Hyperbolically twisted Alexander polynomials of knots

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Setup:

- Knot: $K = S^1 \hookrightarrow S^3$
- Exterior: $M = S^3 - \tilde{N}(K)$

A basic and fundamental invariant of $K$ is its

*Alexander polynomial* (1923):

$$\Delta_K(t) = \Delta_M(t) \in \mathbb{Z}[t, t^{-1}]$$
Universal cyclic cover: corresponds to the kernel of the unique epimorphism $\pi_1(M) \to \mathbb{Z}$. 
$A_M = H_1(\mathcal{M};\mathbb{Q})$ is a module over $\Lambda = \mathbb{Q}[t^{\pm 1}]$, where $\langle t \rangle$ is the covering group.

As $\Lambda$ is a PID,

$$A_M = \prod_{k=0}^{n} \Lambda / (p_k(t))$$

Define

$$\Delta_M(t) = \prod_{k=0}^{n} p_k(t) \in \mathbb{Q}[t, t^{-1}]$$

Figure-8 knot:

$$\Delta_M = t - 3 + t^{-1}$$
Genus:

\[ g = \min \text{ (genus of } S \text{ with } \partial S = K) \]
\[ = \min \text{ (genus of } S \text{ gen. } H_2(M, \partial M; \mathbb{Z})) \]

Fundamental fact:

\[ 2g \geq \deg(\Delta_M) \]

Proof: Note \( \deg(\Delta_M) = \dim_{\mathbb{Q}}(A_M) \). As \( A_M \) is generated by \( H_1(S; \mathbb{Q}) \cong \mathbb{Q}^{2g} \), the inequality follows.
$\Delta(t)$ determines $g$ for all alternating knots and all fibered knots.

Kinoshita-Terasaka knot: $\Delta(t) = 1$ but $g = 2$.

Idea: Improve $\Delta_M$ by looking at $H_1(\widetilde{M}; V_{\rho})$ for the system of local coefficients coming from a representation $\alpha: \pi_1(M) \to \text{GL}(V)$. [Lin 1990; Wada 1994,...]

Twisted Alexander polynomial: $\tau_{M,\alpha} \in \mathbb{F}[t^{\pm 1}]$
Technically, it’s best to define $\tau_{M,\alpha}$ as a torsion, a la Reidemeister/Milnor/Turaev.

Genus bound: When $\alpha$ is irreducible and non-trivial:

$$2g - 1 \geq \frac{1}{\dim V} \deg(\tau_{M,\alpha}) \quad \text{(⋆)}$$

Proof:

$$\deg(\tau_{M,\alpha}) = \dim H_1(\tilde{M}; V_\alpha)$$

$$\leq \dim H_1(S; V_\alpha) = (\dim V) \cdot |\chi(S)|$$

**Thm (Friedl-Vidussi, using Agol and Wise)**

*If $M$ is hyperbolic, then there exists some $\alpha$ where (⋆) is sharp.*

Idea: By Wise, $\pi_1(M)$ is virtually special, hence RFRS. By Agol, there exists a finite cover of $M$ where the lift of $S$ is a limit of fiberations. Use $\alpha$ associated to this cover.
Assumption: $M$ is hyperbolic, i.e.

$$\bar{M} = \mathbb{H}^3 / \Gamma \quad \text{for a lattice } \Gamma \leq \text{Isom}^+ \mathbb{H}^3$$

Thus have a faithful representation

$$\alpha: \pi_1(M) \to \text{SL}_2 \mathbb{C} \leq \text{GL}(V) \quad \text{where } V = \mathbb{C}^2.$$ 

Hyperbolic Alexander polynomial:

$$\tau_M(t) \in \mathbb{C}[t^{\pm 1}] \quad \text{coming from } H_1(\bar{M}; V_\alpha).$$

Examples:

- **Figure-8:** $\tau_M = t - 4 + t^{-1}$
- **Kinoshita-Terasaka:**

\[
\tau_M \approx (4.417926 + 0.376029i)(t^3 + t^{-3})
- (22.941644 + 4.845091i)(t^2 + t^{-2})
+ (61.964430 + 24.097441i)(t + t^{-1})
- (-82.695420 + 43.485388i)
\]
Basic Properties:

