For $n \geq 5$, there does not exist an algorithm which solves:

**ISSPHERE:** Given a triangulated $M^n$ is it homeomorphic to $S^n$?

**Thm** (Geometrization + many results)

There is an algorithm to decide if two compact 3-mflds are homeomorphic.

**Today:** How hard are these 3-manifold questions? How quickly can we solve them?
**Decision Problems**: Yes or no answer.

**Sorted**: Given a list of integers, is it sorted?

**SAT**: Given $p_1, \ldots, p_n \in \mathbb{F}_2[x_1, \ldots, x_k]$ is there $x \in \mathbb{F}_2^k$ with $p_i(x) = 0$ for all $i$?

**Unknotted**: Given a planar diagram for $K$ in $S^3$ is $K$ the unknot?

**Invertible**: Given $A \in M_n(\mathbb{Z})$ does it have an inverse in $M_n(\mathbb{Z})$?

**P**: Decision problems which can be solved in polynomial time in the input size.

**Sorted**: $O(\text{length of list})$

**Invertible**: $O(n^{3.5} \log(\text{largest entry})^{1.1})$

**NP**: Yes answers have proofs that can be checked in polynomial time.

**SAT**: Given $x \in \mathbb{F}_2^k$, can check all $p_i(x) = 0$ in linear time.

**Unknotted**: A diagram of the unknot with $c$ crossings can be unknotted in $O(c^{11})$ Reidemeister moves. [Lackenby 2013]

**coNP**: No answers can be checked in polynomial time.

**Unknotted**: Yes, assuming the GRH [Kuperberg 2011].

**Conj**: **Unknotted** is in $P$. 
**KnotGenus**: Given a triangulation $T$, a knot $K \subset T^{(1)}$, and a $g \in \mathbb{Z}_{\geq 0}$, does $K$ bound an orientable surface of genus $\leq g$?


**Conj (AHT)** If $b_1(T) = 0$, then **KnotGenus** is in coNP.

[AHT] **KnotArea** is NP-complete.

[Dunfield-Hirani 2011] **KnotArea** is in P when $b_1(T) = 0$.

Is **KnotGenus** in P when $b_1 = 0$?

Is the homeomorphism problem for 3-manifolds in NP?

What about deciding hyperbolicity? or being an L-space?

Computing Khovanov homology and $\hat{HFK}$ are in EXPTIME. Just computing the Jones polynomial is #P-hard, but the Alexander polynomial can computed in poly time.
Normal surfaces meet each tetrahedra in a standard way:

and correspond to lattice points in a finite polyhedral cone in \( \mathbb{R}^{7t} \) where \( t = \# T \):

[Haken 1961] There is a minimal genus surface bounding \( K \) in normal form whose vector is fundamental (e.g. on a vertex ray). Hence KnotGenus is decidable.

[Hass-Lagarias-Pippenger 1999] A fund surface has coordinates \( O(\exp t^2) \).

[AHT 2006] KnotGenus is in NP.

Certificate: A vector \( x \) in \( \mathbb{Z}^{7t} \) with entries with a most \( O(t^2) \) digits.

Check:

1. That \( x \) represents a normal surface \( S \).
2. That \( \chi(S) \leq 1 - 2g \).
3. That \( S \) connected and orientable.
4. That \( \partial S \) is as advertised.

All can be done in time polynomial in \( t \) but need a very clever idea for (3) and (4).
[Kuperberg 2011] Assuming GRH, **UNKNOTTING** is in **coNP**.

**Certificate:** $\rho: \pi_1(S^3 - K) \to \text{SL}_2\mathbb{F}_p$

where $\log p$ is $O(\text{poly}(\text{crossings}))$.

**Check:** The following imply $\pi_1$ is not cyclic and so $K$ is knotted.

1. Relators for $\pi_1$ hold, so $\rho$ is a rep.
2. A pair of generators have noncommuting images.

Proof that such a rep exists uses algebraic geometry/number theory and:

[**Kronheimer-Mrowka 2004**]

When $K \subset S^3$ is nontrivial, there is a rep

$\pi_1(S^3 - K) \to \text{SU}_2$ with nonabelian image.

"In theory, there is no difference between theory and practice. But, in practice, there is."

-Jan L. A. van de Snepscheut

**Mystery:** In practice, many 3-mfld questions are easier than the best theoretical bounds indicate.

**Q:** How big a knot can we compute the genus for?

**Q:** Where do we even get big knots from?

There are more 100 crossing prime knots than there are atoms in the Earth!

Here’s a sneak peak of joint work with Malik Obeidin, based on one natural model of random knot.
Personal best:
crossings: 126
genus: 27
fibered: No
time: 7 minutes
hyperbolic volume: 223.6132847441086613
tetrahedra: 243
slope = 5.105
intercept = -10.037
r = 0.937
slope=2.331
intercept=-6.098
r=0.962