Surfaces in finite covers of 3-manifolds:
The Virtual Haken Conjecture

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This talk available at http://dunfield.info/
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Natural place to start: studying surfaces $\Sigma^2$ in $M^3$. Need to ignore things like:

Convention: All manifolds are orientable.

**Def.** A surface $\Sigma \neq S^2$ embedded in $M^3$ is incompressible if $\pi_1(\Sigma) \to \pi_1(M)$ is 1-1.
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Recall that $\pi_1(M)$ is the group of loops in $M$, up to homotopy:

Ex. $\pi_1(S^3) = 1$.

$\pi_1(T^3) = \mathbb{Z}^3$, where $T = S^1 \times S^1 \times S^1 = \mathbb{R}^3 / \mathbb{Z}^3$.

$\pi_1(W) = \langle a, b \mid a^2 b^2 a^2 b^{-1} a b^{-1} = b^2 a^2 b^2 a^{-1} b a^{-1} = 1 \rangle$.

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**Compressible:**

**Incompressible:** For $\Sigma = S^1 \times S^1 \times \{\text{pt}\} \subset T^3$, the map on $\pi_1$ is: $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Similarly, $\Sigma \times \{\text{pt}\}$ is an incompressible surface in $M^3 = \Sigma \times S^1$.

**Def.** A compact $M^3$ is Haken if it is irreducible and contains an incompressible surface.

Irreducible: Every embedded $S^2$ bounds a ball, that is, $M$ is not a connected sum.

An arbitrary $M^3$ is of the form $M_1 \# M_2 \# \cdots \# M_n$ where the $M_k$ can’t be further decomposed.

If $M$ is Haken, then $\pi_1(M)$ is infinite since $\pi_1(\Sigma) \leq \pi_1(M)$ and $\Sigma$ is among: 

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$\pi_1$ condition is not sufficient: Given a knot $K$ in $S^3$, Dehn surgery creates infinitely many compact 3-manifolds via $M = X \cup_{\phi} (S^1 \times D^2)$

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Closely related question: Does \( M \) contain an immersed incompressible surface? Equivalently, does \( \pi_1(M) \) contain the fundamental group of some surface?
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A lot of evidence for this conjecture including:

- True for all the manifolds coming from the figure-8 knot. [D-Thurston 2003].
- Weaker results for surgery on any knot, e.g. [Cooper-Long 1997, Cooper-Walsh 2006].
- True for all 11,000 examples in a census of simple 3-manifolds. In one case, a cover of degree 5,050 was needed! [D-Thurston 2003].
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**Rest of talk:**

- Make the conjecture weaker and prove it.
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**Real point of talk:**

- Role of geometry is crucial for this seemingly topological question (Thurston/Perelman).
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Conj. Let $M$ be an irreducible compact 3-manifold. If $\pi_1(M)$ is infinite, then $M$ has a non-trivial finite cover.

Equivalently, $\pi_1(M)$ has a subgroup $H$ with $1 < [\pi_1(M) : H] < \infty$.

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Geometrization (Thurston/Perelman):
A compact $M^3$ can be cut along spheres and incompressible tori into pieces which admit geometric structures. That is, each piece admits a homogeneous Riemannian metric modeled on one of

$$E^3, S^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \text{Nil, Sol, } \widetilde{\text{SL}_2\mathbb{R}}.$$ 

Ex: $T^3$ is Euclidean as $= E^3/\mathbb{Z}^3$, whereas $S^2 \times S^1$ has a $S^2 \times \mathbb{R}$ geometry.

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From now on, $M$ will be a hyperbolic 3-manifold, i.e. one with a metric of constant sectional curvature $-1$. Equivalently, $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}^+(\mathbb{H}^3) = \text{Möbius}(\hat{\mathbb{C}}) = \text{PSL}_2(\mathbb{C})$.

Here $\mathbb{H}^3 = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ with the metric where $ds_{\mathbb{H}^3} = 2/(1 - |x|^2)ds_{\mathbb{E}^3}$.
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Thm (Perelman 2003). Let $M$ be a compact 3-manifold. If $\pi_1(M)$ is infinite, then $M$ has a non-trivial finite cover. Equivalently, $\pi_1(M)$ has a finite-index proper subgroup.

Proof. Reduce to the case when $M$ is hyperbolic. As $M$ is compact, $\pi_1(M)$ is finitely generated and also $\pi_1(M) \leq \text{PSL}_2(\mathbb{C})$. A finitely generated group of matrices has many finite index subgroups by [Mal’tsev 1940s]. Idea: For $\text{PSL}_2(\mathbb{Z})$ we build the needed subgroup $\Lambda$ by considering:

$$1 \to \Lambda \to \text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/(p\mathbb{Z})) \to 1.$$

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**Generalizations:**

[Lubotzky 1995] Many more subgroups than just the congruence ones.

[D-Thurston 2006] Studied random Heegaard splittings. For a finite simple group $Q$, the number of $Q$-covers is Poisson distributed with mean

$$\mu = |H_2(Q;\mathbb{Z})|/|\text{Out}(Q)|.$$  

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![Diagram](image.jpg)
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Conj. $M^3$ compact hypebolic. Then $M$ has a finite cover $N$ where $H_2(N;\mathbb{Z}) \cong H^1(N;\mathbb{Z}) \neq 0$.

Equivalently, $\pi_1(M)$ has a finite-index subgroup $H$ where $H \rightarrow \mathbb{Z}$.

A tower of regular finite covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$$

exhausts $M$ if $\bigcap \pi_1(M_n) = 1$.

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**Thm (Calegari-D 2006).** There exists an $M$ with exhaustion $M_n$ where $H^1(M_n) = 0$ for all $n$.

Proof conditional on Langlands for $GL_2$ and the Generalized Riemann Hypothesis!

Thankfully, Boston-Ellenberg (2006) were able to analyze these examples unconditionally, using our picture:
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