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N. Dunfield, *Volume change under drilling: theory vs. experiment.* Appendix to the paper of Agol, Storm, and W. Thurston, math.DG/0506338

N. Dunfield, S. Gukov, and J. Rasmussen. *The superpolynomial for knot homologies* math.GT/0505662

Does a random tunnel-number one 3-manifold fiber over the circle?

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joint with
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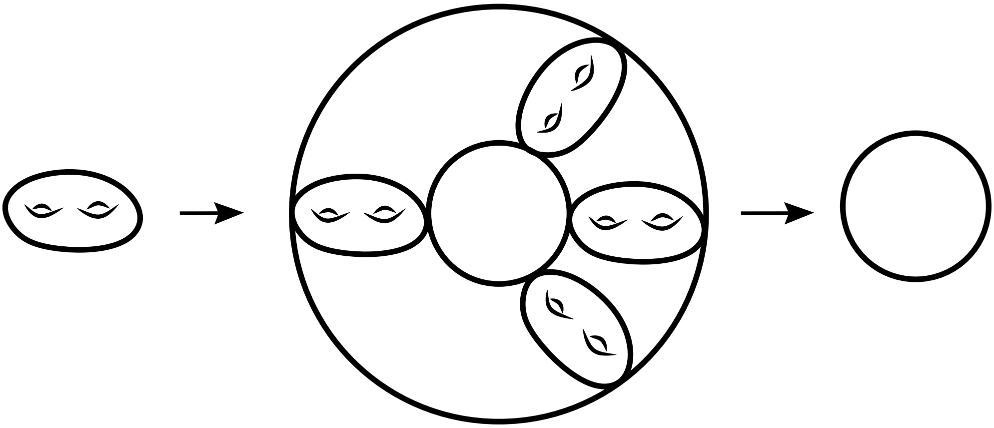
Slides available at
www.its.caltech.edu/∼dunfield/preprints.html

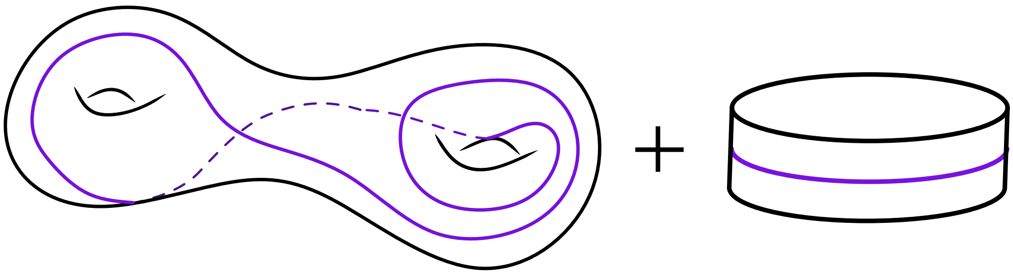
3-manifolds which fiber over S^1:



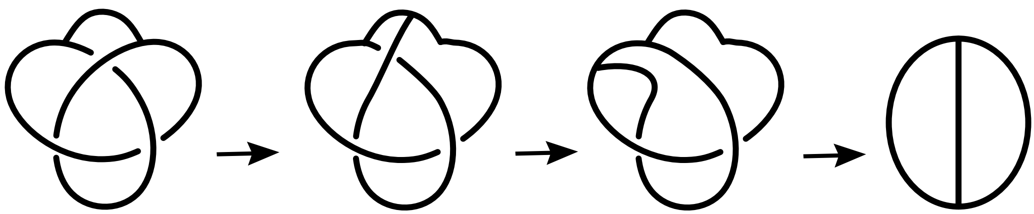
Conj. (W. Thurston) *M a compact 3-manifold whose boundary is a union of tori. If M is irreducible, atoroidal, and has infinite* π_1, *then M has a finite cover which fibers over* S^1.

Main Q: How common are 3-manifolds which fiber over S^1? Does a "random" 3-manifold fiber?

Tunnel-number one: $M = H \cup (D^2 \times I)$ along $\gamma \subset \partial H$.

