Integration w.r.t. a prob. dist. "motive."

Have prob. dist. on \( \mathbb{R} \) (e.g., coming from a density fn.)

\[ F(t) = P\{X \leq t\} = P\{(-\infty, t]\} \] distribution function increasing \( \geq 1 \)

For \( \alpha : [a, b] \to \mathbb{R} \) will define

\[ \int_a^b \alpha(x) \, dF(x) \quad "= \int_a^b \alpha(x) \, dx \text{ for density fn.}" \]

\[ "= \sum \alpha(x_i) \, p_i \text{ for discrete.}" \]

Recall Riemann sums: For \( \int_a^b \alpha(x) \, dx \) we consider

\[ \mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n = b\} \]

Let \( \Delta x_i = x_i - x_{i-1} = \text{len} \{x_{i-1}, x_i\} \) Pick \( x_i^* \) in \([x_{i-1}, x_i]\)

\[ S(\alpha, \mathcal{P}, x_i^*) = \sum_{i=1}^{n} \alpha(x_i^*) \Delta x_i \]

Do for \( \int_a^b \alpha(x) \, dF(x) \): Let \( \Delta Fx_i = P\{[x_{i-1}, x_i]\} = F(x_i) - F(x_{i-1}) \)

\[ S_F(\alpha, \mathcal{P}, x_i^*) = \sum_{i=1}^{n} \alpha(x_i^*) \Delta Fx_i \]

Def. (Riemann-Stieltjes)

\[ \int_a^b \alpha(x) \, dF(x) \text{ exists and equals } A \]

iff \( \forall \varepsilon > 0 \exists \text{ a part } \mathcal{P} \text{ s.t. } \forall \mathcal{P}' \subset \mathcal{P} \)

we have

\[ |S_F(\alpha, \mathcal{P}', x_i^*) - A| < \varepsilon \]

for every choice of \( x_i^* \)’s.

(For more, see Apostol, Math. Anal. Ch. 7.)
Properties:

1. Suppose \( F \) has a cdf \( f \) (i.e., comes from a density \( f \)).

Then \( \int_a^b \Delta F = \int_a^b f \, dx \).

Idea: \( \Delta F_i = F(x_i) - F(x_{i-1}) = f(x_i^-) \Delta x_i \)

Thus \( S_F(x, \beta, x') = S(x, f, \beta, x') \)

Step func: \( F = \sum_{i=1}^n \alpha_i \)

Suppose \( \alpha \) is cdf at \( c \), then \( \int_0^b \Delta F = \alpha(c) \)

By:

\[
S_F = \alpha(x_i^*) \cdot \Delta F_i \cdot x_i = \alpha(x_i^*) \Delta x_i
\]

Uniform defn: Let \( X: \Omega \to \mathbb{R} \) be a random var.

Then \( \mathbb{E}(X) = \int_{-\infty}^{\infty} x \, dF_X(x) \) (provided \( \int_{-\infty}^{\infty} |x| \, dF_X(x) < \infty \))

\[
\mathbb{E}(X) = \int_{-\infty}^{\infty} x \, dF_X(x) = \sum_{i=1}^{\infty} x_i \cdot p_i
\]

Convolution: \( X, Y \) are vars

\[
F_{X+Y}(t) = F_x * F_y \overset{def}{=} \int_{-\infty}^{\infty} F_x(t-x) \, dF_y(x)
\]

\[
= \int_{-\infty}^{\infty} F_y(t-x) \, dF_x(x)
\]
Joint pmf

\[ X \text{ int valued; } P(s) = \sum_{k=0}^{\infty} P_X(s^k) = \text{E}(s^X) \]

\[ \text{independent } \Rightarrow P_{X,Y} = P_X P_Y \quad \text{E}(s^{X+Y}) = \text{E}(s^X s^Y) = \text{E}(s^X) \text{E}(s^Y) \]

Moment gen fn: of \( X \) is \( \psi_X(t) = \text{E}(e^{tX}) = \text{E}(s^X) \quad s = e^t \)

provided this expectation exists for all \( t \) in a

\[ \text{Problem: If } \psi_X \text{ exists, then } \text{E}(1X^k) < \infty \quad \forall k. \]

\[ \psi_X(0) = \text{E}(X^k)(0) \]

Solution: Characteristic Functions

Def: \( X \) random vari. The char fn is

\[ \phi_X(t) = \text{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) \]

Note: 1) How we have complex valued assn. (so what?)

2) \( \phi_X(t) \) exist for any \( t \):

\[ \int_{-\infty}^{\infty} |e^{itx}| dF_X(x) = 1. \]

Ex: \( X = c \Rightarrow \phi_X(t) = e^{itx} \)

Ex: \( X \) with normal dist

\[ \phi_X(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{itx - x^2/2} dx = e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-(x - it)^2} dx = e^{-\frac{t^2}{2}} \]

\[ = 1 \]
Example: \( X = \sum_{i=1}^{\infty} \frac{1}{i} \) and equal prob.

\[
\varphi_X = \int e^{itx} dF_X = \frac{1}{2} (e^{-it} + e^{it}) = \cos t
\]

Properties:
1. \( \varphi_X(-t) = \varphi_X(t) \)
2. \( \text{if indep } X, Y \) then \( \varphi_{X+Y} = \varphi_X \varphi_Y \)
3. \( \varphi_X \) is uniformly cont.

Proof:
\[
|\varphi_X(t+h) - \varphi_X(t)| = |E(e^{i(t+h)x} - e^{itx})| \\
= |E(e^{it}(e^{ihx} - 1))| \leq E(|e^{it}|)E(|e^{ihx}|) \\
= E(e^{ihx} - 1)
\]

Since \( E(e^{ihX} - 1) \) is small as \( h \to 0 \),

\[
|e^{ihx} - 1| \to 0 \text{ pointwise}
\]

and \( |e^{ihx} - 1| \leq 2 \) for all \( X \).

So:
\[
\int g(x) dF_X = \int_{-\infty}^{-N} g(x) dF_X + \int_{|x|>N} g(x) dF_X
\]

for fixed \( N \),

this is small as \( h \to 0 \).

Example of:

Lebesgue Dominated convergence thm. 

\[
N \to \infty \\
\int_{|x|>N} \to 0
\]

Next time: \( \varphi_X \) determines \( F_X \)

\( X \to Y \) in dist \( \iff \varphi_{X_n} \to \varphi_Y \) pointwise.

Extracting moments from the \( \varphi_x \).