- \( \tau_M \) is an unambiguous element of \( \mathbb{C}[t^{\pm 1}] \) with \( \tau_M(t) = \tau_M(t^{-1}) \).
- The coefficients of \( \tau_M \) lie in \( \mathbb{Q}(\text{tr}(\Gamma)) \) and are often algebraic integers.
- \( \tau_M(\zeta) \neq 0 \) for any root of unity \( \zeta \).
- \( \overline{\tau_M} = \tau_M(t) \)
- \( M \) amphichiral \( \Rightarrow \tau_M(t) \in \mathbb{R}[t^{\pm 1}] \).
- Genus bound:

\[
4g - 2 \geq \deg \tau_M(t)
\]

For the KT knot, \( g = 2 \) and \( \deg \tau_M(t) = 6 \) so this is sharp, unlike with \( \Delta_M \).
Knots by the numbers:

313,231 number of prime knots with at most 15 crossings. [HTW 98]

22 number which are non-hyperbolic.

8,834 number where $2g > \deg(\Delta_M)$.

7,972 number of non-fibered knots where $\Delta_M$ is monic.
Basic Properties:

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8,834  number where $2g > \deg(\Delta_M)$.

7,972  number of non-fibered knots where $\Delta_M$ is monic.

0  number where $4g - 2 > \deg(\tau_M)$.

0  number of non-fibered knots where $\tau_M$ is monic.

Conj. $\tau_M$ determines the genus and fibering for any hyperbolic knot in $S^3$.

Computing $\tau_M$: Approximate $\pi_1(M) \to SL_2\mathbb{C}$ to 250 digits by solving the gluing equations associated to some ideal triangulation of $M$ to high precision.
Genus and fibering for most of these knots was previously unknown; Haken-style normal surface algorithms are impractical in this range, various tricks were used.

Q. How can we prove this conjecture?

Not known to be true for infinitely many non-fibered knots!

If conjecture and GRH are true, then knot genus is in $\text{NP} \cap \text{co-NP}$.
**Approach 1:** Deform the representation

Can consider other reps to $\text{SL}_2 \mathbb{C}$, understand how $\tau_{M,\alpha}$ varies as you move around the character variety:

**Example:** $m037$, $X_0 = \mathbb{C} \setminus \{-2, 0, 2\}$

$$\tau_{X_0}(t) = \frac{(u + 2)^4}{16u^2} \left( t + t^{-1} \right) + \frac{(u + 2)(u^4 + 4u^3 - 8u^2 + 16u + 16)}{8 (u - 2)u^2}$$

Can sometimes connect this universal polynomial to $\Delta_M$.

Ideal points corresponding to $S$: not helpful.
**Approach 2:** Use adjoint representation

\[ \text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C}) \to \text{Aut}(\mathfrak{sl}_2) \leq \text{SL}_3\mathbb{C} \]

to get \( \tau_M^{\text{adj}} \) (Dubois-Yamaguchi).

**Point:**
\[ T_{[\alpha]} X(\pi_1(M)) = H^1(M, (\mathfrak{sl}_2)^{\text{adj}} \circ \alpha) \]

8,834 knots where \( 2g > \deg(\Delta_M) \).

knots where \( 6g - 3 > \deg(\tau_M^{\text{adj}}) \).

7,972 non-fibered with \( \Delta_M \) monic.

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- 8,834 knots where \( 2g > \text{deg}(\Delta_M) \).
- 8,252 knots where \( 6g - 3 > \text{deg}(\tau_M^{adj}) \).
- 7,972 non-fibered with \( \Delta_M \) monic.
- 0 non-fibered with \( \tau_M^{adj} \) monic.
**Approach 1: Deform the representation**

Can consider other reps to $\text{SL}_2 \mathbb{C}$, understand how $\tau_{M,\alpha}$ varies as you move around the character variety:

Example: $m037$, $X_0 = \mathbb{C} \setminus \{-2, 0, 2\}$

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Ideal points corresponding to $S$: not helpful.
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8,834 knots where \( 2g > \text{deg}(\Delta_M) \).
8,252 knots where \( 6g - 3 > \text{deg}(\tau_M^{\text{adj}}) \).
12 knots where \( 6g - 9 \geq \text{deg}(\tau_M^{\text{adj}}) \).
7,972 non-fibered with \( \Delta_M \) monic.
0 non-fibered with \( \tau_M^{\text{adj}} \) monic.

Geometric isolation phenomena
Approach 3: Gauge theory

[Kronheimer-Mrowka] Instanton Floer homology detects the genus!