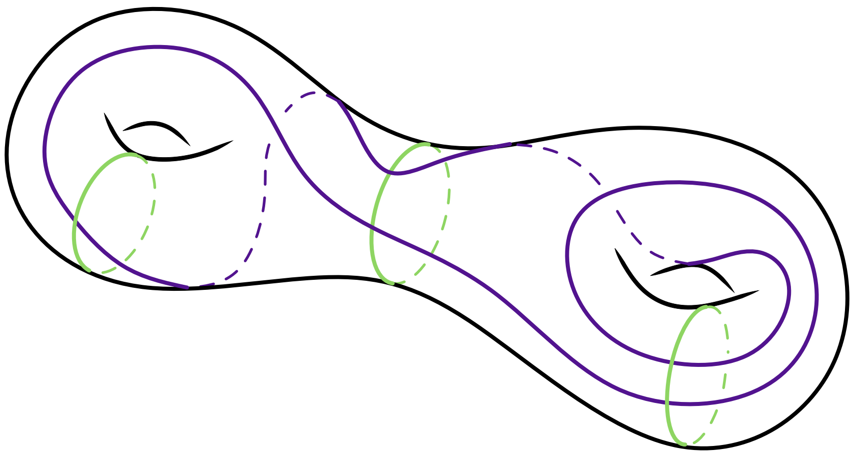


Ex: Complement of a 2-bridge knot in S^3



Key: $\pi_1(M) = \langle \pi_1(H) \mid \gamma = 1 \rangle = \langle a, b \mid R = 1 \rangle$.

Dehn-Thurston coordinates:



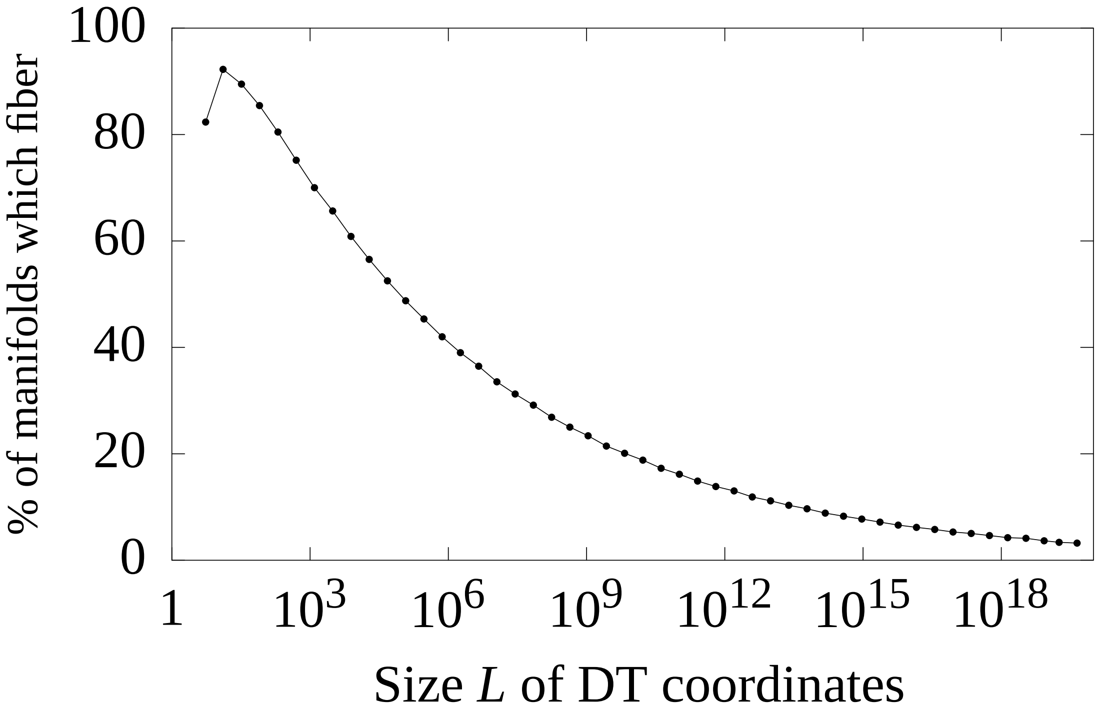
Weights:	a	b	c	;	1	2	2
Twists:	θ_a	θ_b	θ_c	;	0	1	-1

Def. *Let $\mathscr{T}(L)$ be the set of tunnel number one 3-manifolds coming from non-separating simple closed curves with DT coordinates $\leq L$.*

A random tunnel number one 3-manifold of size L is a random element of $\mathscr{T}(L)$.

