Annoying trailers:

SnapPy  http://snappy.computop.org

What is SnapPy?

SnapPy is a user interface to the SnapPea kernel which runs on Mac OS X, Linux, and Windows. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and read the tutorial.

Contents

- Screenshots: SnapPy in action
- Installing and running SnapPy
- Tutorial
- snappy: A Python interface for SnapPea
- Using SnapPy’s link editor
- To Do List
- Development Basics: OS X
- Development Basics: Windows XP

Credits

Written by Marc Culler and Nathan Dunfield. Uses the SnapPea kernel written by Jeff Weeks. Released under the terms of the GNU General Public License.
People who heard this talk also viewed:

- J. Aaber and N. Dunfield
  
  *Closed surface bundles of least volume*

  arXiv:1002.3423
Hyperbolically twisted Alexander polynomials of knots

Nathan M. Dunfield
University of Illinois

Stefan Friedl
Nicholas Jackson
Warwick

Jacofest, June 4, 2010

This talk available at http://dunfield.info/
Math blog: http://ldtopology.wordpress.com/
Setup:

- Knot: \( K = S^1 \hookrightarrow S^3 \)
- Exterior: \( M = S^3 - \tilde{\mathcal{N}}(K) \)

A basic and fundamental invariant of \( K \) its 

**Alexander polynomial** (1923):

\[
\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]
\]
Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$. 
$A_M = H_1(\widetilde{M}; \mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^\pm 1]$, where $\langle t \rangle$ is the covering group.

As $\Lambda$ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda / (p_k(t))$$

Define

$$\Delta_M(t) = \prod_{k=0}^{n} p_k(t) \in \mathbb{Q}[t, t^{-1}]$$

Figure-8 knot:

$$\Delta_M = t - 3 + t^{-1}$$
Genus:

\[ g = \min \left( \text{genus of } S \text{ with } \partial S = K \right) \]
\[ = \min \left( \text{genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z}) \right) \]

Fundamental fact:

\[ 2g \geq \deg(\Delta_M) \]

Proof: Note \( \deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M) \). As \( A_M \) is generated by \( H_1(S; \mathbb{Q}) \cong \mathbb{Q}^{2g} \), the inequality follows.
\( \Delta(t) \) determines \( g \) for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: \( \Delta(t) = 1 \) but \( g = 2 \).

Focus: Improve \( \Delta_M \) by looking at \( H_1(\widehat{M};V) \) for some system \( V \) of local coefficients.
Assumption: \( M \) is hyperbolic, i.e.
\[
\hat{M} = \mathbb{H}^3 / \Gamma \quad \text{for a lattice } \Gamma \leq \text{Isom}^+ \mathbb{H}^3
\]

Thus have a faithful representation
\[
\alpha: \pi_1(M) \to \text{SL}_2 \mathbb{C} \leq \text{Aut}(V) \quad \text{where } V = \mathbb{C}^2.
\]

Hyperbolic Alexander polynomial:
\[
\tau_M(t) \in \mathbb{C}[t^{\pm 1}] \quad \text{coming from } H_1(\hat{M}; V_\alpha).
\]

Examples:

- Figure-8: \( \tau_M = t - 4 + t^{-1} \)
- Kinoshita-Terasaka:

\[
\tau_M \approx (4.417926 + 0.376029i)(t^3 + t^{-3})
- (22.941644 + 4.845091i)(t^2 + t^{-2})
+ (61.964430 + 24.097441i)(t + t^{-1})
- (-82.695420 + 43.485388i)
\]

Really best to define \( \tau_M(t) \) as torsion, a la Reidermeister/Milnor/Turaev.
Basic Properties:

- Can be normalized so $\tau_M(t) = \tau_M(t^{-1})$.
- Then $\tau_M$ is an actual element of $\mathbb{C}[t^{\pm 1}]$, in fact of $\mathbb{Q}(\text{tr}(\Gamma))[t^{\pm 1}]$.
- $\tau_M = \tau_M(t)$
- $M$ amphichiral $\Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}]$.
- $\tau_M(\zeta) \neq 0$ for any root of unity $\zeta$.
- Genus bound:

\[ 4g - 2 \geq \deg \tau_M(t) \]

For the KT knot, $g = 2$ and $\deg \tau_M(t) = 3$ so this is sharp, unlike with $\Delta_M$. 
Knots by the numbers:

313,231 number of prime knots with at most 15 crossings. [HTW 98]

8,834 number where $2g > \deg(\Delta_M)$.

22 number which are non-hyperbolic.
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0 number where $4g - 2 > \deg(\tau_M)$.

Conj. $\tau_M$ determines the genus for any hyperbolic knot in $S^3$.

Computing $\tau_M$: Approximate $\pi_1(M) \to SL_2\mathbb{C}$ to 250 digits by solving the gluing equations associated to some ideal triangulation of $M$ to high precision.
Many properties of $M^3$ are algorithmically computable, including

[Haken 1961] Whether a knot $K$ in $S^3$ is unknotted. More generally, can find the genus of $K$.


[Haken-Hemion-Matveev] Whether two Haken 3-manifolds are homeomorphic.

All of these plus Perelman, Thurston, Casson-Manning, Epstein et. al., Hodgson-Weeks, and others give:

**Thm.** *There is an algorithm to determine if two compact 3-manifolds are homeomorphic.*
Normal surfaces meet each tetrahedra in a triangulation $\mathcal{T}$ of $M$ in a standard way:

and correspond to certain lattice points in a finite polyhedral cone in $\mathbb{R}^{7t}$ where $t = \# \mathcal{T}$:
Meta Thm. *In an interesting class of surfaces, there is one which is normal. Moreover, one lies on a vertex ray of the cone.*

E.g. The class of minimal genus surfaces whose boundary is a given knot.

Problem: There can be exponentially many vertex rays, typically \( \approx O(1.6^t) \) [Burton 2009]. In practice, limited to \( t < 40 \).

[Agol-Hass-Thurston 2002] Whether the genus of a knot \( K \subset M^3 \) is \( \leq g \) is NP-complete.

[Agol 2002] When \( M = S^3 \) the previous question is in co-NP.
Practical Trick: Finding the simplest surface representing some $\phi \in H^1(M; \mathbb{Z}) \cong H_2(M, \partial M; \mathbb{Z})$.

Take a triangulation with only one vertex (cf. Jaco-Rubinstein, Casson). Then $\phi$ comes from a unique 1-cocycle, which realizes $\phi$ as a piecewise affine map $M \to S^1$.

Power of randomization: Trying several different $\mathcal{T}$ usually yields the minimal genus surface.
Basic Fact: If $M$ fibers over the circle then $\tau_M$ is monic, i.e. lead coefficient $\pm 1$.

Current focus: For 15 crossing knots, does $\tau_M$ determine whether $M$ fibers?

By Gabai can reduce to the case of closed manifolds.

**Practical Trick:** Proving that $N = M \setminus \Sigma$ is $\Sigma \times I$.

Start with a presentation for $\pi_1(N)$ coming from a triangulation, and then simplify that it using Tietze transformations. With luck (i.e. randomization), one gets a one-relator presentation of a surface group. This gives $N \cong \Sigma \times I$ by [Stallings 1960].
[Dunfield-Ramakrishnan 2008] Used this when $|\mathcal{T}| > 130$.


Future work: Considering $\tau_M$ as a function on the character variety.

Generic goals:

- Explain why genus bounds of $\tau_M$ are as good as those of $\Delta_M$.
- Use ideal points associated to Seifert surfaces to show nonfibered implies $\tau_M$ is nonmonic.
- Genus info?
Happy Birthday

Bus!