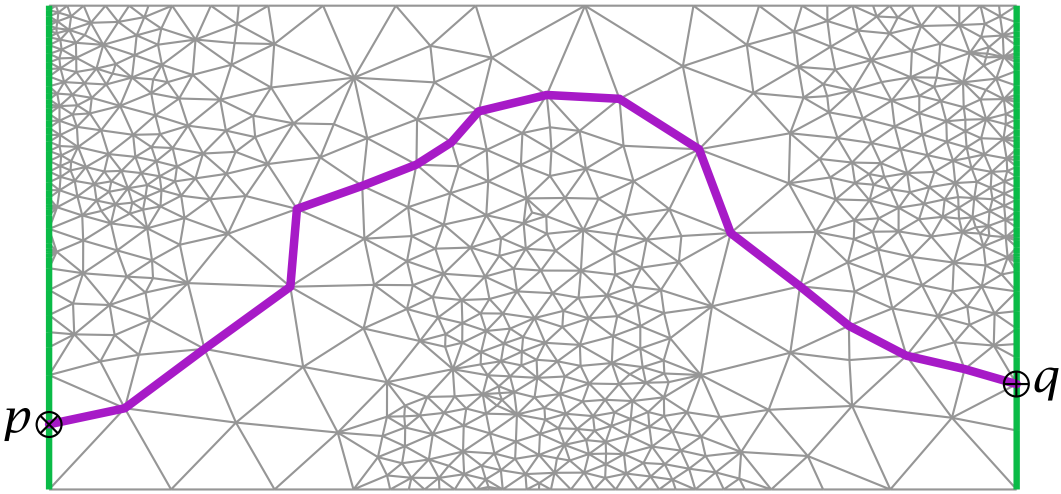
Thm (D-H) *When X is relatively torsion free in dimension n and $H_n(X;\mathbb{Z}) = 0$, then the OBCP for $C_{n-1}(X;\mathbb{Z})$ can be solved in polynomial time.*

Compare

Thm (D-H) *When $H_2(Y;\mathbb{Z}) = 0$, the Least Spanning Area problem for a knot K can be solved in polynomial time.*

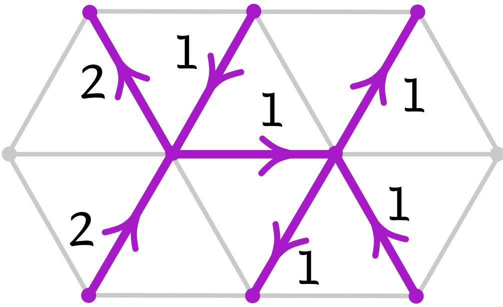
Desingularization: a toy problem

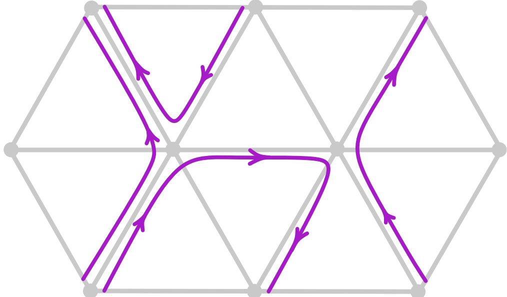
In a triangulated rectangle X, find the shortest embedded path in the 1-skeleton joining vertices p and q.



Consider $b = q - p \in C_0(X;\mathbb{Z})$, which is 0 in $H_0(X;\mathbb{Z})$. Let $c \in C_1(X;\mathbb{Z})$ be a solution to the OBCP for b.

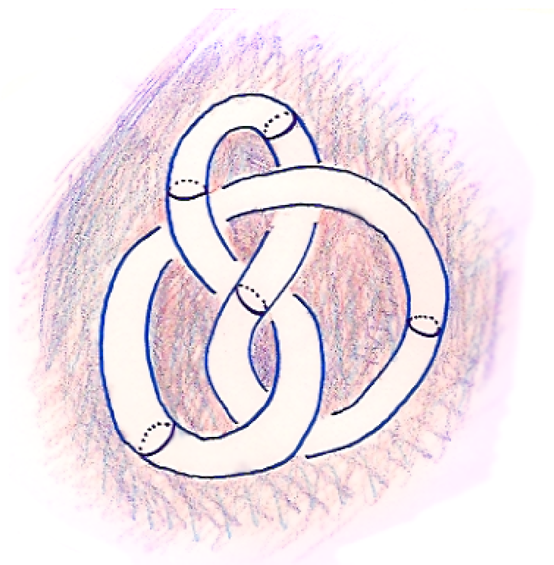
Claim: c corresponds to an embedded simplicial path.





Rest of desingularization

1. Pass to the exterior of the knot K.



2. Introduce a relative version of the Optimal Bounding Chain Problem.