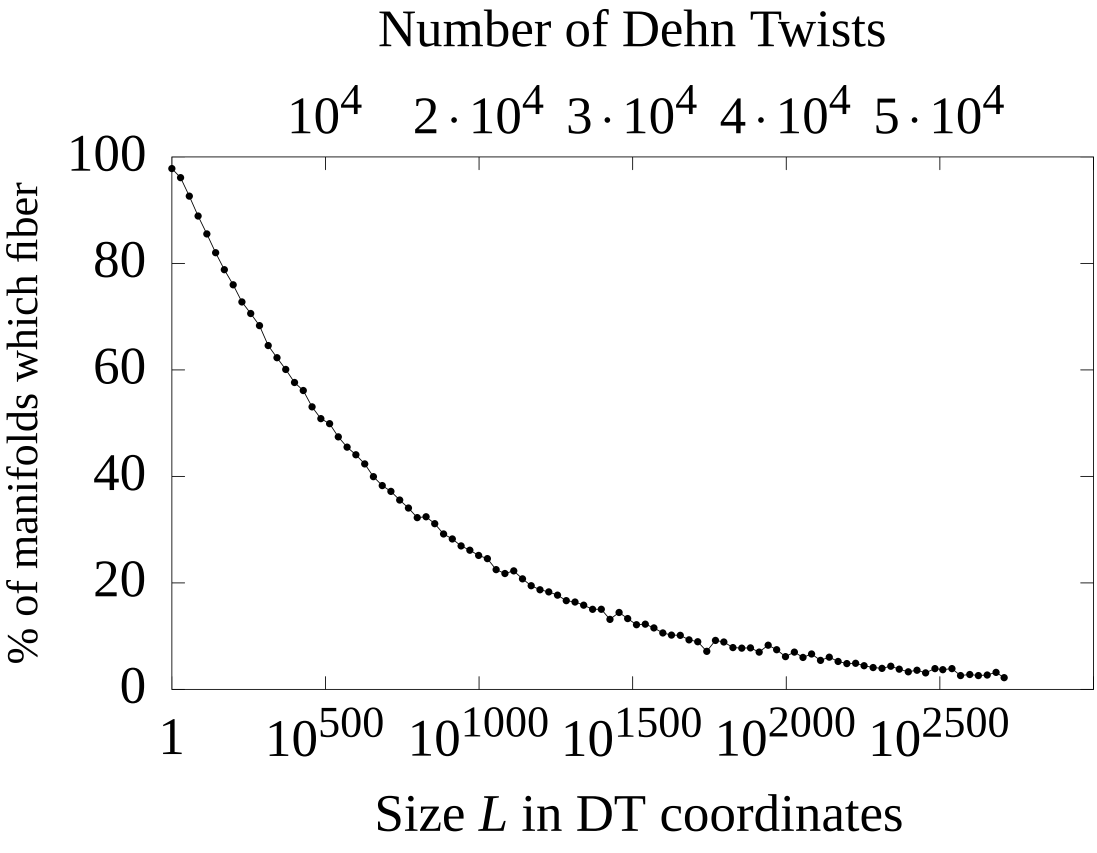
Interested in asymptotic probabilities as $L \longrightarrow \infty$.

Thm (Dunfield - D. Thurston 2005) *Let M be a tunnel number one 3-manifold chosen at random by picking a curve in DT coordinates of size $\leq L$. Then the probability that M fibers over the circle goes to* 0 *as* $L \to \infty$.



Mapping class group point of view

Fix generators of $\mathscr{MCG}(\partial H)$ and a base curve γ_0. Apply a random sequence of generators to γ_0.



Conj *With this $\mathscr{MCG}$ notion, the probability of fibering over S^1 is also* 0.

Proof ingredients:

Stallings 1962: *Determining if a 3-manifold fibers is an algebraic problem about* $\pi_1(M)$.

