The Least Spanning Area of a Knot and the Optimal Bounding Chain Problem

Nathan M. Dunfield
University of Illinois, Mathematics

Anil N. Hirani
University of Illinois, Computer Science

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Knot in $\mathbb{R}^3$: Smooth embedding of $S^1$ in $\mathbb{R}^3$.

Spanning surface: Any knot in $\mathbb{R}^3$ is the boundary of a smooth orientable embedded surface $S$.

Knot Genus: What is the least genus of such an $S$?

Least Spanning Area: What is the least area of such an $S$?

Both questions are decidable [Haken 1960, Sullivan 1990].
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More generally, consider a closed orientable 3-manifold $Y$ containing a knot $K$.

- $Y$ is given as a simplicial complex $T$ with areas (in $\mathbb{N}$) assigned to each 2-simplex.
- $K$ is a loop of edges in $T$.
- Consider spanning surfaces which are “made out of” 2-simplices of $T$.


Thm (D-H) When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.

 Conj When $H_2(Y;\mathbb{Z}) = 0$, Knot Genus can be solved in polynomial time.
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**Agol-Hass-Thurston (2002)** For general $Y$ the Knot Genus and Least Spanning Area problems are $NP$-hard.

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**Conj** When $H_2(Y; \mathbb{Z}) = 0$, Knot Genus can be solved in polynomial time.
Algorithm uses linear programming.

**Thm (D-H)** When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.

**Approach:**

1. Consider the related Optimal Bounding Chain Problem, where $S$ is a union of 2-simplices of $\mathcal{T}$ but perhaps not a surface.

2. Reduce to an instance of the Optimal Homologous Chain Problem that can be solved in polynomial time by [Dey-H-Krishnamoorthy 2010].

3. Desingularize the result using two topological tools.
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**Homology:** $X$ a finite simplicial complex, with $C_n(X; \mathbb{Z})$ the free abelian group with basis the $n$-simplices of $X$.

Boundary map: $\partial_n : C_n(X; \mathbb{Z}) \to C_{n-1}(X; \mathbb{Z})$

Homology:

$$H_n(X; \mathbb{Z}) = \frac{\ker(\partial_n)}{\text{image}(\partial_{n-1})} = \frac{\{n\text{-dim things without boundary}\}}{\{boundaries of (n+1)\text{-dim things}\}}$$

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Example: $H_1(\text{torus}) = \mathbb{Z}^2$.

A knot $K$ in an orientable 3-manifold $Y$ gives an element of $H_1(Y; \mathbb{Z})$; when this is zero, $K$ has a spanning surface by Poincaré-Lefschetz duality. Thus if $H_1(Y; \mathbb{Z}) = 0$, e.g. $Y = S^3$ or $\mathbb{R}^3$, then every knot has a spanning surface.

Assign a “volume” to each $n$-simplex in $X$, which gives $C_n(X; \mathbb{Z})$ an $\ell^1$-norm.

**Optimal Homologous Chain Problem (OHCP)**

Given $a \in C_n(X; \mathbb{Z})$ find $c = a + \partial_{n+1}x$ with $\|c\|_1$ minimal.

**Optimal Bounding Chain Problem (OBCP)**

Given $b \in C_{n-1}(X; \mathbb{Z})$ which is 0 in $H_{n-1}(X; \mathbb{Z})$, find $c \in C_n(X; \mathbb{Z})$ with $b = \partial nc$ and $\|c\|_1$ minimal.
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**Thm (D-H)** Both OHCP and OBCP are NP-hard.

OHCP with mod 2 coefficients is NP-complete by [Chen-Freedman 2010].

**Dey-H-Krishnamoorthy (2010)** When $X$ is relatively torsion free in dimension $n$, then the OHCP for $C_n(X; \mathbb{Z})$ can be solved in polynomial time.

Key: Applies when $X$ is an orientable $n + 1$ manifold.

**Thm (D-H)** When $X$ is relatively torsion free in dimension $n$ and $H_n(X; \mathbb{Z}) = 0$, then the OBCP for $C_{n-1}(X; \mathbb{Z})$ can be solved in polynomial time.

Compare

**Thm (D-H)** When $H_2(Y; \mathbb{Z}) = 0$, the Least Spanning Area problem for a knot $K$ can be solved in polynomial time.
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Compare

Thm (D-H) *When $H_2(Y; \mathbb{Z}) = 0$, the Least Spanning Area problem for a knot $K$ can be solved in polynomial time.*

Desingularization: a toy problem

In a triangulated rectangle $X$, find the shortest embedded path in the 1-skeleton joining vertices $p$ and $q$.

Consider $b = q - p \in C_0(X; \mathbb{Z})$, which is 0 in $H_0(X; \mathbb{Z})$. Let $c \in C_1(X; \mathbb{Z})$ be a solution to the OBCP for $b$.

Claim: $c$ corresponds to an embedded simplicial path.
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Rest of desingularization

1. Pass to the exterior of the knot $K$.

2. Introduce a relative version of the Optimal Bounding Chain Problem.
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