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N. Dunfield, *Volume change under drilling: theory vs. experiment*. Appendix to the paper of Agol, Storm, and W. Thurston, math.DG/0506338

N. Dunfield, S. Gukov, and J. Rasmussen. *The superpolynomial for knot homologies* math.GT/0505662
Does a random tunnel-number one 3-manifold fiber over the circle?

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Slides available at
www.its.caltech.edu/∼dunfield/preprints.html
3-manifolds which fiber over $S^1$:

Conj. (W. Thurston) $M$ a compact 3-manifold whose boundary is a union of tori. If $M$ is irreducible, atoroidal, and has infinite $\pi_1$, then $M$ has a finite cover which fibers over $S^1$.

Main Q: How common are 3-manifolds which fiber over $S^1$? Does a “random” 3-manifold fiber?
**Tunnel-number one:** $M = H \cup (D^2 \times I)$ along $\gamma \subset \partial H$.

**Ex:** Complement of a 2-bridge knot in $S^3$

**Key:** $\pi_1(M) = \langle \pi_1(H) \mid \gamma = 1 \rangle = \langle a, b \mid R = 1 \rangle$. 
Dehn-Thurston coordinates:

Weights: \( a \ b \ c \); 1 2 2
Twists: \( \theta_a \ \theta_b \ \theta_c \); 0 1 -1

**Def.** Let \( \mathcal{T}(L) \) be the set of tunnel number one 3-manifolds coming from non-separating simple closed curves with DT coordinates \( \leq L \).

A random tunnel number one 3-manifold of size \( L \) is a random element of \( \mathcal{T}(L) \).

Interested in asymptotic probabilities as \( L \to \infty \).
Thm (Dunfield - D. Thurston 2005) Let $M$ be a tunnel number one 3-manifold chosen at random by picking a curve in DT coordinates of size $\leq L$. Then the probability that $M$ fibers over the circle goes to 0 as $L \to \infty$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Percentage of manifolds which fiber vs. size $L$ of DT coordinates.}
\end{figure}
Mapping class group point of view

Fix generators of $\text{MCG}(\partial H)$ and a base curve $\gamma_0$. Apply a random sequence of generators to $\gamma_0$.

Conj With this $\text{MCG}$ notion, the probability of fibering over $S^1$ is also 0.
Proof ingredients:

**Stallings 1962:** Determining if a 3-manifold fibers is an algebraic problem about $\pi_1(M)$.

**Ken Brown 1987:** If $\pi_1(M) = \langle a, b \mid R = 1 \rangle$, there is an algorithm to solve this algebraic problem.


A “magic” splitting sequence which guarantees that $M$ doesn’t fiber.

Given a general $M$, does it fiber?

Consider $\phi \in H^1(M, \mathbb{Z})$, can $\phi$ represent a fibration?

Consider $\phi_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$.

**Stallings:** $M$ irreducible. Then $\phi$ can be represented by a fibration iff $\ker \phi_*$ is finitely generated.
Consider \( G = \langle a, b \mid R = 1 \rangle \), a quotient of the free group \( F = \langle a, b \rangle \).

Unless \( R \in [F, F] \), have \( H^1(G, \mathbb{Z}) = \mathbb{Z} \).

Think of \( H^1(F, \mathbb{R}) \) as linear functionals on this cover:

\[
\tilde{R} \text{ lift of } R = b^2abab^{-1}ab^{-1}ab^{-1}a^{-2}.
\]

\( H^1(G, \mathbb{R}) \) is generated by \( \phi \) which is projection orthogonal to the line joining the endpoints of \( \tilde{R} \).
Brown: \( G = \langle a, b \mid R = 1 \rangle \). \( \ker \phi \) is finitely generated iff the number of global extrema of \( \phi \) on \( \tilde{R} \) is 2.

\[
R = b^2abab^{-1}ab^{-1}ab^{-1}a^{-2}
\]

infinitely gen (non-fibered) \hspace{1cm} \( R' = Ra \)

infinitely gen (fibered)
Consider $G = \langle a, b \mid R = 1 \rangle$, where $R$ is chosen at random from among all words of length $L$.

**Q:** What is the probability that $G$ “fibers”?

**A:** Experimentally, the probability is 94% (based on $R$ of length $10^8$).

**Thm (DT)** $p_L =$ probability of fibering for $R$ of length $L$. Then $p_L$ is bounded away from 0 and 1 independent of $L$:

$$0.0006 < p_L < 0.975$$
Fix $\phi : F \to \mathbb{Z}$. Let $w = x_1x_2\cdots x_n$ be a word in $F = \langle a, b \rangle$. The box $B(w)$ of $w$ records:

- $\phi(w)$
- The max and min of $\phi$ on a subwords $x_1x_2\cdots x_k$ and whether those maxes and mins are repeated.

**Brown’s Criterion** \( G = \langle a, b \mid w = 1 \rangle, \phi : G \to \mathbb{Z} \). Then $\ker \phi$ is finitely generated iff $B(w)$ is marked on neither the top or the bottom.

$$B(w_1w_2) = B(w_1) \times B(w_2)$$
Train tracks:

With weights, gives a multicurve:

Given $\gamma \subset \partial H$ in DT coordinates, then $\gamma$ is also carried by some standard initial train track $\tau_0$.

Problem: Given $\gamma$ carried by $\tau_0$ (in terms of weights) does $M$ fiber?
Simpler question: is $\gamma$ connected? Can use train track splitting to answer:
To compute the element $w$ of $\pi_1(H)$ represented by $\gamma$, label the edges of the train track by words in $w$ and follow along like this:

![Diagram of a train track with labels a, b, c, d, e, a·e, e·d]

Can compute related things by applying a morphism to these labels, e.g. the class of $\gamma$ in $H_1(H, \mathbb{Z})$. To apply Brown’s Criterion, we label the train tracks with the corresponding boxes.

**Stability:** If at some intermediate stage all the boxes are marked top and bottom then $M$, is not fibered.

But why do we get marked boxes in the first place?
Key Lemma: If the following magic splitting sequence occurs, then at the last stage all boxes are marked. Hence $M$ is not fibered.
Let $\gamma$ be a non-separating simple closed curve on $\partial H$ carried by $\tau_0$ with weight $\leq L$.

**Thm (DT)** *The probability that $M_\gamma$ fibers over $S^1$ goes to 0 as $L \to \infty$.***

By the key lemma, it is enough to show that the magic splitting sequence occurs somewhere in the splitting of $(\tau_0, \gamma)$ with probability $\to 1$ as $L \to \infty$. This follows from:

**Kerckhoff 1985:** *Suppose we don’t require that $\gamma$ be connected or non-separating. Then any splitting sequence of complete train tracks that can happen, happens with probability $\to 1$ as $L \to \infty$.***

**Mirzakhani 2003:** *Let $\Sigma$ be a closed surface of genus 2. Let $C$ be the set of all non-separating simple closed curves on $\Sigma$. Then as $L \to \infty$

$$\frac{\#\{\gamma \in C \mid \text{weight} \leq L\}}{\#\{\text{All multicurves w/ coor} \leq L\}} \to c \in \frac{\mathbb{Q}^+}{\pi^6}$$