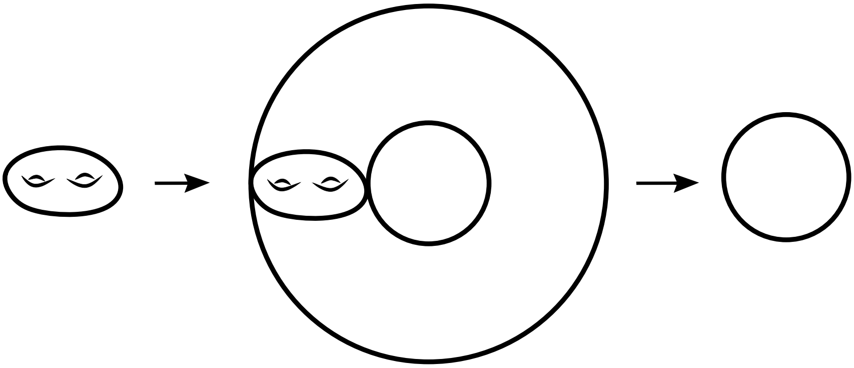
Ken Brown 1987: *If* $\pi_1(M) = \langle a, b \mid R = 1 \rangle$, *there is an algorithm to solve this algebraic problem.*

Our adaptation of Brown's algorithm to train tracks, in the spirit of Agol-Hass-W. Thurston (2002). Train tracks labeled with "boxes", which transform via splitting sequences.

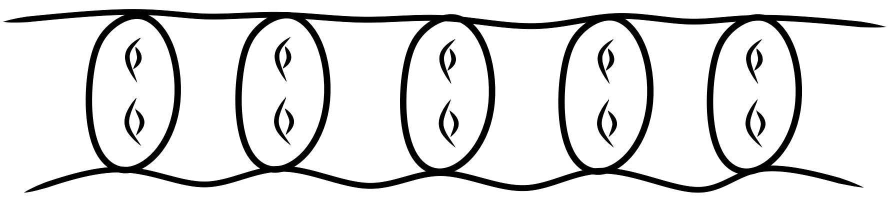
A "magic" splitting sequence which guarantees that M doesn't fiber.

Work of Kerckhoff (1985) and Mirzakhani (2003) completes the proof.

Given a general M, does it fiber?



Consider $\phi \in H^1(M, \mathbb{Z})$, can ϕ represent a fibration?



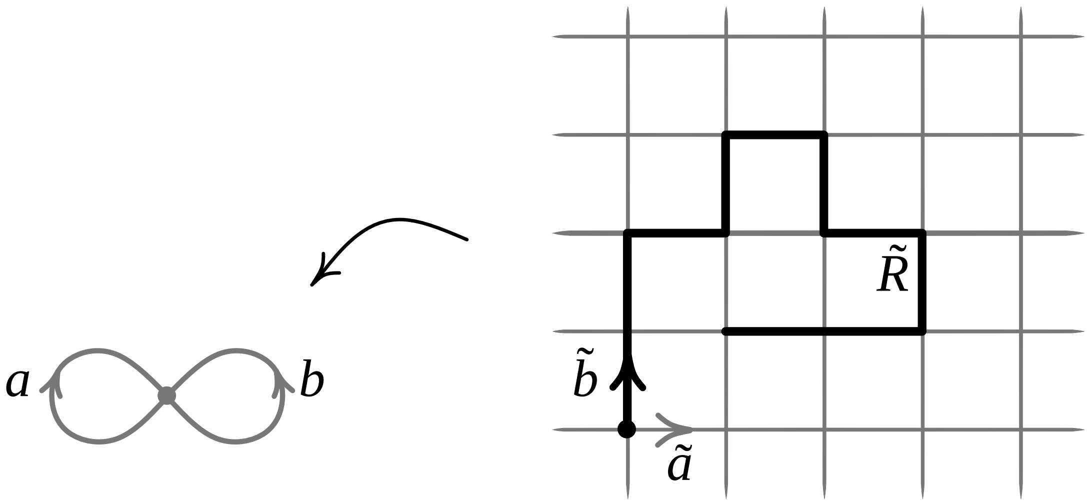
Consider $\phi_* \colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$.

Stallings: *M irreducible. Then ϕ can be represented by a fibration iff $\ker \phi_*$ is finitely generated.*

Consider $G = \langle a, b \mid R = 1 \rangle$, a quotient of the free group $F = \langle a, b \rangle$.

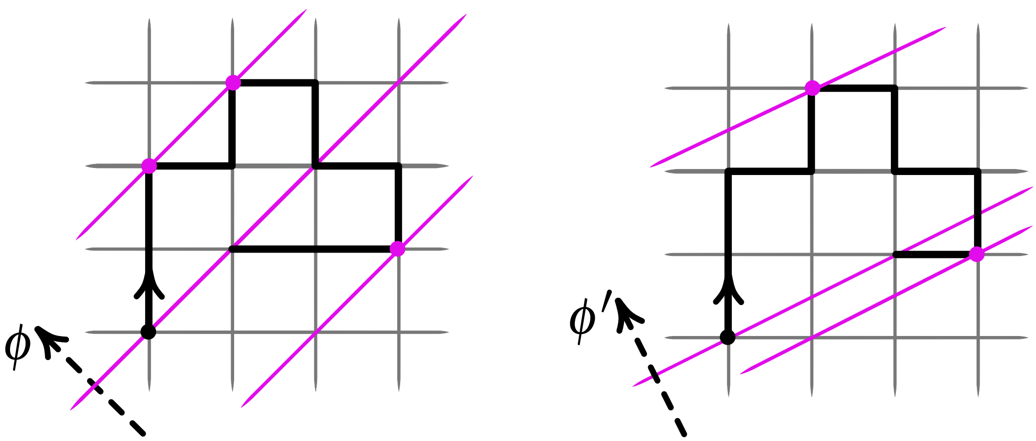
Unless $R \in [F,F]$, have $H^1(G,\mathbb{Z}) = \mathbb{Z}$.

Think of $H^1(F,\mathbb{R})$ as linear functionals on this cover:



$\widetilde{R}$ lift of $R = b^2abab^{-1}ab^{-1}ab^{-1}a^{-2}$.
$H^1(G,\mathbb{R})$ is generated by ϕ which is projection orthogonal to the line joining the endpoints of $\widetilde{R}$.

Brown: $G = \langle a, b \mid R = 1 \rangle$. $\ker\phi$ *is finitely generated iff the number of global extrema of* ϕ *on* $\widetilde{R}$ *is 2.*



$R = b^2abab^{-1}ab^{-1}ab^{-1}a^{-2}$ $\quad$ $R' = Ra$

infinitely gen (non-fibered) $\quad$ finitely gen (fibered)

Consider $G = \langle a, b \mid R = 1 \rangle$, where R is chosen at random from among all words of length L.

Q: What is the probability that G "fibers"?

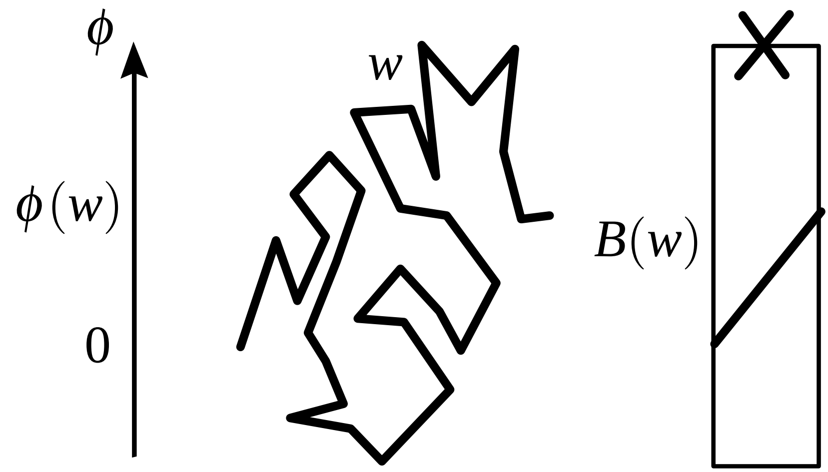
A: Experimentally, the probability is 94% (based on R of length 10^8).

Thm (DT) *p_L = probability of fibering for R of length L. Then p_L is bounded away from 0 and 1 independent of L:*

$$0.0006 < p_L < 0.975$$

Boxes: Fix $\phi\colon F \to \mathbb{Z}$. Let $w = x_1 x_2 \cdots x_n$ be a word in $F = \langle a, b \rangle$. The box $B(w)$ of w records:

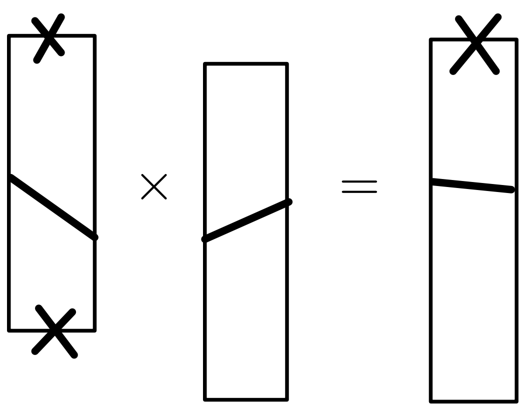
- $\phi(w)$
- The max and min of ϕ on a subwords $x_1 x_2 \cdots x_k$ and whether those maxes and mins are repeated.



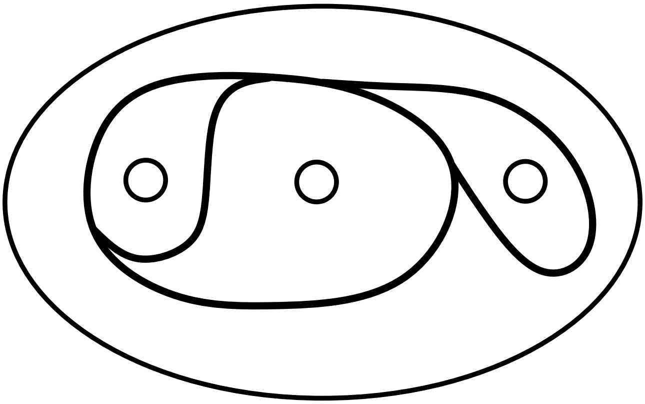
Brown's Criterion $G = \langle a, b \mid w = 1 \rangle, \phi\colon G \to \mathbb{Z}$. *Then* $\ker \phi$ *is finitely generated iff* $B(w)$ *is marked on neither the top or the bottom.*

$B(w_1 w_2) =$

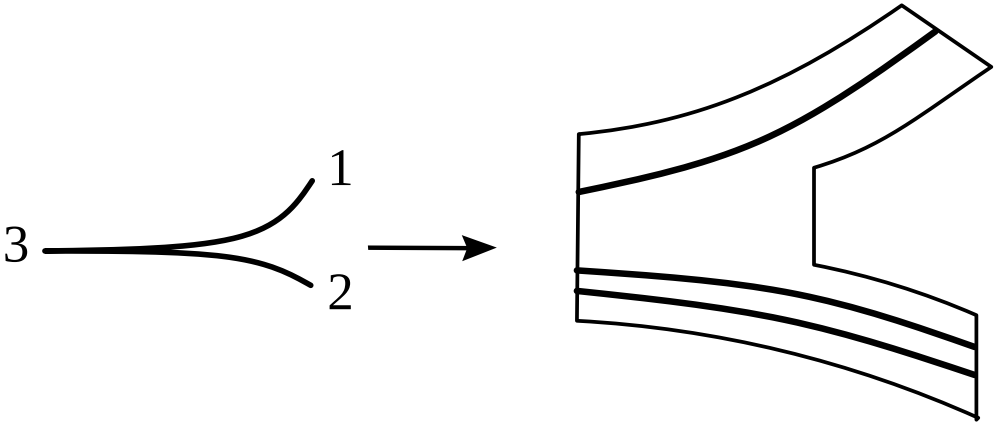
$B(w_1) \cdot B(w_2)$



Train tracks:



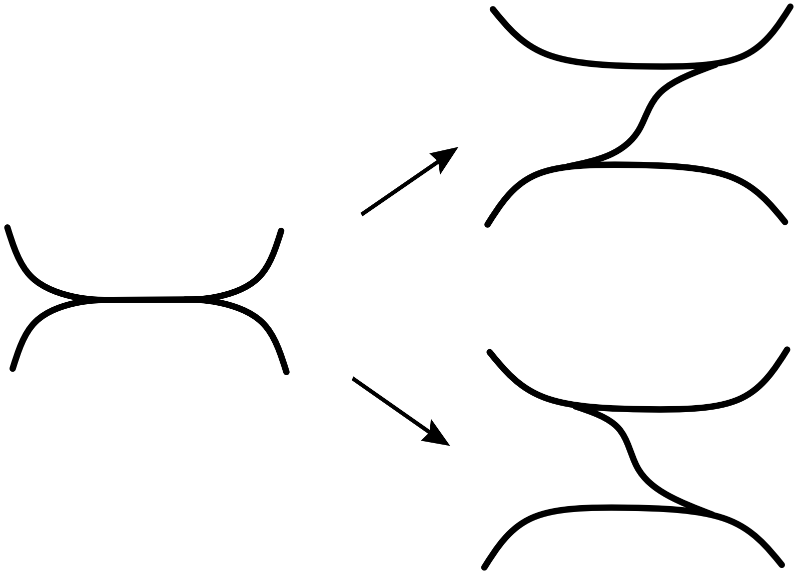
With weights, gives a multicurve:

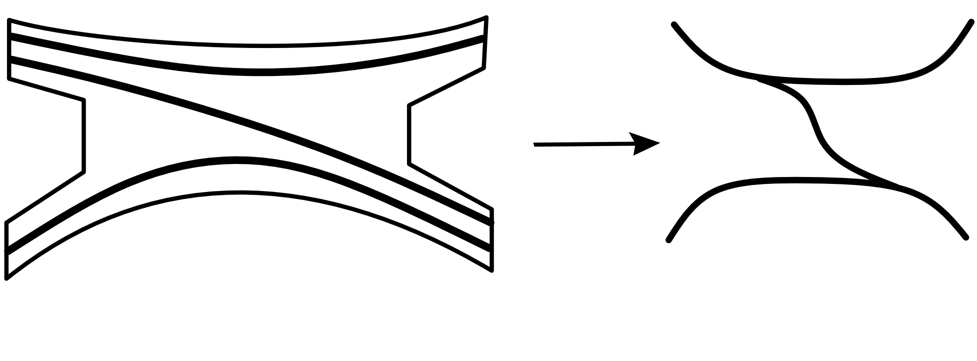


Given $\gamma \subset \partial H$ in DT coordinates, then γ is also carried by some standard initial train track τ_0.

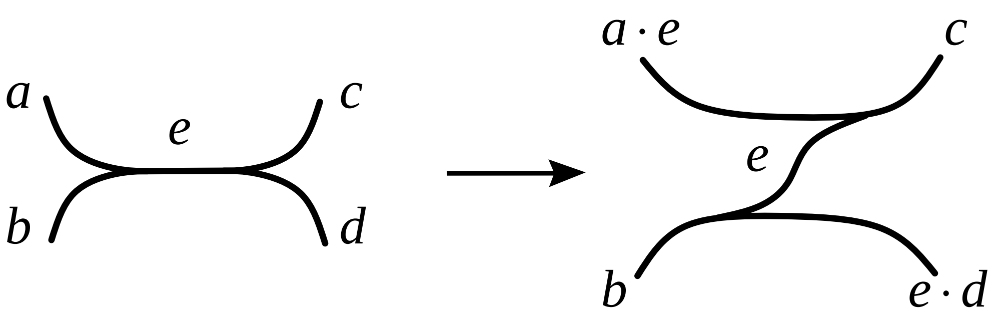
Problem: Given γ carried by τ_0 (in terms of weights) does M fiber?

Simpler question: is γ connected? Can use train track splitting to answer:





To compute the element w of $\pi_1(H)$ represented by γ, label the edges of the train track by words in w and follow along like this:

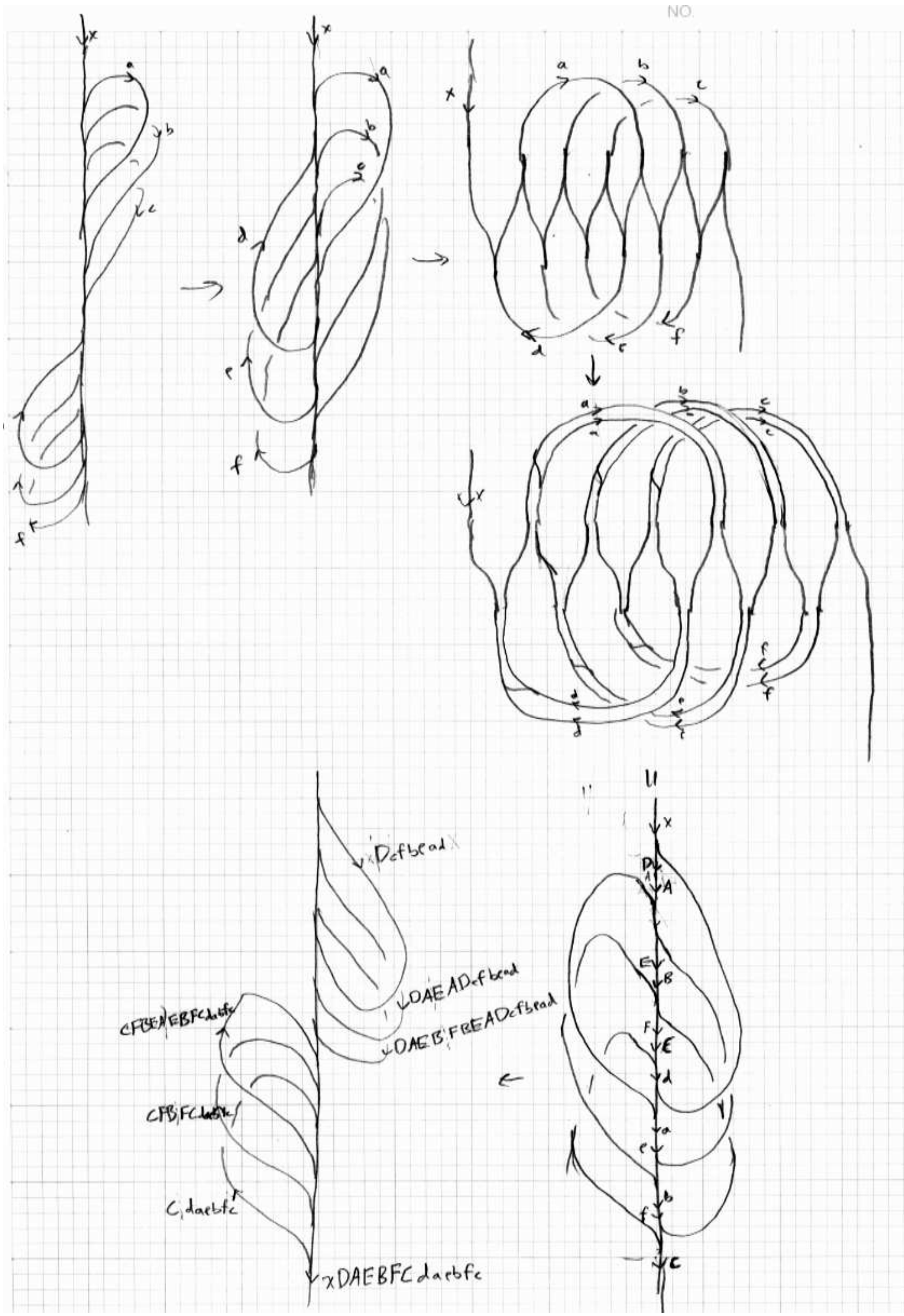


Can compute related things by applying a morphism to these labels, e.g. the class of γ in $H_1(H, \mathbb{Z})$. To apply Brown's Criterion, we label the train tracks with the corresponding boxes.

Stability: If at some intermediate stage all the boxes are marked top and bottom then M, is not fibered.

But why do we get marked boxes in the first place?

Key Lemma: If the following magic splitting sequence occurs, then at the last stage all boxes are marked. Hence M is not fibered.



Let γ be a non-separating simple closed curve on ∂H carried by τ_0 with weight $\leq L$.

Thm (DT) *The probability that M_γ fibers over S^1 goes to 0 as $L \to \infty$.*

By the key lemma, it is enough to show that the magic splitting sequence occurs somewhere in the splitting of (τ_0, γ) with probability $\to 1$ as $L \to \infty$. This follows from:

Kerckhoff 1985: *Suppose we don't require that γ be connected or non-separating. Then any splitting sequence of complete train tracks that can happen, happens with probability $\to 1$ as $L \to \infty$.*

Mirzakhani 2003: *Let Σ be a closed surface of genus 2. Let C be the set of all non-separating simple closed curves on Σ. Then as $L \to \infty$*

$$\frac{\#\{\gamma \in C \mid \text{weight} \leq L\}}{\#\{\text{All multicurves w/ coor} \leq L\}} \to c \in \frac{\mathbb{Q}_+}{\pi^6}$$